Chaos and Monte-Carlo approximations of the Flip-Annihilation process

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Abstract

The flip-annihilation process is a one-dimensiona local interaction process in discrete time. Its components are enumerated by integer numbers and every component has two states, minus and plus. At every time step two transformations occur. The first one, called flip, turns every minus into plus with probability β. The second one, called annihilation, whenever a plus is a left neighbor of a minus, both disappear with probability α. This process is ergodic for $\beta > \alpha/2$ and non-ergodic for $\beta < \alpha^2/250$. We conjecture that there is some transition curve, which we call the *true curve*, which separates the areas of ergodicity and non-ergodicity of this process from each other. The rigorous estimates leave a large gap between them and the present work' s purpose is to obtain some closer, albeit non-rigorous, approximations of the true curve. We do it in two ways, a chaos approximation and a Monte Carlo simulation. Thus we obtain two curves, which are much closer to each other than the rigorous estimations. Also we fill in, albeit only numerically, another shortcoming of the rigorous estimation $\beta < \alpha^2/250$, namely that it leaves us uncertain whether the true curve has a zero or positive slope at the point $\alpha = \beta = 0$.

Introduction

Tradition of

statistical physicists.

$D = 1 \Rightarrow$

no phase transition

Based on this tradition

and

computationals simulations:

Positive Rates Conjecture

PRC: All non-degenerate

1-D cellular automata with

uniform local interaction have only

one invariante measure.

Gács, 2001.

After 15 years refuted the PRC.

Model with $\approx 2^{100}$ states

Gray, 2001.

Comments Gács's results.

simple models do not refute the PRC

Toom, 2002.

Proposed a new class of 1-D processes.

