A NEW VERSION OF TOOM'S PROOF

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ABSTRACT. There are several proofs now for the stability of Toom's example of a two-dimensional stable cellular automaton and its application to fault-tolerant computation. Simon and Berman simplified and strengthened Toom's original proof: the present report is simplified exposition of their proof.

1. INTRODUCTION

For a 2-dimensional cellular automaton, the set $\mathbf{C}$ of *sites* is the set $\mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$ where $\mathbf{Z}_m$ is the set of integers modulo the potentially infinite modulus m. Cellular automata in other dimensions are defined similarly. In a space-time vector (x, t), we will always write the space coordinate first. A *configuration* is a function $\xi(x)$ assigning a state to each site from a finite set $\mathbf{S}$ of states. An *evolution* is a partial funcion $\eta(x, t)$ assigning a state to each site at each time within the interval of interest. In discrete-time cellular automata, the only kind discussed here, state transitions occur only at integer times.

A (one-dimensional) *deterministic cellular automaton* is determined by a local transition rule *Trans* (): we can denote it by

$$CA \ (Trans \ ).$$

An evolution η in one dimension is a *trajectory* if

$$\eta(x, t) = Trans \ (\eta(x - B, t - T), \eta(x, t - T), \eta(x + B, t - T))$$

holds for all x, t. Cellular automata in several dimensions are defined similarly. Given a configuration ξ over the space $\mathbf{C}$ and a transition function, there is a unique trajectory η with the given transition function and the initial configuration $\eta(\cdot, 0) = \xi$.

A *random evolution* is a pair (μ, η) where μ is a measure over some measurable space Ω and $\eta(\omega)$ is a measurable function from Ω to the set of evolutions. Generally, we speak about a fixed random evolution η and refer to μ as Prob. We will say that a random evolution (μ, η) is a trajectory of the ε-*perturbation*

$$CA \ _\varepsilon(Trans \ )$$

of the transition function *Trans* if the following holds. For all x, t, r_{-1}, r_0, r_1, if $\eta(x + j, t - 1) = r_j \ (j = -1, 0, 1)$ and $\eta(x', t')$ is otherwise fixed arbitrarily for all $t' < t$ and for all $x' \neq x$, $t' = t$, then the conditional probability of $\eta(x, t) = Trans \ (r_{-1}, r_0, r_1)$ is at least $1 - \varepsilon$.

A simple stable two-dimensional deterministic cellular automaton given by Toom in [3] can be defined as follows. First we define the neighborhood

$$H = \{(0, 0), (0, 1), (1, 0)\}.$$

The transition function is, for each cell x, a majority vote over the three values $x + g_i$ where $g_i \in H$.

As in [2], let us be given an arbitrary one-dimensional transition function $Trans$ and the integers N, T. We define the three-dimensional transition function $Trans'$ as follows. The interaction neighborhood is $H \times \{-1, 0, 1\}$ with the neighborhood H defined above. The rule $Trans'$ says: in order to obtain your state at time $t+1$, first apply majority voting among self and the northern and eastern neighbors in each plane defined by fixing the third coordinate. Then, apply rule $Trans$ on each line obtained by fixing the first and second coordinates.

For a finite or infinite m, let $\mathbf{C}$ be our 3-dimensional space that is the product of $\mathbf{Z}_m^2$ and a 1-dimensional (finite or infinite) space $\mathbf{A}$ with $N = |\mathbf{A}|$. For a trajectory ζ of $Trans$ on $\mathbf{A}$, we define the trajectory ζ' of $Trans'$ on $\mathbf{C}$ by $\zeta'(i, j, n, t) = \zeta(n, t)$.

Let ζ' be a trajectory of $Trans'$ and η a trajectory of $CA_\varepsilon(Trans')$ such that $\eta(0, w) = \zeta'(0, w)$.

Theorem 1.1. *Suppose $\varepsilon < \frac{1}{32 \cdot 12^8}$. If $m = \infty$ then*

$$\mathrm{Prob}\Big\{ \eta(w, t) \neq \zeta'(w, t) \Big\} \leq 24\varepsilon.$$

If m is finite then

$$\mathrm{Prob}\Big\{ \eta(w, t) \neq \zeta'(w, t) \Big\} \leq 24tm^2 N (2 \cdot (12)^2 \varepsilon^{1/12})^m + 24\varepsilon.$$

The proof we give here is a further simplification of the simplified proof of [1].

