All Local Operators Are Continuous and Fine

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INTRODUCTION

The bulk of modern theory of interacting particle systems is based on the assumption that the set of components, also called the space, does not change in the process of interaction; usually it is $\mathbb{Z}^d$ or $\mathbb{R}^d$, where d is dimension. Elements of this space, also called sites, may be in different states (e.g. 1 and 0, often interpreted as presence vs. absence of a particle), but the sites themselves do not appear or disappear in the process of functioning. This assumption is not the only possible one and seems to be motivated partially by mathematical convenience. Here we present another approach.

In several areas of knowledge including the theory of information transmission, molecular biology, historical linguistics and others we deal with long sequences of symbols, which are subject to a large class of local random transformations. The fact that the lengths of the files may well change under these transformation, did not receive due attention for a long time. Indeed, taking this fact into consideration needs an adequate theory, which started to develop less than twenty years ago. According to our knowledge, [1, 2] and a few related papers (dealing with continuous time processes) are the earlies works in this direction, but they deal mostly with processes involving finite sequences, whose definitions present no conceptual difficulties.

According to our knowledge, [8, 9, 10, 3, 4, 5, 6] are the first publications dealing with infinite sequences ([8] with continuous time, the others with discrete time), but mostly with special cases. According to our knowledge, [7] till now is the only rigorous definition of processes with infinite sequences. The attention of [7] is concentrated on *substitution operators* which substitute some finite combinations of letters by some other finite combinations of letters in a discrete time.

The purpose of our study is to introduce a more general class of such transformations than those of which we were aware till now and to describe their properties.

BASIC DENOTATIONS

We assume that all our symbols belong to some finite set $\mathcal{A}$, which we call the *alphabet*. Elements of $\mathcal{A}$ are called *letters*. Finite sequences of letters are called *words*. The length of a word W (that bis the number of letters in it) is denoted by $|W|$. There is the empty word, denoted by Λ, whose length is zero.

As it is usual in mathematical theory we use infinite sequences of letters along with large finite ones. Thus the space of our process is $\mathbb{Z}$, the set of integer numbers. We call the set $\Omega = \mathcal{A}^{\mathbb{Z}}$ *configuration space*. Its elements, that is bi-infinite sequences of letters, are called *configurations*. Every *configuration* $x \in \Omega$ is determined by its *components* $x_v \in \mathcal{A}$ for all $v \in \mathbb{Z}$.

Cylinders in Ω are defined in the usual way. We denote by $\mathcal{M}$ the set of normalized measures on the σ-algebra generated by cylinders. By convergence in $\mathcal{M}$ we mean convergence on all cylinders. Any map $P : \mathcal{M} \to \mathcal{M}$ is called an *operator*. Let us introduce a special class of operators, which we call *local operators*.

WORDS AND RANDOM WORDS AND THEIR CONCATENATION

For convenience we assume that $\mathcal{A}$ contains no commas or brackets. Given any words $W_1, \ldots, W_n$, we call their *concatenation* and denote by

$$concat(W_1, \ldots, W_n)$$

the word obtained by writing the words $W_1, \ldots, W_n$ one after another in that order in which they are listed without brackets and commas.

Let us denote by $\mathcal{D}$ the set of words in the given alphabet. By a random word we mean a probability distribution on $\mathcal{D}$ concentrated in a finite subset of $\mathcal{D}$.

Concatenation of random words.
Suppose that we have n random words $W_1, \ldots, W_n$. For every $i = 1, \ldots, n$ we denote by $X_i^1, \ldots, X_i^{k_i}$ the possible values of W_i. By concatenation of $W_1, \ldots, W_n$ we mean a random word, which takes $k_1 \times \cdots \times k_n$ possible values, namely for any

$$j_1 \in \{1, \ldots, k_1\}, \ldots, j_n \in \{1, \ldots, k_n\}$$

it takes the value $concat(X_1^{j_1}, \ldots, X_n^{j_n})$ with the probability

$$\text{Prob}\,(W_1 = X_1^{j_1}) \times \cdots \times \text{Prob}\,(W_n = X_n^{j_n}).$$

DEFINITION OF OPERATOR

We choose a non-negative integer number r called *range* (range of interaction). We take any function f from $\mathcal{A}^{2r+1}$ to the set of random words and define an operator from $\mathcal{M}$ to $\mathcal{M}$ as follows.

First we define how our operator acts on deterministic words. Given a word L, consider two cases.

If $|L| < 2r+1$, then $P(L)$ is concentrated in the empty word.

If $|L| \geq 2r+1$, then we denote $L = (l_1, \ldots, l_n)$ and define $P(L)$ as the random word, which is the concatenation

$$P(L) = concat\,(f(l_1, \ldots, l_{2r+1}), \ldots, f(l_{n-2r}, \ldots, l_n))\,.$$

Now let us define how P acts on random words - just by linearity. Given a random word W, let us denote by $X_1, \ldots, X_n$ its possible values and define $P(W)$ as

$$\text{Prob}\,(W = X_1) \cdot P(X_1) + \cdots + \text{Prob}\,(W = X_n) \cdot P(X_n).$$

And finally we define how P acts on uniform measures.
We denote by $\mathcal{D}(\mathcal{A})$ and call *dictionary* the set of words in a given alphabet $\mathcal{A}$. Any map from $\mathcal{D}(\mathcal{A})$ to $\mathbb{R}$ is called a *pseudo-measure*. Since any uniform measure is determined by its values on all words, any uniform measure may be considered a pseudo-measure.

Let us attribute to every word W a positive number $\mathcal{P}(W)$ called its *weight* so that the sum of weights of all words is finite.

