School Math in USA and Russia

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My name is Andrei Toom. I am a matematician of Russian origin interested in mathematical education. I graduated and got a Ph. D. from Moscow University and participated in several educational projects in Russia including mathematical olympiads and circles and School by Correspondence. Also I have done research and taught mathematics in Russia, USA and Brazil. Today I shall compare school mathematics in Russia and USA.

To live in USA is better than in Russia in several respects; indeed, I have voted with my feet. But in one respect Russia is much ahead of USA: in the quality of mathematical education. I do not mean that Russia is an exception: there are several countries, where mathematical education is reasonably good, including some East-European countries and some countries at the Far East. Let us start with the most undeniable task of the school: Mental and paper-and-pencil calculations.

Mental and paper-and-pencil calculations in Russia

From time immemorial it was a commonplace in all countries that one of the most important duties of the school was to teach children mental and p-p calculations. In many countries, including Russia, this conviction is still valid in spite of omnipresence of electronics. By the end of elementary school all Russian children are well trained in numerical calculations. Even when calculators became available, mastery of mental and p-p calculations by the end of elementary school remained *sine qua non* for every student. And of course memorization of multiplication table at least to 10×10 was part of it. There are many number puzzles, which improve children's mastery of numbers. Just a few examples:

Problem 1 *Write a three-digit number twice without any space, so that you get a six-digit number. This number surely is a multiple of seven. Why?*

Problem 2 *Calculate* $\sqrt{0,999999999}$ *with nine digits after the decimal point.*

Problem 3 *Is there a number, which has more than one digit and equals the product of its digits ?*

You can easily find more puzzles at different places including Kordemsky's famous book. English translation is available [7].

Mental and paper-and-pencil calculations in America

Let me start with a personal recollection. My daughter was twelve years old when she came to USA. She started to attend a public middle school and one of my first surprises in America was that my daughter's teacher tried to foist a clculator into every student's hand to do all calculations, even the simplest ones. Mental and p-p calculations were considered obsolete and provincial. Calculator became a symbol of modern times and teachers who allowed their students to calculate mentally or on paper felt obsolete and inadequate. My wife and I were horrified. With our Russian mentality we deemed mental and p-p calculations essential for healthy development.

Private schools might be immune against the calculators fad, but we had no money to pay a private school. So my daughter had to stay where she was, but I decided to warn her; every time when I saw a calculator in her hand, I called her a "victim of American education". She got my point and tried to calculate mentally as much as possible. By the end of the school she was one of a few students who could calculate mentally. Most of the others needed a calculator whenever they wanted to multiply seven by nine or three by eight. What was the cause of this fad which made American teachers eradicate mental and p-p calculations? To answer this question, let us speak about "Standards" of mathematical education in both countries.

Russian Standards

In Russia educational decisions are issued by the Ministry of Education and they are obligatory for all those who are involved in public education. Let us concentrate our attention on two important Russian documents published under one cover. They are about school mathematical education [4, first part, p. 111-123]. These documents pertain to all children, not to some special ones. Here is the first one:

The standard of the general education in mathematics (p. 111). It includes sections: **(1) Arithmetics, (2) Algebra, (3) Geometry, (4) Elements of Logic, Combinatorics, Statistics and Probability Theory.** Each section includes several subsections.

For example, the section **Geometry** includes subsections: **Initial notions and theorems of geometry, Triangle, Quadrangle, Polygons, Circumference and circle, Measurement of geometric quantities, Vectors, Geometric transformations, Constructions with ruler and compass.**

Each subsection includes several items. For example, the subsection **Triangle** includes the items:

• Right-angled, acute-angled and obtuse-angles triangles. The altitude, median, bisector, midline of a triangle. Isosceles and equilateral triangles; necessary and sufficient conditions of an isosceles triangle.

• Sufficient conditions of congruence of triangles. The triangle inequality. Sum of angles of a triangle. External angles of a triangle. Dependence between magnitudes of sides and angles of a triangle.

• Thales' theorem. Similarity of triangles; coefficient of similarity. Sufficient conditions of similarity of triangles.

• Pythagorean theorem. Sufficient conditions of congruence of right-angled triangles. Sine, cosine, tangens, cotangens of an acute angle of a rectangular triangle and of angles from 0^o to 180^o; reduction to acute angle. Solution of right-angled triangles. The main trigonometric identity. Formulas connecting sine, cosine, tangens, cotangens of one and the same angle. The cosine and the sine theorem; examples of their use to compute elements of a triangle.

• The remarcable points of a triangle: points of intersection of midperpendiculars, bisectors, medians. The Euler circle.

This document is followed by a six-page additional document (same volume, p. 118-123) "Requirements to the level of preparation of students". The previous document speaks almost nothing about problems, but the second document fills this gap. It starts with the following preamble:

> **In result of the study of mathematics a student must know/understand**
> *the essence of the notion of mathematical proof, examples of proofs;*
> *the essence of the notion of algorithm; examples of algorithms;*
> *how mathematical formulas, equations and inequalities are used; examples of their use to solve mathematical and practical problems;*
> *how functions defined mathematically may describe real dependencies; to present examples of such a description;*
> *how needs of practice brought the mathematical science to necessity of enlargement of the notion of number;*

the probabilistic character of many laws of the surrounding world; examples of statistical relations and conclusions;
how geometry appeared from practical tasks of land survey; examples of geometric objects and statements about them, important for practice;
the meaning of idealization, which allows to solve problems of reality by mathematical methods, examples of errors resulting from idealization.

Then follow sections with the same titles as the sections of the first document, but the first document concentrated on theory and the second document states necessity of solving problems and argumentation. For example, the section "Geometry" includes several items including:
- to solve geometrical problems based on already studied properties of figures and relations between them, using additional constructions, algebraic and trigonometric tools, ideas of symmetry;
- to perform proofs in the form of arguments in solving problems, using already studied theorems, finding possibilities to use them.

These documents provide the "obligatory minimum"; advanced regions or districts or even individual schools may upgrade it, but nobody may degrade it. It is not perfect, but in any case every teacher or administrator knows where he or she is. What I want to emphasize and what I especially like about Russian standards is exactly this quality, for which some American educators scornfully call Standards written in this style "laundry lists". Let us look at some American standards to see how they are different from Russian.

American Standards

There is no exact counterpart of the Russian Standards in USA. The simplest reason is that in USA there is no Ministry of Education. Instead there is the triumvirate consisting of the federal Department of Education or DoE for short, the National Science Foundation or NSF for short and the National Council of Teachers of Mathematics or NCTM for short. Roughly speaking, the business of DoE is administration, the business of NSF is finance and NCTM defines to a large extent, what is going on in the classrooms. Also all states and even smaller units have their local educational officials. It is they who usually decides, which textbooks wii be adopted by local schools in every year.