The particles may

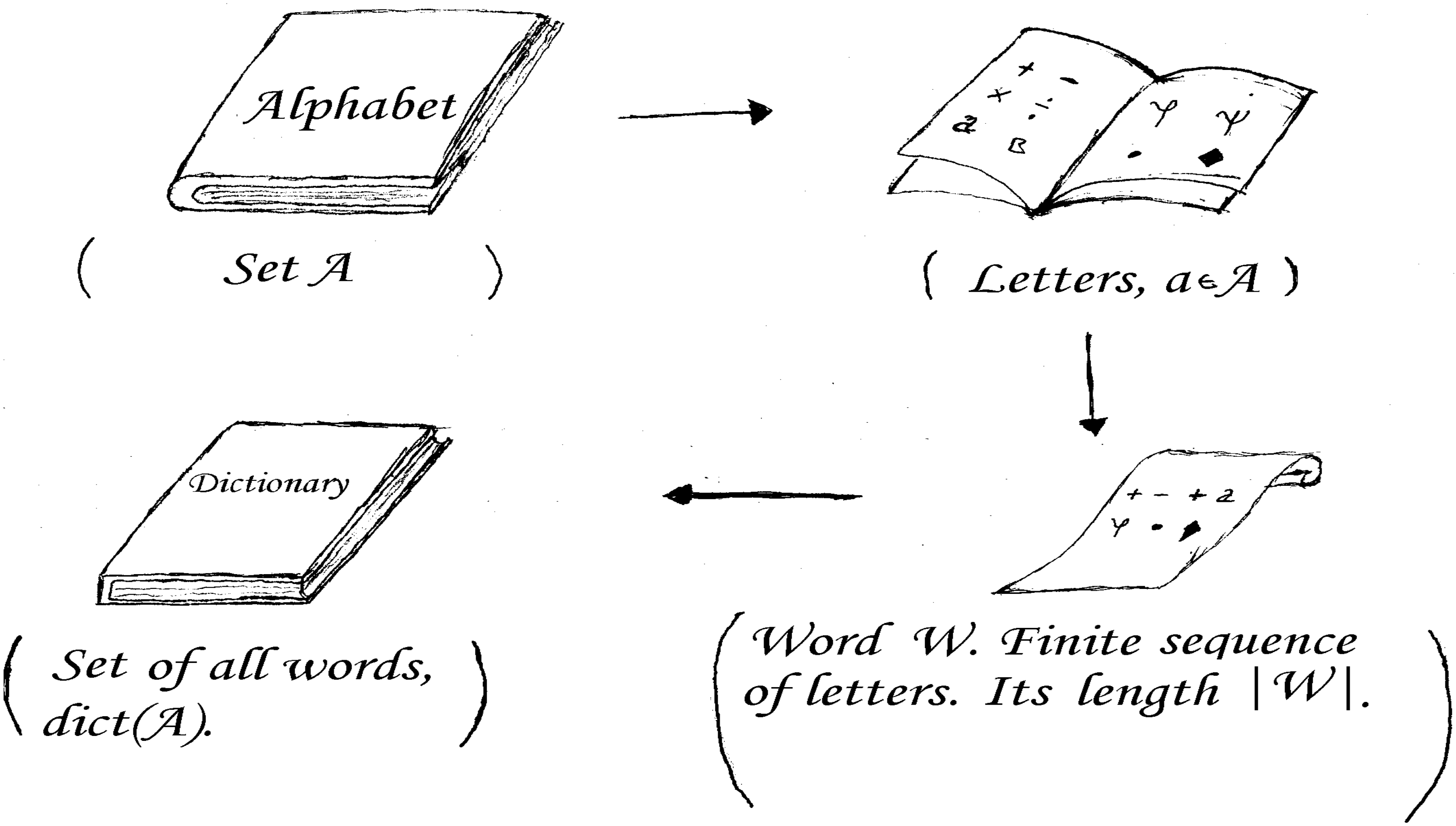
appear and disappear

Toom, 2004.

A process of this class.

An analog of non ergodicity.

Definitions



We define *dict(A)* $= \bigcup_{m=0}^{\infty} \mathcal{A}^m$, where $\mathcal{A}^m$ is the set of words with length m. We call the bi-infinite product $\mathcal{A}^{\mathbb{Z}} = \ldots \mathcal{A} \times \mathcal{A} \times \mathcal{A} \ldots$ *configuration space.* Any *configuration* $x \in \mathcal{A}^{\mathbb{Z}}$ is a bi-infinite sequence $x = (x_i)$ of *components* $x_i, \quad i \in \mathbb{Z}$. We call a *thin cylinder* any set on the form

$$\mathcal{C} = \{x \in \mathcal{A}^{\mathbb{Z}} : x_i = a_i \text{ for all } i \in [m, n]\} \text{ for all } i. \quad (1)$$

where $a_i \in \mathcal{A}$. We consider normalized measures on $\mathcal{A}^{\mathbb{Z}}$, i. e., on the $\sigma-$algebra generated by the thin cylinders. A measure μ in $\mathcal{A}^{\mathbb{Z}}$ is called *uniform* if it is invariant under all translations. For any uniform measure μ we may use the following notations for any word $W = (a_1, \ldots, a_n)$:

$$\mu(W) = \mu(a_1, \ldots, a_n) = \mu(x_{i+1} = a_1, \ldots, x_{i+n} = a_n).$$

We denote by $\mathcal{M}_{\mathcal{A}}$ the set of uniforms measures on $\mathcal{A}^{\mathbb{Z}}$. By convergence in $\mathcal{M}_{\mathcal{A}}$ we mean convergence on all the words of the alphabet $\mathcal{A}$. We say that $\mu \in \mathcal{M}_{\mathcal{A}}$ is *invariant* for P if $\mu P = \mu$ and say that P is *ergodic* if $\lim_{t\to\infty} \nu P^t$ exists and is one and the same for all ν. Otherwise, we call P *non-ergodic*.

Description of process and Toom's theorems

Here the alphabet $\mathcal{A}$ has only two elements, which we denote $\ominus$ and $\oplus$ and call *minus* and *plus*. We define two operators acting on $\mathcal{M}_{\{\ominus,\oplus\}}$ with the parameters $0 < \alpha < 1$ and $0 < \beta < 1$:

$$\text{flip} : \ominus \xrightarrow{\beta} \oplus \qquad \text{annihilation:}(\oplus, \ominus) \xrightarrow{\alpha} \Lambda \text{ (empty word)}$$

We denote by $\delta_\ominus$ and $\delta_\oplus$ the measures concentrated in the configuration "all $\ominus$" and "all $\oplus$" respectively. In [4] for all natural t we denote

$$\mu_t = \delta_\ominus(\text{Flip}_\beta \text{Ann}_\alpha)^t \tag{2}$$

(First acts the operator Flip_β and then Ann_α).