Then to every pseudo-measure μ we attribute a norm

$$\|\mu\| = \sum_{W \in \mathcal{D}(\mathcal{A})} \mathcal{P}(W) \cdot \mu(W).$$

Let us denote by $\mathcal{M}'$ the set of pseudo-measures, whose norm is finite. Evidently, $\mathcal{M}'$ is a normed linear space. Having this norm, we can define a distance on $\mathcal{M}'$: $dist(\mu,\ \nu) = \|\mu - \nu\|$. Now we can define convergence in $\mathcal{M}'$: a sequence (μ_i) tends to λ if the sequence $dist(\mu_i,\ \lambda)$ tends to zero. Notice that (μ_i) tends to λ if and only if $(\mu_i(W))$ tends to $\lambda(W)$ for every word W.

We say that a word $W = (a_1, a_2, \ldots, a_k)$ *appears at a place* i in a word $V = (b_1, b_2, \ldots, b_m)$ if $b_{i+1} = a_1,\ b_{i+2} = a_2, \ldots,\ b_{i+k} = a_k$.

Let us denote by $quant(W|C)$ the *quantity* of different places, at which a word W appears in a word V.

After that we define the *frequency* of W in V as

$$freq(W|V) = \frac{quant(W|V)}{|V|}.$$

Given a deterministic word V, we define the corresponding pseudo-measure V' as follows: for any

$$V'(W) = \begin{cases} freq(W|V) & \text{if } |W| \leq |V|, \\ 0 & \text{if } |W| > |V|. \end{cases}$$

Analogously, given an arbitrary random word X, whose possible values are words $W_1, \ldots, W_k$, the corresponding pseudo-measure by definition is

$$X' = \sum_{i=1}^{k} \text{Prob}\,(X = W_i) \cdot W_i'.$$

We say that a sequence of random words tends to a pseudo-measure μ if the pseudo-measures corresponding to them tend to μ.

We say that a sequence of random words (X_n) is Cauchy if

$$\forall\, \varepsilon > 0\ \exists\, k : \forall\, m > k, n > k : dist(X_m',\ X_n') < \varepsilon.$$

Given a measure μ, we take a sequence of random words W_n, which tends to μ, take the sequence $P(W_1),\ P(W_2),\ P(W_3), \ldots$ and prove that it is Cauchy, therefore converges to some measure, which is one and the same for all sequences which tend to μ. this measure we accept as $P(\mu)$. We call *local* all operators $P : \mathcal{M} \to \mathcal{M}$ obtainable in this way.

DECLARATIONS OF THE THEOREMS WITH SOME COMMENTS

Theorem 1. *Every substitution operator as defined in [7] is local (but not vice versa).*

Hint. Let us take any substitution operator $G \xrightarrow{\rho} H$ where G is self-avoiding. Excluding some trivial cases, we may assume that G is non-empty. Let us denote $G = (g_1, \ldots, g_n)$, where $n \geq 1$. All we need to define a local operator is to choose the value of r and define a random word $W(x)$ for every $(2r+1)$-tuple $(x_{-r}, \ldots, x_r)$.

We choose $r = n - 1$, where $n = |G|$, and define f by the following two rules.

1-st rule: if the word $(x_{-r}, \ldots, x_r)$ can be represented as a concatenation $concat(A, G, B)$, where $|A| > 0$, then $f(x_{-r}, \ldots, x_r)$ is concentrated in the empty word.

2-d rule: if the word $(x_{-r}, \ldots, x_r)$ can be represented as a concatenation $concat(G, B)$, then $f(x_{-r}, \ldots, x_r)$ may take two values: H with probability ρ and G with probability $1 - \rho$.

Since G is self-avoiding, this definition is not self-contradictory.

It is easy to prove that thus defined operator is identical with the substitution operator $G \xrightarrow{\rho} H$.

Theorem 2. *Every local operator is continuous in the sense defined in [10].*

Remember that every local operator acts on measures and words. For any local operator P and any $\mu \in \mathcal{M}$ we define *extension* $Ext(\mu|P)$ as the limit of the fraction $|P(W)|/|W|$, when $W \to \mu$.

Theorem 3. *For any local operator P and any uniform measure the definition of extension is consistent.*

Theorem 4. *For any local operator P, any $\mu,\ \nu \in \mathcal{M}$ and any real number $L \in [0,\ 1]$*

$$P(L \cdot \mu + (1 - L) \cdot \nu) = \tilde{L} \cdot P(\mu) + (1 - \tilde{L}) \cdot P(\nu),$$

where

$$\tilde{L} = \frac{L \cdot Ext(\mu|P)}{L \cdot Ext(\mu|P) + (1 - L) \cdot Ext(\nu|P)}.$$

Theorem 4 is a technical introduction into Theorem 5, about which we shall speak now. For any $\mu,\ \nu \in \mathcal{M}$ we denote by $convex(\mu,\ \nu)$ their convex hull, that is by definition

$$convex(\mu,\ \nu) = \{k \cdot \mu + (1 - k) \cdot \nu : 0 \leq k \leq 1\}\,.$$

We call an operator $P : \mathcal{M} \to \mathcal{M}$ *fine* if for any $\mu,\ \nu \in \mathcal{M}$

$$\lambda \in convex(\mu,\ \nu) \Longrightarrow P(\lambda) \in convex(P(\mu),\ P(\nu)).$$

Theorem 5. *Every local operator is fine.*

Theorem 5 is a direct consequence of Theorem 4. All the other theorems can be proved like analogous theorems in [7].

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