Of course, NCTM is not government and formally speaking its recommendations are not obligatory, but the system of financing used by NSF and approved by DoE is such that friends of NCTM are financed much better than its enemies. From time to time NCTM publishes books expressing its opinions about what should take place at the school lessons of mathematics. Let us concentrate on three of these issues.

1980: "Agenda for Action".

Since the USSR launched "Sputnik" in 1957, American educators were under pressure to improve the public education. In 1980 NCTM published a 30-page document called "An Agenda for Action" [1], which should better be called An Agenda for Inaction. On its first page we find a list of recommendations, which is a bizarre mixture of suggestions, some of which look sound (at first sight), some are unclear and some are outright unsound. Let us quote and comment on them.

Recommendation 1: problem solving be the focus of school mathematics in the 1980s.

This seemed sound or at least acceptable to me at first. However, the subsequent developments showed that this recommendation in fact meant "let us not teach theory". More than that, as soon as the authors of Agenda and their friends escaped from the Scylla of theory, they started to avoid the Charybdis of problems. To this end they claimed that the existing problems were too bad for them and proclaimed creation of new, much better problems, which never materialized.

Recommendation 2: basic skills in mathematics be defined to encompass more than computational facility.

This actually meant: let us not teach computational facility and nobody ever clarified what exactly beyond computational facility should be encompassed.

Recommendation 3: mathematics programs take full advantage of the power of calculators and computers at all grade levels".

This recommendation is the most harmful of all recomemndations of "Agenda". We have seen consequences of it.

Recommendation 4: stringent standards of both effectiveness and efficiency be applied to the teaching of mathematics.

This sounds great, but in fact is too vague. We shall see, what kind of "standards" were offered by NCTM in the next devades.

Recommendation 5: the success of mathematics programs and student learning be evaluated by a wider range of measures than convetional testing.

The same criticism: too vague.

Recommendation 6: more mathematics study be required for all students and a flexible with a greater range of options be designed to accomodate the diverse needs of the student population.

This recommendation caused the greatest harm. First, a good deal of students were sent to pseudo-programs like "consumer math". Second, no teacher could assume that all of his students already know something. So interaction between subjects was impossible. For example, a teacher of science had to completely exclude mathematics from his lessons.

Recommendation 7: mathematics teachers demand of themselves and their colleagues a high level of professionalism.

This seems good, but what is "high level"?

Recommendation 8: public support for mathematics instruction be raised to a level commensurate with the importance of mathematical understanding to individuals and society.

This is vague and useless.

1989 "Standards"

In 1989 NCTM piblished "Curriculum and Evaluation Standards for School Mathematics" [16], the first volume of the three-volume publication expressing its vision at that time.

First of all, I must say that "standards" is a difficult reading for a mathematician who has got used to expect exact meanings. It is written in a very fancyful manner, many words have strange meanings or seem to have no definite meanings at all. For example, chapters are called "standards", which gives impression that there are some standards there.

But if you apply effort and concentrate, you notice that this looseness is not only in *how* it is written, but also in what it *recommends*. This document is written by people, for whom all mathematics is but a disordered collection of interchangeable appendices to their vague generalities.

Although there is no clear general declaration to this end, in fact the "Standards" time and again recommend to use calculators without any warning against their overuse. Influential educators explained that use of calculators would save the student's time to develop "high-level thinking skills", but nobody ever described these skills in detail.

Now about the three chapters "Mathematics as reasoning" in the "standards": they don't even mention any of the famous facts like that the set of prime numbers is infinite or none rational number has its square equal to 2. What is there? The high-school chapter starts with tampering with calculator! If you don't believe me, look by yourself. What about the Pythagorean theorem, it is mentioned in the "standards" on pp. 113-114 with a well-known picture, which can be used to prove it, but it is only proposed to use this picture to "discover this relationship through exploration". The possibility to prove this important theorem is not even mentioned and the very idea of proof is avoided throughout the document.

You will find no "laundry lists", that is lists of topics to study in these "Standards". Instead their authors prefer to turn heads of school teachers by pompous suggestions. This is an example, a quote from the "Standards" [16, p. 157]:

> *Prior to the work of the ancient Greeks (e.g. Thales and Pythagoras) geometrical ideas were tied directly to the solution of real-world problems. Hence, the subsequent abstraction and formalization of these ideas, which evolved into the subject of geometry as we know it, has always had many applications in the real world. More recently, fractal geometry, which originated in the mid-1970s with the pioneering work of Benoit Mandelbrot, has provided useful models for analyzing a wide variety of phenomena, from changes in coastlines to chaotic fluctuations in commodities prices.*

This text would be nice **in addition** to Russian style standards, but not **instead** of them. American "Standards" go straight from ancient Greece to fractals without even mentioning all those concrete topics, which were listed in Russian standards; this is too mundane for them. In fact they offer a bright dream instead of modest reality. In result many teachers naively tried to teach fractals knowing only that they produce periodic patterns. Why wall-paper is not a fractal, they could not explain. Most of them had no idea that the notion of dimension had any role in this play. My friend, an American teacher of mathematics, attended a meeting where a guest from NCTM suggested that all teach fractals. My friend (whose competence was above average) was the only one who dared to admit that he was not competent enough to teach fractals. "But this is very simple - said the guest - look, this is a sheet of paper. Now I crumple it and it is a fractal!" At the same time the 1989 "Standards" recommended to decrease attention to really teachable topics: "Euclidean geometry as a complete axiomatic system", "Two-column proofs" etc (p. 127).

Absence of fractals in Russian standards does not prevent Russian children from their study as much as they are interested. Browsing Google for the word "fractal" in Russian, I found 649000 references in 0.44 seconds. But Russians make a clear distinction between what is obligatory for all and what is advanced and good only for enthusiasts.

Another example of turning heads of American teachers: on p. 157 the "standards" recommend to "develop an understanding of an axiomatic system through investigating and comparing various geometries", which is widely interpreted as a suggestion to teach non-Euclidean geometry, which is impossible to do having eliminated almost all the logical structure.

Another example: on p. 169 of "Standards" we find this: "College-intending students should become familiar with such distributions as the normal, Student's t, Poisson, and chi square". You will find nothing similar in Russian standards. Indeed, introduction into calculus is present in a more advanced version of Russian standards (same book, second part), but the normal distribution, Student's t, Poisson, and chi square need much more as a base than introduction into calculus. It seems that the authors of the 1989 "Standards" were just too ignorant to comprehend what they were suggesting.

Ignorance of powerful leaders grossly desorient American teachers. They invest all their ambitions into unrealistic projects and neglect realistic ones. It does not come to their minds that they could teach more complicated word problems or prove some theorems. Their usual ambition is to teach calculus in high school and doing this they often cause more harm than good. Teaching Calculus in an American university I had a hard time with some of those students who had had Calculus in the school. They had memorized a few formulas and thought that it was Calculus.