Theorems [4](improved in [5])

Theorem 1. *For all natural t the frequency of $\oplus$ in the measure μ_t does not exceed $250 \cdot \beta/\alpha^2$.*

Theorem 2. *If $2 \cdot \beta > \alpha$, the measure μ_t tends to $\delta_\oplus$ when $t \to \infty$.*

Chaos approximation

In general, estimations obtained by rigorous proofs, are rough. So, it is common to use approximations. We define the chaotic operator $\mathcal{C} : \mathcal{M} \to \mathcal{M}$, which mixes randomly all components. In other words, for each measure $\mu \in \mathcal{M}$, $\mu\mathcal{C}$ is a product-measure with the same frequencies of letters. We approximate the process μP^t by the process $\mu(\mathcal{C}P)^t$. Thus, instead of the original process, which is complicated, we study the evolution of densities of letters. This is called *chaos approximation*.

Theorem 3. The chaos approximation $\mathcal{C}$ Flip_β Ann_α is ergodic if $\beta > \beta^*(\alpha)$ and is not ergodic if $\beta \leq \beta^*(\alpha)$, where

$$\beta^*(\alpha) = \begin{cases} \dfrac{4 - \alpha - 2\sqrt{4 - 2\alpha}}{\alpha} & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Thus for the chaos approximation we know exactly the curve dividing ergodicity and non-ergodicity: it is the continuous curve $\beta = \beta^*(\alpha)$: it starts at the origin with the slope $1/8$, grows smoothly and reaches $3 - 2\sqrt{2} \approx 0.17$ at $\alpha = 1$. The graph of this curve is labeled "Chaos" in the figure 1.

In fact, we can describe completely the limit behavior of this dynamical system:

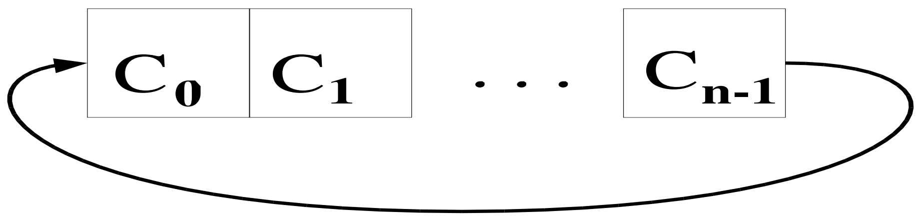
$$\text{If } \Delta < 0, \text{ then } \lim_{t\to\infty} f^t(x_0) = p_3 = 1 \text{ for all } x_0.$$

$$\text{If } \Delta = 0, \text{ then } \lim_{t\to\infty} f^t(x_0) = \begin{cases} p_1 = p_2 & \text{if } x_0 \leq p_1 = p_2, \\ p_3 = 1 & \text{if } x_0 > p_1 = p_2. \end{cases}$$

$$\text{If } \Delta > 0, \text{ then } \lim_{t\to\infty} f^t(x_0) = \begin{cases} p_1 & \text{if } x_0 < p_2, \\ p_2 & \text{if } x_0 = p_2, \\ p_3 = 1 & \text{if } x_0 > p_2. \end{cases}$$

Finite Approximation and Monte Carlo simulation

We have simulated the functioning of the following approximation of the model defined by Toom. This approximation is a Markov chain with a countable set Ω of states, called *circulars*. This circulars are finite sequences, which are composed by terms $\oplus$ and $\ominus$, but we imagine these sequences to have circular form.



We could use words instead of circulars, but this would necessitate specials definitions at their ends.

We denote by $|C|$ the number of components of a circular C. Its components are indexed by remainders modulo $|C|$. In our simulations the initial circular C always consisted of 1000 minuses. We call $quant(W|C)$ the quantity of places, where a word W appears in the circular C.

Now we define the frequency of W in C, as $quant(W|C)$, divided by number of components in C and denote

$$freq(W|C) = \frac{quant(W|C)}{|C|}. \tag{3}$$

In [4], $s(\alpha, \beta)$ denoted the supremum of density of $\oplus$ at the measure μ_t for all natural t. We estimate the quantity $s(\alpha, \beta)$ by

$$\overline{s(\alpha, \beta)} = \max\{freq(\oplus|C^t) : t = 0, \ldots, 100,000\}.$$

In some cases, to obtain better estimations, we did several independent experiments and took an average of them.