Let $Noise$ be the set of space-time points v where η does not obey the transition rule of $Trans'$. Let us define a new process ξ such that $\xi(w, t) = 0$ if $\eta(w, t) = \zeta'(w, t)$, and 1 otherwise. Let

$$Corr(a, b, u, t) = \mathrm{Maj}(\xi(a, b, u, t), \xi(a+1, b, u, t), \xi(a, b+1, u, t)).$$

Then for all points $(a, b, u, t+1) \notin Noise(\eta)$, we have

$$\xi(a, b, u, t+1) \leq \max(Corr(a, b, u-1, t), Corr(a, b, u, t), Corr(a, b, u+1, t)).$$

Now, Theorem 1.1 can be restated as follows:

Suppose $\varepsilon < \frac{1}{32 \cdot 12^8}$. If $m = \infty$ then

$$\mathrm{Prob}\Big\{ \xi(w, t) = 1 \Big\} \leq 24\varepsilon.$$

If m is finite then

$$\mathrm{Prob}\Big\{ \xi(w, t) = 1 \Big\} \leq 24tm^2 N (2 \cdot (12)^2 \varepsilon^{1/12})^m + 24\varepsilon.$$

2. Proof using small explanation trees

If $m < \infty$ let $\mathbf{C}' = \mathbf{Z}^3$ be our *covering space*, and $\mathbf{V}' = \mathbf{C}' \times \mathbf{Z}$ our covering space-time. There is a projection $proj(u)$ from $\mathbf{C}'$ to $\mathbf{C}$ defined by

$$proj(u)_i = u_i \bmod m \qquad (i = 1, 2).$$

This rule can be extended to $\mathbf{C}'$ identically. We define a random process ξ' over $\mathbf{C}'$ by

$$\xi'(w, t) = \xi(proj(w), t).$$

The set $Noise$ is extended similarly to $Noise'$. Now, if $proj(w_1) = proj(w_2)$ then $\xi'(w_1, t) = \xi'(w_2, t)$ and therefore the failures at time t in w_1 and w_2 are not independent.

In figures, we generally draw space-time with the time direction going down. Therefore, for two neighbor points u, u' of the space $\mathbf{Z}$ and integers a, b, t, we will call *arrows*, or *vertical edges* the following kind of (undirected) edges:

$$\{(a, b, u, t), (a, b, u', t-1)\}, \{(a, b, u, t), (a+1, b, u', t-1)\},$$
$$\{(a, b, u, t), (a, b+1, u', t-1)\}.$$

We will call *forks*, or *horizontal edges* the following kinds of edges:

$$\{(a, b, u, t), (a+1, b, u, t)\}, \{(a, b, u, t), (a, b+1, u, t)\},$$
$$\{(a+1, b, u, t), (a, b+1, u, t)\}.$$

We define the graph $\mathbf{G}$ by introducing all possible arrows and forks. Thus, a point is adjacent to 6 possible forks and 6 possible arrows: the degree of $\mathbf{G}$ is at most

$$r = 12.$$

(If the space is $d+2$-dimensional instead of 3, then $r = 6(d+1)$.) We use the notation $Time\,((w, t)) = t$.

Lemma 2.1 (Explanation Tree). *Let u be a point outside Noise' with $\xi'(u) = 1$. Then there is a subtree Expl (u, ξ') of $\mathbf{G}$ rooted at u called an explanation of u such that if n nodes of Expl belong to Noise$'$ then the number of edges of Expl is at most $4n - 4$.*

This lemma will be proved in the next section. To use it in the proof of the main theorem, we need some easy lemmas on trees whose nodes have weights 0 or 1, with the root having weight 0. The *redundancy* of such a tree is the ratio of its number of edges to its weight. The set of nodes of weight 1 of a tree T will be denoted by $F(T)$. A subtree of a tree is a subgraph that is a tree.