The 1989 Standards were widely proclaimed across the country. The president of NCTM declared that all criticisms of these Standards were mean-spirited and self-serving. (Like in Andersen's tale "The Emperor's New Clothes".) So it became urgent for all the teachers in the country to teach to the Standards. But the bulk of these Standards was very vague; the only clear recommendation was to have children use calculators in the classroom and the more the better. At least this is the message which many teachers got from it. Several years later it became clear that this fad had created a disaster. NCTM started to pretend that the "Standards" had not really promoted calculators so much. Indeed, the 1989 "Standards" are so vague that it is impossible to say for sure what exactly they promote. Several years later my daughter noticed that the teachers started to forbid use of calculator even when it was appropriate. But for many students it was too late.

Problems by type

Page 126 of the 1989 "Standards" [16] is devoted to *topics to receive increased attention* and the first topic is **The use of real-world problems to motivate and apply theory.**

Page 127 is devoted to *topics to receive decreased attention.* and the first topic is **Word problems by type, such as coin, digit, and work.**

It is natural to compare these two recommendations. When I read the 1989 "Standards" first time, I was confused. I thought: 'Do coins exist in real life ? It seems that they do. Then why should they receive decreased attention if real-world problems should receive increased attention ? And what about work? It also seems to exist in real life. And if coin problems should receive decreased attention, what to do with problems about paper money or checks or money orders ?'

I shared my doubts with some American colleagues and they kindly explained to me (in private, not in public) that the phrase 'by type' meant the widely used but uncreative manner of teaching, when the teacher starts by describing in detail a certain method and then the students solve many almost identical problems using exactly this method. Taught in this way students can solve problems of a certain type, but often get lost when given even a slightly different problem. I understand this explanation, but I still do not understand what is written on those pages. If 'by type' means some way or manner of teaching, then how can it be listed as 'topic'? And what do coin, digit and work here? After publication of these "Standards", many teachers refused to teach coin, digit and work problems and wanted instead to teach pizza problems. The more problems, such as coin, digit or work, are purged from the curriculum, the more uniform and monotonic become those few problems that remain. The main problem is not with the word problems, but with the poor preparation of American teachers.

Real-world problems

Now let us try to understand, what is a "real-world problem". There is only one problem explicitly called "real-world" in the whole document [16]. In fact, it is called REAL-WORLD PROBLEM SITUATION. Here it is on p. 139:

Problem 4 *In a two-player game, one point is awarded at each toss of a fair coin. The player who first attains n points wins a pizza. Players A and B commence play; however, the game is interrupted at a point at which A and B have unequal scores. How should the pizza be divided fairly? (The intuitive division, that A should receive an amount in proportion to $A's$ score divided by the sum of $A's$ score and $B's$ score has been determined to be inequitable.)*

THIS IS FOLLOWED BY "PROBLEM FORMULATION":

"Consider the situation with the following data: The winning score is $n = 10$; when the interruption occurs, the score is $A : B = 8 : 7$. The pizza will be divided in proportion to each player's probability of winning the game."

This "problem formulation" is equivalent to that described by Pascal in his letter to Fermat on August 24, 1654 and it is used in introductory textbooks of probability, e.g. in the excellent books by Chung [3, p. 26-28] and Snell [15, p. 3-5]. Both Chung and Snell refer to Pascal's letter and provide interesting historical background. On the other hand, the "standards" completely omits all historical details, which makes all the situation far-fetched. I asked many sympatizers of "standards" to refer to any of their acquaintances who ever played this game and received none positive answer. So in which sense it is "real-world"?

If the authors were serious about real-world equity or fairness, they might recommend to divide pizza equally or give a bigger piece to the more hungry player, but certainly **not** to divide food by gambling. Administrators of orphanages would be horrified by this idea; they care that each pupil consumes all the food he needs for his health and not gambles it away. Further, what is the mathematical meaning of the word "inequitable"? If there is no such meaning, then how could it be "determined" that some division is "inequitable"? What does the word "determined" mean? The author is trying to herd the readership towards the well-known solution without explaining in which sense this solution is correct, which is anti-mathematical, even anti-rational. Further, the author pretends that the requirement to divide the pizza in proportion to each player's probability of winning the game appears only in the "Problem formulation", but in fact it is implicitly present from the very beginning, because otherwise it can not be "determined" that to share the pizza in proportion of the players' scores is "inequitable". This assumption is implicit, but unstated, which is anti-pedagogical. Even leaving all this aside, educational value of this example is very doubtful. It may seem to be an advantage to include probability into the high school curriculum, but in fact the "Standards" avoid any theory: it just tampers with one numerical example, which is a step backwards by comparison with any reasonable version of the traditional curriculum.

So what does the phrase "real-world problems" really mean ? Browsing through 1989 "Standards", I found some statements about these mysterious beings. On p. 76 (middle-school part) it is said:

> *The nonroutine problem situations envisioned in these standards are much broader in scope and substance than isolated puzzle problems. They are also very different from traditional word problems, which provide contexts for using particular formulas or algorithms but do not offer opportunities for true problem solving.*

What a nonsence! With their narrow experience the authors pretend to set standards! Are they aware of the rich resourses of excellent traditional word problems around the world? Let us read further:

> *Real-world problems are not ready-made exercises with easily processed procedures and numbers. Situations that allow students to experience problems with "messy" numbers or too much or not enough informations or that have multiple solutions, each with different consequences, will better prepare them to solve problems they are likely to encounter in their daily lives.*

Pay attention that the author uses future tense. This means that he or she has never actually used such problems in teaching and never observed influence of this usage on his or her students' daily lives. He or she has not even invented such problems because he or she does not present any of them. Nevertheless, he or she is quite sure that these hypothetized problems will benefit students. What a self-assurance!

After such a pompous promise, the author owes us an explanation. He or she ought to give several examples of these magic problems. Indeed, we find a problem on the same page, just below the quoted statement. Here it is:

Problem 5 *Maria used her calculator to explore this problem: Select five digits to form a two-digit and a three-digit number so that their product is the largest possible. Then find the arrangement that gives the smallest product.*

It is not said whether the five selected digits should be different from each other. If these digits may coincide, the answer is easy: 99×999. Let us assume that all the five digits must be different. But this problem has **none** of the qualities attributed to "real-world problems" on the same page: there is neither too much nor not enough information and there are no multiple solutions, each with different consequences. Does it contain *messy numbers*, remains unknown because the author never explained what is a messy number.

Now, what is expected from students in connection with this problem? It seems that the author expects Maria to try several cases, to choose that one which provides the greatest product and to declare that it is the answer. But what if the right choice never happened to come to her mind? This is very bad pedagogics. Also let us notice that Maria is expected only to "explore" this problem rather than to solve it. Where I came from, exploration is only the first stage towards a complete solution. Do the authors expect Maria ever to attain a complete solution? Do they want children to **solve** problems or just to tamper for a while?