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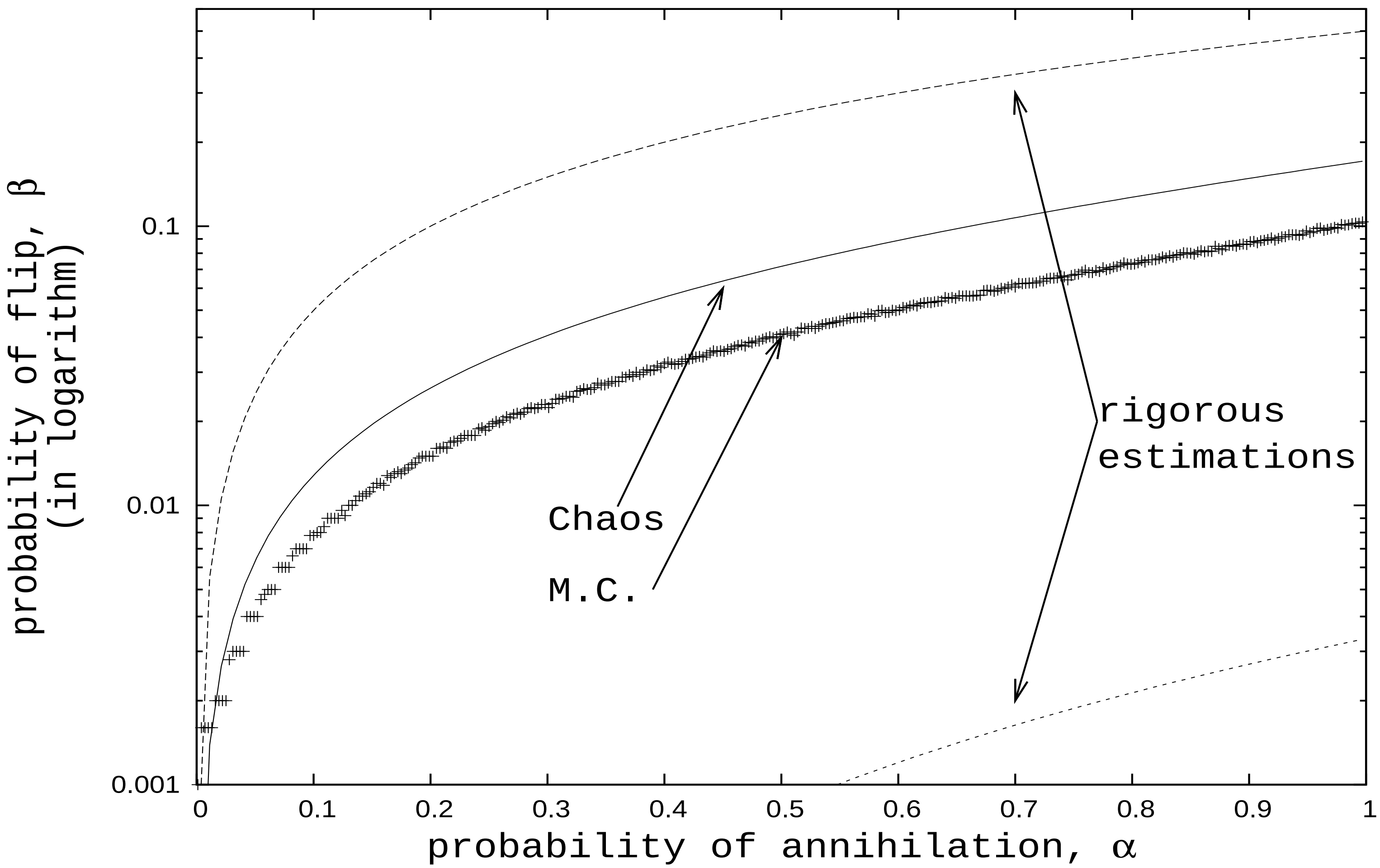

Figure 1: *This graph shows both rigorous estimations and the two approximations of* $\mathsf{true}(\alpha)$*: the chaos approximation(Chaos) and the Monte Carlo approximation (M. C.). Every point of the latter curve was obtained as an average of* 5 *independent experiments. The vertical scale is logarithm.*