Lemma 2.2. *Let T be a weighted tree of total weight $w > 3$ and redundancy λ. Then there is a subtree of total weight w_1 with $w/3 < w_1 \leq 2w/3$, and redundancy $\leq \lambda$.*

Proof. Let us order T from the root r down. Let T_1 be a minimal subtree below r with weigth $> w/3$. Then the subtrees immediately below T_1 all weigh $\leq w/3$. Let us delete as many of these as possible while keeping T_1 weigh $> w/3$. At this point, the weight w_1 of T_1 is $> w/3$ but $\leq 2w/3$ since we could subtract a number $\leq w/3$ from it so that w_1 would become $\leq w/3$ (note that since $w > 3$) the tree T_1 is not a single node.

Now T has been separated by a node into T_1 and T_2, with weights $w_1, w_2 > w/3$. Since the root of a tree has weight 0 by definition the possible weight of the root of T_1 stays in T_2 and we have $w_1 + w_2 = w$. The redundancy of T is then a weighted average of the redundancies of T_1 and T_2, and we can choose the one of the two with the smaller redundancy: its redundancy is smaller than that of T. $\qquad\square$

Theorem 2.3 (Tree Separator). *Let T be a weighted tree with weight w and redundancy λ, and let $k < w$. Then T has a subtree with weight w' such that $k/3 < w' \leq k$ and redundancy $\leq \lambda$.*

Proof. Let us perform the operation of Lemma 2.2 repeatedly, until we get weight $\leq k$. Then the weight w' of the resulting tree is $> k/3$. $\qquad\square$

Lemma 2.4 (Tree Counting). *In a graph of maximum node degree r the number of weighted subtrees rooted at a given node and having k edges is at most $2r \cdot (2r^2)^k$.*

Proof. Let us number the nodes of the graph arbitrarily. Each tree of k edges can now be traversed in a breadth-first manner. At each non-root node of the tree of degree i from which we continue, we make a choice out of r for i and then a choice out of $r - 1$ for each of the $i - 1$ outgoing edges. This is r^i possibilities at most. At the root, the number of outgoing edges is equal to i, so this is r^{i+1}. The total number of possibilities is then at most r^{2k+1} since the sum of the degrees is $2k$. Each point of the tree can have weight 0 or 1, which multiplies the expression by 2^{k+1}. $\qquad\square$

Proof of Theorem 1.1. Let us consider each explanation tree a weighted tree in which the weight is 1 in a node exactly if the node is in *Noise'*. For each n, let $\mathcal{E}_n$ be the set of possible explanation trees *Expl* for u with weight $|F(\textit{Expl})| = n$. First we prove the theorem for $m = \infty$, i.e. *Noise'* = *Noise*. If we fix an explanation tree *Expl* then all the events $w \in \textit{Noise'}$ for all $w \in F = F(\textit{Expl})$ are independent from each other. Therefore the probability of the event $F \subset \textit{Noise'}$ is at most ε^n. Therefore

$$\text{Prob}\Big\{\xi(u) = 1\Big\} \leq \sum_{n=1}^{\infty} |\mathcal{E}_n|\varepsilon^n.$$

By the Explanation Tree Lemma, each tree in $\mathcal{E}_n$ has at most $k = 4n - 4$ edges. By the Tree Counting Lemma, we have

$$|\mathcal{E}_n| \leq 2r \cdot (2r^2)^{4n-4},$$

Hence

$$\text{Prob}\{\xi(u) = 1\} \leq \frac{2r}{16r^8} \sum_{n=1}^{\infty} (16r^8\varepsilon)^n = \frac{2r\varepsilon}{1 - 16r^8\varepsilon}.$$

In the case $\mathbf{C} \neq \mathbf{C}'$ this estimate bounds only the probability of $\xi'(u) = 1$, $|\textit{Expl}(u, \xi')| \leq m$, since otherwise the events $w \in \textit{Noise'}$ are not necessarily independent for $w \in F$. Let us estimate the probability that an explanation *Expl* (u, ξ') has m or more nodes. It follows from the Tree Separator Theorem that *Expl* has a subtree T with weight n' where $m/12 \leq n' \leq m/4$, and at most m nodes. Since T is connected no two of its nodes can have the same projection. Therefore for a fixed tree of this kind, for each node of weight 1 the events that they belong to *Noise'* are independent. Hence for each tree T of these sizes, the probability that T is such a subtree of *Expl* is at most $\varepsilon^{m/12}$. To get the probability that there is such a subtree we multiply by the number of such subtrees. An upper bound on the number of places for the root is tm^2N. An upper bound on the number of trees from a given root is obtained from the Tree Counting Lemma. Hence

$$\text{Prob}\Big\{|\textit{Expl}(u, \xi')| > m\Big\} \leq 2rtm^2N \cdot (2r^2\varepsilon^{1/12})^m.$$