One colleague noticed that the book still contains some problems described in general on page 76. Indeed, there are, but in another document. Here is one of them:

Problem 6 *You have 10 items to purchase at a grocery store. Six people are waiting in the express lane (10 items or fewer). Lane 1 has one person waiting, and lane 3 has two people waiting. The other lanes are closed. What check-out line should you join? [16, p. 212].*

I have never read any report about usage of this problem. Also I have never read any solution of this problem. Irresponsibility again!

It must be said that the "standards" have a wide popularity among American educators. I think it is because the vague feelings of the authors are close to the vague feelings of their audience. All of them feel that their teaching is too rigid, mechanical, uninspired and want to make it more flexible, more human, but they are not competent enough to keep mathematics on this way: as they move towards more human approach, they lose mathematics.

In fact, the description of "real-world" problems quoted above explains the authors' inability to provide examples of them. Since these perishable fruits are not ready-made, they cannot be found in a book, because any book is ready-made. Although the "standards" is an exceptionally careless document, it was revised once before publication (according to its preface), so it cannot contain "real-world problems" because a problem revised at least once becomes ready-made and cannot be real-world any more. Real-world problems just happen in the actual course of daily life, at least this is what I conclude from the "standards". What the authors lose from sight is that a mathematician, used to concentrate on abstractions, is not the best person to deal with such events. An experienced handyman or surgeon or life-saver or fireman or police officer would be much more heplful. It may be a good idea to teach children to cope with emergency situations, but it is not mathematics.

Another relevant statement can be found on page 157 of "standards" (high-school part): "Prior to the work of the ancient Greeks (e.g. Thales, Pythagoras), geometric ideas were tied directly to the solution of real-world problems." Thus problems solved thousands years ago, when there was no theory, were "real-world problems". But page 126 recommends to use them to motivate and apply theory. How can problems posed and solved in the absense of theory motivate and apply theory?

Now let us compare what is written on pages 76 and 157. Remember that in the ancient Egypt farmers paid taxes depending on the area of their plots of land. So measurement of ground was a very responsible job. Now imagine, how would a peasant of ancient Egypt react if the government official in charge of measuring area of his farm would obtain several answers, each with different consequences? Does he want a bribe? - might guess the peasant. You bet, that awkward official would be hastily removed lest he might provoke a rebellion.

Throughout all these years of bitter arguments about what this or that phrase of "standards" really meant, its authors never interfered with explanations. It looks like they wrote the "standards" in such a somnambulic state of mind that afterwards could not explain rationally what did they mean. Nevertheless this irresponsible document is awed by American educrats. "Principles and Standards for School Mathematics" or PSSM for short, the next vision of NCTM, published eleven years later, writes about the "standards": "Since their release, they have given focus, coherence, and new ideas to efforts to improve mathematics education [14, p. ix].

Wars in American mathematical education

The triumvirate of DoE, NSF and NCTM is powerful, but not omnipotent. Sometimes it faces disobedience. In October of 1999 the US Department of Education headed by Richard W. Riley approved ten K-12 mathematics programs by calling five of them "exemplary" and the other five "promising". This decision was based on conclusions of an Expert Panel, most members of which had neved published a research article in mathematics.

This is not an accident. Although there are plenty of bright mathematicians in USA, for a long time they were not invited to participate in making important decisions about education. Sincerely speaking, they did not especially object. Like all people, mathematicians are prone to avoid extra work and sometimes say: "Why should I bother myself with public education? There are special people to care about it." So they did for a long time in America (but not in Russia). However, this time some of them decided to act. On November 18, 1999 The Washington Post published a letter signed by 200 mathematicians and other scientists urging Riley to withdraw his department's approval. NCTM immediately expressed a complete support for Riley's decision, which is understandable because the Expert Panel based its criteria partially on NCTM publications including the 1989 "Standards". Riley answered to the mathematicians' letter by reaffirming his position. Thus the American public observed an open confrontation between mathematicians and scientists on one side and educational officials and leaders on the other. What was it about?

The most evident point of confrontation is whether children should be taught paper-and-pencil arithmetical algorithms or use calculators instead of that. The difference of opinions can be illustrated by two quotes, both included into the mathematicians' letter. One is from an article written by Steve Leinwand, a member of the Expert Panel (and one of directors of NCTM), entitled "It's Time To Abandon Computational Algorithms" and published on February 9, 1994, in Education Week on the Web. This article clamed:

> *It's time to recognize that, for many students, real mathematical power, on the one hand, and facility with multidigit, pencil-and-paper computational algorithms, on the other, are mutually exclusive. In fact, it's time to acknowledge that continuing to teach these skills to our students is not only unnecessary, but counterproductive and downright dangerous.*

This is just a personal opinion, but this person was selected by the government to make an important decision about mathematical education. The other quote is from a report made by a committee formed by the American Mathematical Society (AMS) to represent its views to NCTM [2]:

> *We would like to emphasize that the standard algorithms of arithmetic are more than*

just 'ways to get the answer' – that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials.

Pay attention that this statement was made only in 1997 and published only in 1998, which was too late because the "standards" had recommended the usage of calculators instead of paper and pencil already in 1989 and at that time AMS seemed to support it because the preface to the 1989 "Standards" declared (p. vi): "The following mathematical science organizations join with the National Council of Teachers of Mathematics in promoting the vision of school mathematics described in the Curriculum and Evaluation Standards for School Mathematics", followed by an impressive list including AMS. What did this "promoting the vision" mean? Neither AMS nor NCTM ever made a public statement about it. When the "standards" started to be implemented in some classrooms and mathematicians became aware of what was going on there, they became horrified and only then, probably, some of them looked attentively under the cover of "standards". Notices of AMS published several letters urging AMS to withdraw its endorsement (whatever it meant), but there was no comment from the headquarters of AMS.

Principles and Standards of School Mathematics

The nest book of the sequence of "Standards" was issued in 2000. It is called "Principles and Standards for School Mathematics". We shall call it PSSM for short [14]. Like in 1989 "Standards", chapters of PSSM are also called "standards", which leads to the same confusion.)

PSSM consists of eight chapters.

Chapter 1 presents a general vision.

Chapter 2 lists the main principles for School Mathematics, namely The Equity Principle, The Curriculum Principle, The Teaching Principle, The Learning Principle, The Assessment Principle, The Techology Principle. After that:

Chapter 3 deals with all ages.

Chapter 4 deals with Pre-K-2.

Chapter 5 deals with grades 3-5.

Chapter 6 deals with grades 6-8.

chapter 7 deals with grades 9-12.

All the chapters from 3 to 7 have similar structure. Each one of them consists of the following sections: Number and Operations, Algebra, Geometry, Measurement, Data Analysis and Probability, Problem Solving, Reasoning and Proof, Communication, Connections, Representation.