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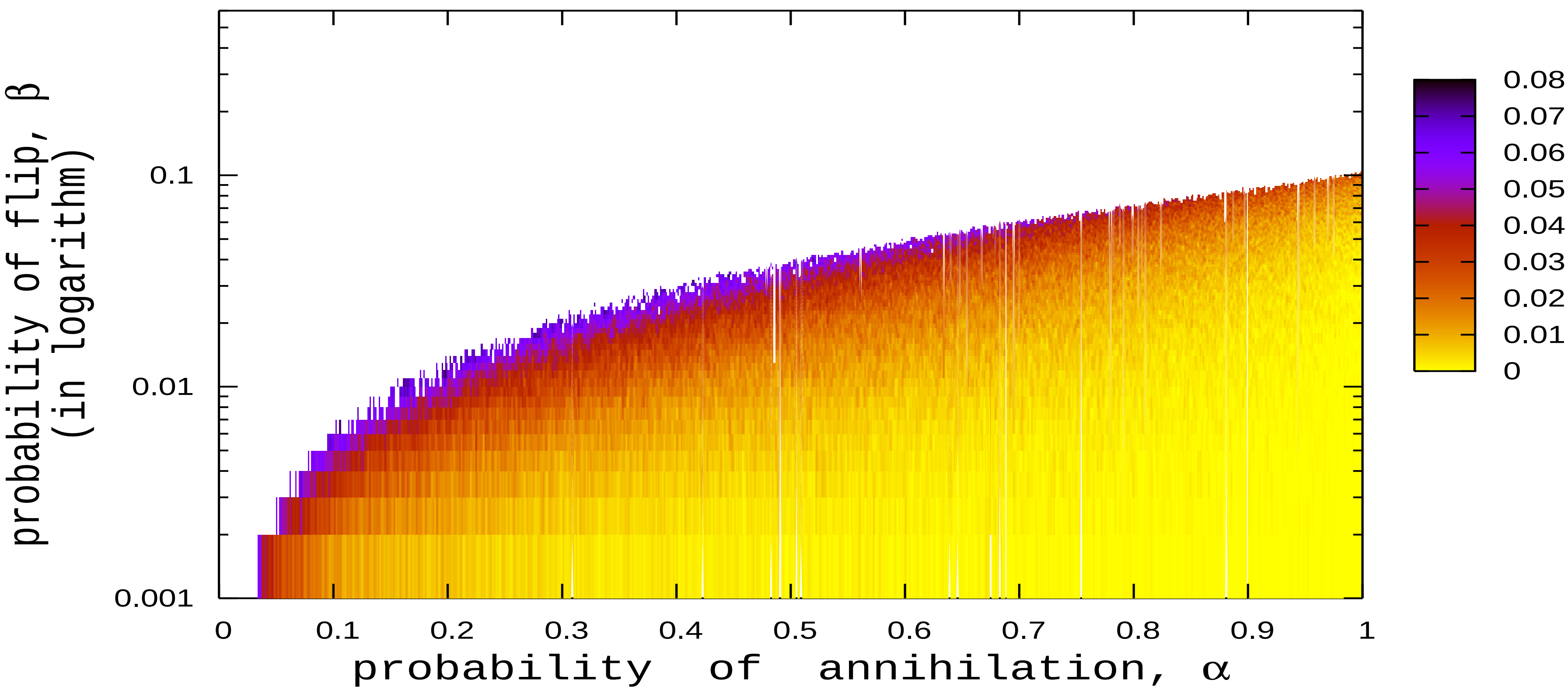

Figure 2: *Here we used colors to represent the values of* $\overline{s(\alpha,\ \beta)}$ *in the area, where the process is suggested to be non-ergodic. The color box on the right side shows how colors from yellow to black represent the values of* $\overline{s(\alpha,\ \beta)}$. *For better visualization, we excluded the values greater than* 0.08, *which constitute less than* 1% *of all data.*

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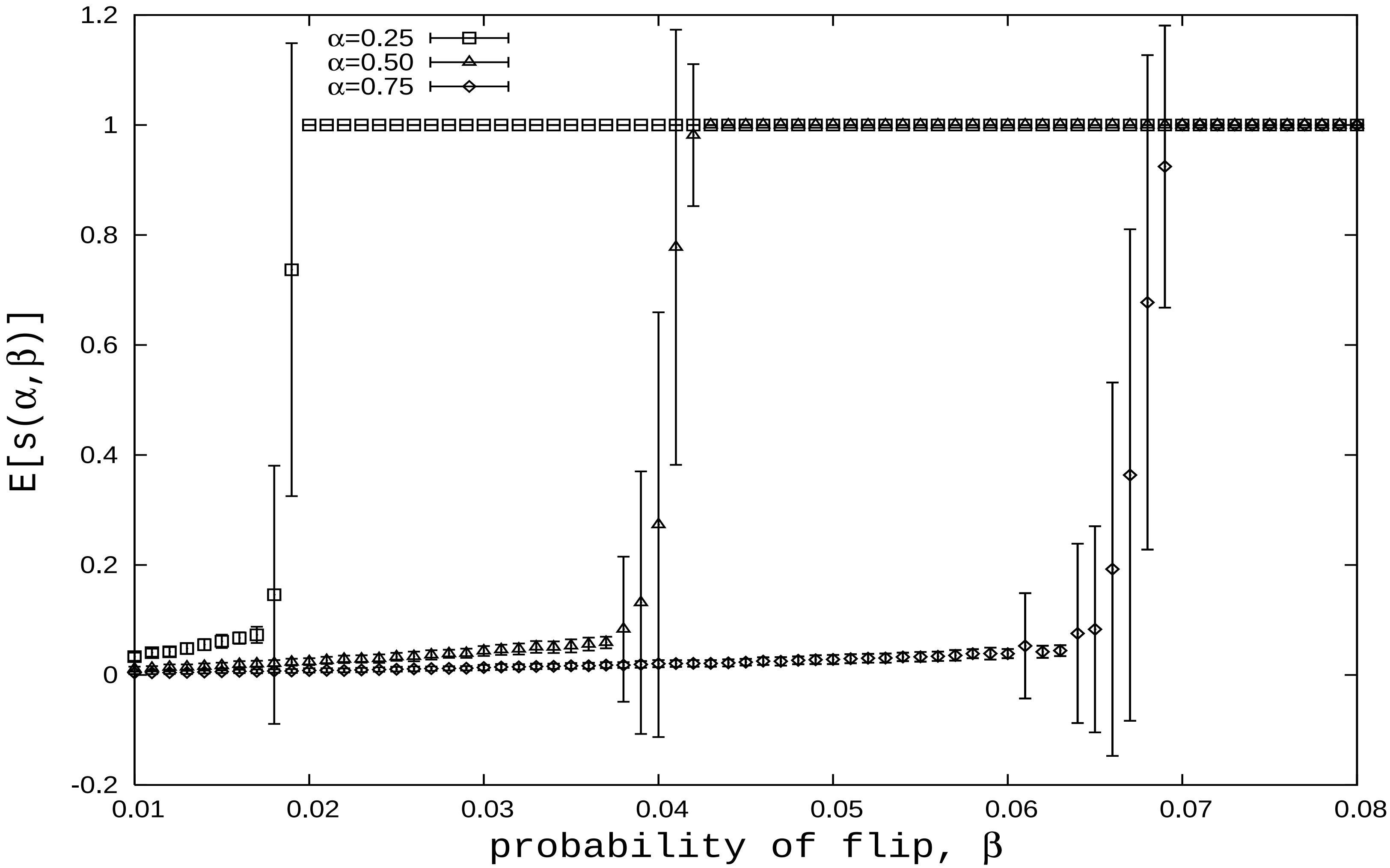

Figure 3: *Behavior of* $\overline{s(\alpha,\beta)}$ *for three values of* α, *namely* $\alpha = 0.25,\ 0.5$ *and* 0.75. *In each case it grows sharply near the critical value. For each value of* β *we made* 100 *independent experiments. Error bars represent the standart deviation.*