It is evident that the Russian standards are much easier to comprehend and use than American ones. Even a very busy teacher can find time to browse the 13 pages of the Russian "Standards" and easily conclude what to teach. Say, triangles are well represented, but complex numbers are completely omitted. You may like or dislike it, but at least you know where you are.

To deal with American "Standards" is by far not so easy. The PSSM has 402 pages and no index. In addition, material is put into chapters in fanciful ways and chapters are given strange titles, so if you want to find, say, a proof of the Pythagorean theorem (if it is there), you have to browse all the 402 pages of the book. I looked for that proof in the sections "Geometry" and "Reasoning and Proof", but failed. I looked for it in the chapters with strange titles, which allow to put there everything, like "Communication", "Connections" and "Representation", but failed again. Only after a time-consuming search I found it in the section "Algebra" in the chapter devoted to 9-12 grades (p. 301) with a caption "An algebraic explanation of a visual proof of the Pythagorean theorem." I still don't know, what PSSM says (if any) about quadratic equations or properties of triangles or trigonometry or logarithms or complex numbers or combinatorics or several other topics because I have no time to browse it again and again. So PSSM, althogh issued ten years ago, never was thoroughly criticized because nobody has enough time for that. An efficient way to avoid criticisms! The only style of teaching compatible with these "Standards" is chaotic, when a teacher browses the chapter corresponding to the grade, finds something appropriate and brings it to the class.

Another failure of PSSM - almost complete absence of problems. PSSM has several chapters named "Problem Solving", on pp. 52, 116, 182, 256, 334, but every one of these chapters contains at most one or two problems and some vague discourses. To give you some taste of the level and style of this book, let me quote one of its problems (p. 257)

Problem 7 *Over the past few weeks, the American Movie Corporation has introduced two new kinds of candy at the concession stands in movie theaters in town. For three weeks, two theaters have offered Apple Banana Chews. For two weeks, five other theaters sold Mango Orange Nips. Only one of the* *two types of candy was sold at each theater, and all the theaters showed the same movies and had roughly the same attendance each week during the introductory period. During that period, 660 boxes of Apple Banana Chews and 800 boxes of Mango Orange Nips were sold. Suppose you have been hired by the company to help them determine which candy sold better. Use the information to decide which type of candy was more popular, and carefully and completely explain the basis for your answer.*

After that the author pompously writes:

> *This problem can help students see the need to go beyond superficial approaches and to dig deeply into their understanding of ratios and rates.*

There is not one word about use of this problem in a real classroom with real students. So all this pomp is based on nothing but the author's imagination! Speaking seriously, this problem is boring because the three-word names of the candies are cumbersome. But some modern American educators think that such lengthy names make the problem more attractive. (Fat chance!) No attempt is made to make a general conclusion or represent it algebraically. Instead the author recommends:

> *Teachers should regularly ask students to formulate interesting problems based on a wide variety of situations, both within and outside mathematics.*

Thus the author proposes to waste the limited time of mathematics lessons on chatter about irrelevant themes.

Common Core State Standards

In the last months a group of American educators proposed what they called "Common Core State Standards" or CCSS for short in English language and Mathematics. I shall comment only the Mathematics part. I must admit that this version of "Standards" is better than all American school math "Standards" mentioned above. Now we find many concrete topics placed in a reasonable order, so a prospective author of a textbook will have some idea, what to write.

Without discussing these "Standards" in detail, I shall only point at their most vulnerable point: after reading them, I still have no clear idea, which problems will children solve. Russian Standards quoted above also contain no concrete problems, but in Russia there is a strong tradition and a large pool of good problems, while in USA this tradition has been artificially demolished. Even the CCSS is not completely free from this fad. For example, they put Statistics before Probability and do not mention Combinatorics at all in that context, although it is well-known that Statistics is based on Probability, which in its turn is based on Combinatorics.

Are there no authors of good problems in America? Of course, there are: it is sufficient to mention those who organize mathematical olympiads and various competitions. But it seems that none of them was invited to co-author CCSS (or any previous version of "Standards" mentioned above). For many years American educrats are on non-speaking terms with authors of good problems. It seems that even the managers of CCSS failed to change this.

Word problems in Russia

By non-word problems I mean problems, which include only mathematical notations and formulas and some mathematical phrases like "solve the equation". Accordingly, by word problems I mean problems, which contain words, which are not mathematical terms and need to be interpreted mathematically. Word problems are so common in Russia that the mere word "problem" usually means a word problem, while non-word problems are called exercices.

At the K-12 level there is not much room for sophisticated formalisms of professional mathematics, so the greatest variety in school math is provided by word problems. Word problems bring to a classroom a plethora of images, such as coins, buttons, matches and nuts, time and age, work and rate, distance and speed, length, width, perimeter and area, fields, boxes, barrels, balls and planets, price, percentage, interest and discount, volume, mass and mixture, ships and current, planes and wind, pumps and pools etc. etc. It is an invaluable experience for children to discern those formal characteristics of these images, which should be taken into account to solve a problem.

What is at least equally important, in my opinion, is that solving word problems, children have to comprehend and translate into mathematics a multitude of verbs, adverbs and syntactic words indicating actions and *relations* betwen objects, such as put, give, take, bring, fill, drain, move, meet, overtake, more, less, later, earlier, before, after, from, to, between, against, away etc. Although I say "children", I actually mean a wide range of ages, including American college undergraduates, for whom all this may be quite a challenge.

When do Russian children start to solve multi-step problems?

I have spent the first fourty five years of my life in Russia, where presence, even abundance of word problems in mathematical education was always taken for granted.

In Russia children start to solve word problems already in elementary school, long before they start to study algebra. Words problems solved without algebra are called "arithmetical" and they come in a wide range of sophistication. One measure of their difficulty is the number of steps, that is arithmetical operations necessary to solve a problem. According to my observations, Russian children
start to solve one-step problems at the end of the first grade,
start to solve two-step problems at the end of the second grade,
start to solve three-step problems at the end of the third grade,
start to solve four-step problems at the end of the fourth grade
and they solve more sophisticated word problems after that.
Thus, when they start algebra (sixth grade), Russian children already have various arithmetical skills, which help them to comprehend algebra.

When I was in elementary school, a Russian writer Nikolay Nosov published a story [10], which immediately became very popular. The main character, Vitya Maleyev, has just started his fourth grade at school. Last year he failed in math and promised the teacher that he would catch up. So he solves several problems at the third grade level. This is one of these problems (p. 624):

Problem 8 *A boy and a girl collected 24 nuts. The boy collected twice as many nuts as the girl. How many nuts did each collect?*

First Vitya does not know what to do. He draws a picture of a boy and a girl. He draws two pockets on the boy's coat. and one pocket on the girl's apron to express the fact that the boy collected twice as many nuts as the girl. Then he looks at his picture and sees three pockets. Then suddenly a thought "like a lightning" comes to him: to divide the total number of nuts into three parts! He divides 24 by 3 and gets 8, which is the number of nuts in a pocket. Thus the girl has 8 nuts. The boy has twice of 8, that is 16. Vitya adds 8 and 16 and gets 24, so he checks his answer and sees that it is right. Vitya is very excited: this is first time in his life that he solved a problem completely on his own.

In my opinion this is a piece of good mathematical education. One should not think that Vitya is an exceptionally gifted or priviledged child. On the contrary, this episode is similar to what has been typically going on in Russian schools for many decades independently of political regime.

Nowadays arithmetical word problems constitute a respectable part of Russian education. Their presence is explicitly mentioned in the Russian standards and approved by prominenent scientists. For example, Vladimir Arnold [1937-2010], a world-famois Russian mathematician expressed his approval of arithmetical word problems. At one interview [9] he remembered the following problem.

Problem 9 *Two old women started at sunrise and each walked at a constant velocity. One went from A to B and the other from B to A. They met at noon and, continuing with no stop, arrived respectively at B at 4 p.m. and at A at 9 p.m. At what time was the sunrise on that day?*

Arnold remembered that solving this problem independently (in 1949 when he was 12 years old) was his first real mathematical experience, uses the words *revelation* and *feeling of discovery* to describe this experience and said that in Russia his experience was not unusual.

When do American children start to solve multi-step problems?

First a personal memory. My daughter always liked art and humanities more than exact sciences, but like all Russian children she solved various word problems at shool and did not find them especially difficult. When she came to America and went to school, she found that there were three streams of mathematics for her age: slow, middle and fast. As a newcomer, she was put in the middle. But she soonly noticed that all the problems, which they solved, were one-step. She asked to transfer her to the fast group, but found there the same one-step problems. I shared my daughter's experience with an e-mail discussion list devoted to mathematical education. Here are some answers:

> *Somebody asked why word problems were so rare in math textbooks. The reason should be obvious – they scare elementary school teachers to death. (Eric Lee Green, 09/05/1994).*
>
> *As an elementary teacher who works with many other elementary teachers and middle school teachers, I agree with Eric when he says word problems "scare teachers to death" and that provide a lot of frustration for all students. In my discussions with teachers, I*

find that many of them feel this way because they feel unprepared to teach mathematics. Teaching out of the book with the answers right there is much safer than using word problems that require the teacher first, to be able to solve the problem themselves, and second, that there might be other ways to solve it. I have been in workshops for middle school teachers that were supposed to be on methods of teaching math and instead turned into classes on the mathematics involved. I tell teachers about my second graders who subtracted 32-17 by saying 2 minus 7 is -5 and 30 minus 10 is 20 and 20 and -5 is 15. Almost all ask me to explain it again because they aren't comfortable with negative numbers and have a difficult time understanding what the children were doing. (Lynn Nordstrom, 10/05/1994).

Oooo... Word Problems! Why aren't there more of them in the text? Here's a story from the past... The first year I taught Algebra (17 years ago) I was approached by my department head. He told me that the rest of the math department just skipped the word problems because they were too difficult for the students. (Some of my students would like me to take his advice :)) (Mark Priniski, 09/05/1994).

I would say your experience is quite typical. This is in response to the posting about word problems not being done in high school. Hopefully, it's changing. A lot of teachers avoid word problems because they are hard for the students, they are hard to teach, and they take up a good quantity of time. In interest of "covering the curriculum" some things have to be left out. Why not word problems? My personal feeling is that word problems are very important. They show the students how mathematics can be used. Maybe we could work on making better problems for them. A lot of textbook problems are of the "cookbook" variety. My students tell me they "can't do word problems, have never been able to do them, and don't expect to ever be able to do them." I think it is lack of experience. By the way, I teach precalculus. (Sandra Halfacre, 20/03/1995).

These were voices from the real teachers who face real students every day. Somebody may dismiss these observations as anecdotal. But ignorance of American teachers is well documented. Just an example: the article published lately by Patricia Clark Kenschaft in the Notices of AMS [6]. The article is full of eye-opening examples. This is one of them (p. 209): in 1986 Kenschaft visited a K-6 elementary school and discovered that *not a single teacher* knew how to find the area of a rectangle:

"What is the area of a rectangle that is x high and y wide?" I asked. $< \cdots >$ "x plus y?" said two in the front simultaneously. $< \cdots >$ Then all fifty of them shouted together, "x plus y."

Although Kenschaft's main concern is education of black children, she observes that mathematical knowledge of teachers of schools with mostly white students is poor also (p. 210):

My first time in a fifth grade in one of New Jersey's most affluent districts (white, of course), I asked where one-third was on the number line. After a moment of quiet, the teacher called out, "Near three, isn't it?" The children, however, soon figured out the correct answer; they came from homes where such things were discussed. Flitting back and forth from the richest to the poorest districts in the state convinced me that the mathematical knowledge of the teachers was pathetic in both. It appears that the higher scores in the affluent districts are not due to superior teaching in school but to the supplementary informal "home schooling" of children.

Another example: Liping Ma's famous book [8]. In it Ma showed that American school math. teachers knew mathematics worse than their Chinese colleagues and even worse than Chinese high school students. In particular, the following problem was given to various subjects:

Problem 10 *People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?*

$$1\frac{3}{4} \div \frac{1}{2} =$$

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$?

This is what Liping Ma reports in her book [8, pp. 56-72]:

Of the 23 U.S. teachers, 21 tried to calculate $1\frac{3}{4} \div \frac{1}{2}$. Only nine (43%) completed their computations and reached the correct answer. $< \cdots >$ All of the 72 Chinese teachers *computed correct and complete answers to the problem.* < ··· > *Although 43% of the U.S. teachers successfully calculated* $1\frac{3}{4} \div \frac{1}{2}$ *, almost all failed to come up with a representation of division by fractions.* < ··· > *While only one among the 23 U.S. teachers generated a conceptually correct representation for the meaning of the equation, 90% of the Chinese teachers did.*

Restore two numbers from their sum and difference

Let us concentrate on one small class of problems, where two numbers must be restored from their sum and difference. When I was in the fourth grade (ten years old), we solved the following problem (I don't remember exact numbers):

Problem 11 *A plane has two gasoline tanks. The total amount of gasoline in both tanks is 30 liters. The first tank contains 4 liters of gasoline more than the second. How much gasoline is there in each tank?*

A similar problem is included in a modern Russian textbook, also for the fourth grade:

Problem 12 *An ancient artist drew scenes of hunting on the walls of a cave, including 43 figures of animals and people. There were 17 more figures of animals than people. How many figures of people did the artist draw? [5, p. 11]*

A similar problem is included in the 5-th grade Singapore textbook:

Problem 13 *Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive? [13, p. 23]*

The problem is followed by visual hints: there are two horizontal bars named Raju and Samy and the data are shown very clearly in this picture. In addition, there is a girl, from whose head a thought appears in a cloud, saying "Give Raju $100 first and divide the remaining money into two units." Two pages further a similar problem is given to solve independently:

Problem 14 *John is 15 kg heavier than Peter. Their total weight is 127 kg. Find John's weight. [13, p. 25]*

There are many other useful pictures in Singapore textbooks. I believe that solving word problems without algebra with help of various visual images is very good pedagogics.