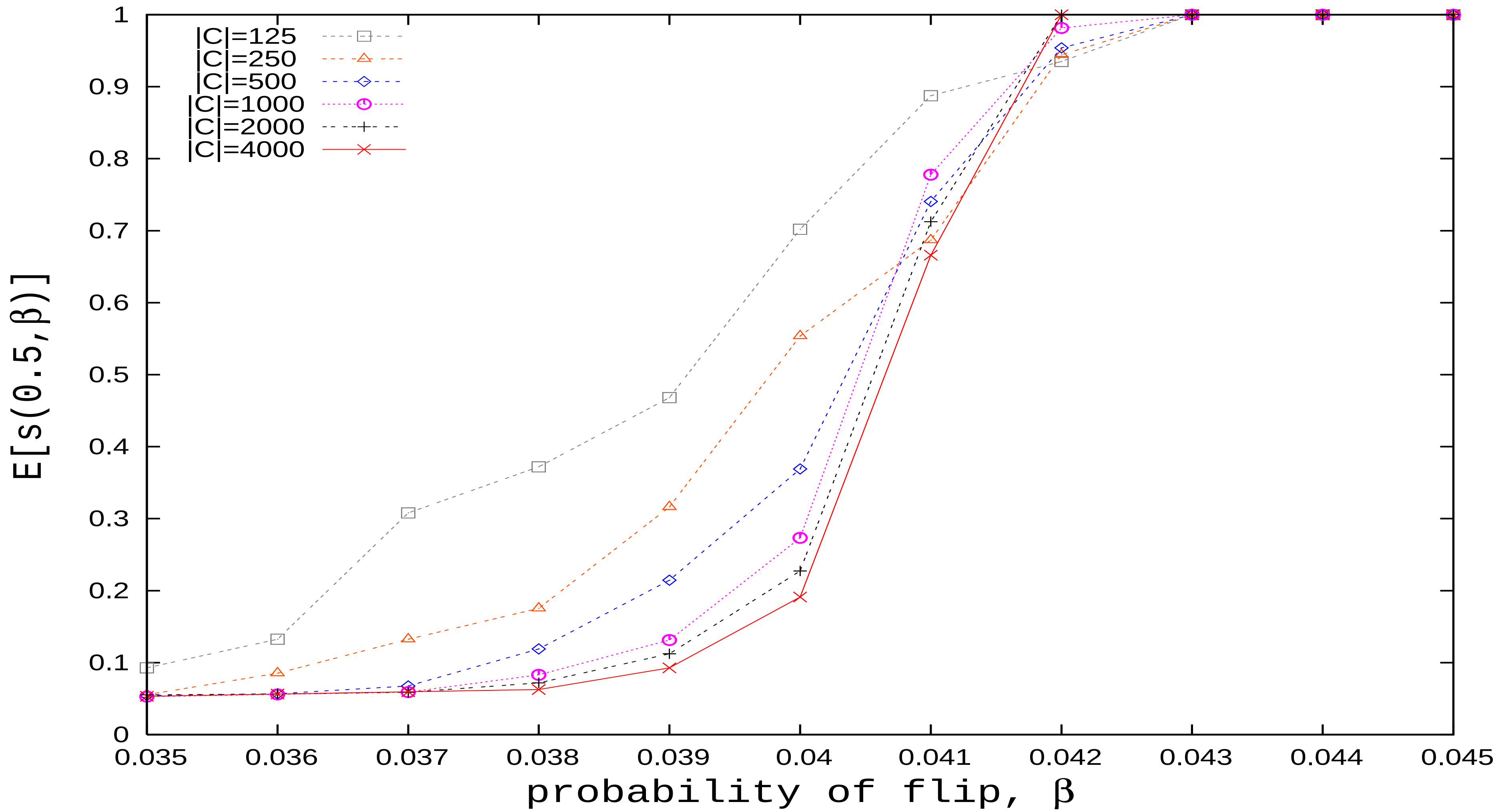

Figure 4: *For* $|C| = 125, 250, 500, 1000, 2000$ *and* 4000 *we displayed the estimations of* $E[s(0.5, \beta)]$ *near critical curve.*

α and β small

For this process, when the probabilities are near zero, we were interested whether the slope of the transition curve at $\alpha = 0$ is positive. We hope that it is positive; in this case, when α and β go to zero, we expected our process to tend to a process with continuous time.

For every fixed j we took α varying from zero to $1/2^j$ with increment of $0,001/2^j$. Then, for each specífic j, there are 1000 pairs, $\{\alpha_i^j, \beta_i^j\}$ with $i = 1, \ldots, 1000$ and, for these pairs we calculated the fits: linear and quadratic, which we denoted,

$$f_L^j(\alpha) = a_j \cdot \alpha \qquad \text{and} \qquad f_Q^j(\alpha) = c_j \cdot \alpha^2 + b_j \cdot \alpha.$$

j	a_j	b_j	c_j
0	0.0911	0.0632	0.0371
1	0.0779	0.0718	0.0165
2	0.0742	0.0737	0.0027
3	0.0729	0.0738	-0.0095
4	0.0730	0.0745	-0.0321
5	0.0729	0.0712	0.0715

Table 1: *The first column displays the coefficients of linear fits, which show convergence to* ≈ 0.0729. *The second and third column display the coefficients of the terms of first and second degree respectively. These coefficients are of quadratic fits. We observe that* b_j *behaves like* a_j *and* c_j *does not converge.*

Acknoledgements

A. D. Ramos cordially thanks the CNPq/MCT and CT-Info, which supported this work. A. Toom was supported by CNPq and PRONEX.

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