Similar problems are solved without algebra in Japan, also in the 4-th grade, according to Stevenson and Stigler's "Learning Gap" [17]. On p. 187 they describe a Japanese teacher explaining the following problem to 4-th graders:

Problem 15 *There is a total of 38 children in class. There are 6 more boys than girls. How many boys and how many girls are there?*

Stevenson and Stigler add that "its solution is generally not taught in US until students take a course in algebra" and write on the next page:

> *With this concrete visual representation* $< \cdots >$ *and careful guidance from the teacher, even fourth-graders were able to understand the problem and its solution.*

What does it actually mean, "until students take a course in algebra"? Even now most American children meet algebra first time in their life at the eighth or ninth grade. Attempts to enlist all eight-graders into Algebra have led to failures and frustrations: it turned out that many of them were not prepared for that. An example: on 9/21/2008 the newspaper "USA today" placed a large headline "Study: Many 8th-graders can't handle algebra" with a detailed article by Greg Toppo, which includes a statement "It's hard to teach a real algebra class if you have kids who don't know arithmetics". This publication provoked many comments from the readers. I like this one:

> *It actually has very little to do with the students and even the teachers. Unfortunately, the main problem are the University Education Professors that are teaching our teachers ineffective and in some cases detrimental instructional practices. What I mean is Math Education Professors are actually teaching instructional techniques for math that simply DO NOT work with many children, especially students who are at risk for failure. The idea that students must "construct" or "discover" math is why this country's math program is an utter failure. Basically, the math education system is revisiting the "whole language" debacle that occurred in reading. History certainly*

repeats itself. Since 1989, the math instruction in elementary and middle schools has completely deemphasized basic facts and computational fluency. Again, Math Educators (University Professors) took the position that it was NOT important and completely ignored it. It is this ridiculous philosophy combined with a spiraling curriculum that put our children far behind children in other countries in the area of mathematics. Until the professors in Math Education start teaching our future teachers how teach math in a systematic and explicit fashion, the US math performance will remain poor and significantly lag behind other countries.

My opinion is that these frustrations come from poor preparation of teachers who can teach neither arithmetics nor multi-step word problems. Children of several countries solve in 4-5 grades by arithmetical means such word problems, which American kids cannot solve even in the eighth grade because they never were taught to solve word problems without algebra.

However, the leaders of American education carefully avoid to admit that teachers should be better prepared and I can explain why. Having admitted this, they would have to admit that schools of education should devote more time to subjects which their students will teach, at the expence of hours devoted to vague generalities. George Polya quoted one prospective teacher say, "The mathematics department offers us tough steak which we cannot chew and the school of education vapid soup with no meat in it".

Again about my daughter, this time in high school. The teacher wrote a word problem on the board and proposed her students to solve it. In fact the teacher had a teachers' edition of the textbook in her hands, where a solution was explained. Following this book, the teacher recommended to denote one quantity by X and write an equation. My daughter noticed that another quantity could be chosen to denote it by a letter; so she solved the problem in a slightly different way and asked the teacher if her solution was correct. But this was beyond the teacher. She could not decide whether my daughter's solution was correct and asked her to solve the problem in the recommended way. In this case, as in many others, the teacher missed an opportunity to compare two solutions of a problem; not because she did not wish well for her students, but because she was constrained by her ignorance.

The Hard-Trivial Paradox

Another strange, but widespread, idea is that word problems are more uniform than non-word ones. For example, the influential “Agenda for Action” recommends [1, p. 3]:

> *The definition of problem solving should not be limited to the conventional “word problem” mode.*

What did the authors mean by the ‘conventional “word problem” mode’? Perhaps, that uninspiring manner of teaching which still plagues classrooms, and which is caused by poor preparation of teachers? Who knows?

Another example: “Summary of changes in content and emphases in 9-12 mathematics” [16] recommends to decrease attention to “word problems by type” and never mentions non-word problems by type or word problems not by type. So in fact this recommendation moves the teachers to avoid any word problems because nobody can tell which of them are by type. This recommendation shows that the authors feel that there is something wrong with teaching word problems, but fail to analyze what exactly is wrong. They say nothing about how to teach them. Or, perhaps, the phrase “by type” means some bad manner of teaching? Who knows...

How did that strange idea of uniformity of word problems come into existence? I think that some teachers and educators, too incompetent to cope with the richness of word problems, reduced them to a few types, and this secondary phenomenon, which went *against* the grain of word problems and came from mere incompetence rather that from potential of word problems, was mistaken more than once for an inalienable feature of word problems. It seems that word problems were almost always taught so badly in USA that most students could not separate word problems from the dreadful manner of teaching. Ralph Raimi is one of those few who made this distinction (personal communication):

> *I was a tractable student and did what I was told, and they told me what boxes to put certain numbers into, for a limited range of problems, few enough to memorize. It was hard going, and I later realized how easy the problems were, but since I was told how to do them, and since I was rewarded with praise, that’s what I did, totally without insight. Nor did the insight emerge as it does in language learning, when one puts words into sentences and inflections on verbs in a sort of continuous process of accretion. In my case algebra did not come to me that way, and what I learned later, that caused me to see how idiotic my high school exercises were, was not rooted in the boxes I had learned*

earlier. The fault was not in the problems, nor in the "type" idea. The fault was in the teaching.

Thorndike and his no-transfer theory

It is only very natural that American educators try to get rid of problems, which the teachers cannot teach. To get a scientific appearance, they use the legacy of Edward L. Thorndike, the classic of American psychology and education. The following is a quote from his book published in 1926 with several co-authors. On p. 137 he discusses what he calls "genuineness". As an example he quotes and discusses the following problem:

> *For example, "In ten years John will be half as old as his father. In twenty years he will be three-fifth as old as his father. How old is John now? How old is his father?" In reality such a problem would only occur in the remote contingency that that someone knowing that John was 10 and his father 30, figured out these future ratios, then forgot the original 10 and 30, but remembered what the future ratios were!*

This quote is just one display of Thorndike's strange theory. In more general terms, Thorndike proposed to atrubute grades from 0 to 10 to problems according to their "genuineness", that is realism. If a problem would "never occur as a problem in life, in whole or in part; nor would anything at all closely like it", it gets a grade zero and should be purged from the textbboks. If a problem could "occur just as it is in every detail, quantities, relations, and all", it gets the maximal grade ten and is welcomed in the textbooks [18, p. 139].

This classification is a part of a strange but widespread, feature of American mathematical education, which completely ignores connections between different parts and levels of mathematical education. The statement of AMS quoted above hits the nail in this respect. It emphasizes that mathematics has intrinsic structure and mathematical education should emphasize and explore all those connections.

According to Thorndike, children cannot transfer their skills and knowledge from classroom to life outside school and therefore the curriculum should be filled with those problems which coincide in all detail with those, which people solve in everyday life. Mathematics as an abstract science does not exist for Thorndike. When I came to America and became aware of arguments based on this theory, I first ignored them as absurdist jokes, so crazy they seemed to me. It took me several years or communication and reading educational literature to accustom myself to the strange fact that arguments of this sort are taken seriously and really influence American curriculum and manner of teaching.

Imagine that prospective teachers of literature in a certain country are made believe through their professional preparation that all fairy tales, fables, fantastic stories are useless. When told a fable, where animals speak to each other, they cannot comprehend and enjoy it in a normal way, as all children do, but exhaust their imagination in figuring out how could it happen in real life: perhaps, animals were especially trained to speak? perhaps, they were made some operation? perhaps, it were disguised people? etc. This is similar to the approach of some American educators towards word problems: they insist that it should be possible for the situation and for the question asked to take place in reality. Actually these educators suffer from some sort of mental deficiency which is not innate but artificially created by their professional preparation.

The value of word problems

As an example of a much more productive approach to age problems than that of Thorndike, let me quote the Russian educator Perelman's book [11], where the second chapter, called "The language of algebra", consists of 25 sections, each devoted to a problem. One of them, called "An equation thinks for us", starts as follows: "If you doubt that an equation is sometimes more prudential than we are, solve the following problem: *The father is 32 years old, the son is 5 years old. How many years later will the father's age be ten times the son's age?*" An equation is made and solved, but the answer is negative: -2. What does this mean? Perelman explains: "When we made the equation, we did not think that the father's age will never be ten times the son's age in the *future* - this relation could take place only in the *past*. The equation turned out more thoughtful and reminded us of our omission." I believe that this comment is really instructive, and constitutes a sufficient reason to discuss such a problem.

Good, interesting and useful word problems often deal with imaginary situations, which don't need to be met in everyday life. Numerical data don't need to be taken from reality. Quantities asked do not need to be unknown or needed in real situations and quantities given do not need to be easily accessible in everyday life. What matters in this case is intrinsic consistency and interesting mathematical structure rather than consistency with or importance for everyday life. Some people would say that such problems are useful as mental gymnastics, and this may be true, but I believe that this reason is insufficient and shall present a more important reason to use such problems.

Their purpose is to convey mathematical meaning, that is to use suitable *concrete* objects to represent or reify *abstract* mathematical notions. Like animals in fables, 'real objects' in these problems should not be taken literally. They are allegories or *mental manipulatives* or *reifications*, which pave children's way to abstractions. For example, coins, nuts and buttons are clearly distinct and countable and for this reason are convenient to represent relations between integer numbers. The youngest children need some real, tangible tokens, while older ones can imagine them, which is a further step of intellectual development. That is why coin problems are so appropriate in elementary school. Pumps and other mechanical appliances are easy to imagine working at a constant rate. Problems involving rate and speed should be (and in Russia are) common already in middle school. Trains, cars and ships are so widely used in textbooks not because all students are expected to go into transportation business, but for another, much more sound reason: these objects are easy to imagine moving at constant speeds and because of this are appropriate as reifications of the idea of uniform movement, which, in its turn, can serve as a reification of linear function. Thus, we can move children further and further on the way of *dereification*, that is development of abstract thinking.

Observations of this sort make me conclude that training students in solving *non-real-world* problems, where data should be accepted at face value, as abstract hypotheses, rather than statements about reality, and conclusions should be deduced from these data, is very important for developing the ability of formal reasoning, which in its turn is essential for success in modern society.

A good word problem should not imitate reality in all detail. It should be as aesthetic as a piece of art. Take Aesop's fable "The Crow and the Fox". On one hand it uses images known to every child. On the other hand, it is cleaned of all irrelevant details. But, from the strange viewpoint of Thorndike and his followers, Aesop's fable is useful only for those who have a chance in some future to perch on a tree branch with a cheeseburger in their mouths.

We are dealing here with some of the most fundamental laws of culture: human culture never describes reality one-to-one. It condenses, idealizes. For example, geographical maps are not, can not and should not be equal to those landscapes which they represent. Creations of the human mind are subject to the drastic law of economy: redundancy should be avoided, every detail must serve the purpose.

Homeschoolers

I have just searched Google for *multi step word problem* as key words and got 715000 references in 0.27 second. So multi step word problems are fairly popular at many places (except departments of education). I especially liked one of them, called "Math word problems - the do's and dont's of teaching problem solving in math", a subdirectory of a page dedicated to home schooling. See `http://www.homeschoolmath.net` . Good luck, homeschoolers!

Calculations by hand in high and higher school

We have discussed importance of mental and p-p calculations in more simple cases. But in Russia mastery of numbers is considered essential not only for everyday life, but also for all the further studies including advanced theory. In the study of logarithms and trigonometry, numerical estimations are deemed an essential condition of understanding. For example, in dealing with logarithms, we notice that $2^{10} = 1024$, which is slightly greater than 10^3 , whence the decimal logarithm of two is slightly greater than $3/10 = 0,3$. Also we may use $3^2 = 9 < 10$ to conclude that $\log_{10} 3$ is slightly less than $1/2$. But we can estimate $\log_{10} 3$ better. First observe that

$$\log_{10} 4 = 2\log_{10} 2 \approx 0,6 \quad \text{and} \quad \log_{10} 8 = 3\log_{10} 2 \approx 0,9.$$

Then we may interpolate

$$\log_{10} 9 \approx \frac{\log_{10} 8 + \log_{10} 10}{2} \approx \frac{0,9+1}{2} = 0,95.$$

Therefore $\log_{10} 3 \approx 0,95/2 = 0,475.$

In trigonometry we may estimate $\sin(1^\circ)$ as follows:

$$\sin(1^\circ) = \sin(\pi/180) \approx \frac{\pi}{180} \approx \frac{3}{180} = \frac{1}{60}$$

$$\tan(1^\circ) = \tan(\pi/180) \approx \frac{\pi}{180} \approx \frac{3}{180} = \frac{1}{60}$$

and

$$\cos(1^\circ) = \sqrt{1-(\sin(1^\circ))^2} \approx 1 - \frac{1}{2}(\sin(1^\circ))^2 \approx 1 - \frac{1}{2}(1/60)^2 = 1 - \frac{1}{7200}.$$

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