$\qquad\square$

3. THE EXISTENCE OF SMALL EXPLANATION TREES

3.1. **Some geometrical facts.** Three linear functionals are defined as follows for $v = (x, y, z, t)$.

$$L_1(v) = -x, \qquad L_2(v) = -y, \qquad L_3(v) = x + y.$$

Notice $L_1(v) + L_2(v) + L_3(v) = 0$. For a set S, we write

$$Size\ (S) = \sum_{i=1}^{3} \max_{v \in S} L_i(v).$$

For every point v, $Size\ (\{v\}) = 0$.

A set $\mathcal{S} = \{S_1, \dots, S_n\}$ of sets is *connected by intersection* if the graph over $\mathcal{S}$ is connected which we obtain by introducing an edge between S_i and S_j whenever $S_i \cap S_j \neq \emptyset$.

A *spanned set* is an object $\mathbf{P} = (P, v_1, v_2, v_3)$ where P is a space-time set, $v_i \in P$. The points v_i are the *poles* of P, and P is its *base set*. We define $Span\ (\mathbf{P})$ as $\sum_{i=1}^{3} L_i(v_i)$.

Lemma 3.1 (Spanned Set Creation). *If P is a set then there is a spanned set (P, v_1, v_2, v_3) on P such that $Span\ (\mathbf{P}) = Size\ (P)$.*

Proof. We assign v_i to a point of the set P in which L_i is maximal. $\qquad\square$

Lemma 3.2 ("Stokes's" Theorem for spanned sets). *Let $\mathbf{L} = (L, u_1, u_2, u_3)$ be a spanned set and $\mathcal{M}$ be a set of subsets of L connected by intersection, with $\{u_1, u_2, u_3\}$ contained in $\bigcup_{M \in \mathcal{M}} M$. Then there is a set $\{\mathbf{M}_1, \dots, \mathbf{M}_n\}$ of spanned sets whose base sets are elements of $\mathcal{M}$, such that the following holds. Let M'_i be the subset of M_i consisting of its poles.*
- (a) *$Span\ (\mathbf{L}) = \sum_i Span\ (\mathbf{M}_i)$.*
- (b) *Each u_i is in one of the M'_j.*
- (c) *The system $\{M'_1, \dots, M'_n\}$ is a minimal system connected by intersection that connects the three poles u_i.*

Proof. Without loss of generality, suppose that set $M_i \subset \mathcal{M}$ contains the point u_i. Let us choose u_i as the i-th pole of M_i. Now leave only those sets of $\mathcal{M}$ that are needed for a spanning tree in the minimum tree connecting M_1, M_2, M_3. Keep deleting points from each set (except u_i from M_i) until every remaining point is necessary for a connection among u_i. There will only be two-and three-element sets. Let us draw an edge between each pair of points if they belong to a common set M'_i. This turns the set $\bigcup_i M'_i$ into a graph. (Actually, this graph can have only two simple forms: a point connected via disjoint paths to u_i or a triangle connected via disjoint paths to u_i.) For each i and j, there is a shortest path between M'_i and u_j. The point of M'_i where this path leaves M'_i will be made the j-th pole of M_i. (This rule puts three poles into each M_i.) Let us show that it puts the same number of each kind of pole into each point different from u_1, u_2, u_3 and therefore, the sum of spans did not change:

$$\sum_i Span\ (\mathbf{M}_i) = Span\ (\mathbf{L}).$$

Let v be a point. Let $H_1, \dots, H_p$ be the sets M'_i such that v is a pole of M'_i. For $k = 1, \dots, p$, let E_k be the set of those $j \in \{1, 2, 3\}$ that the shortest path from

v to u_j goes through H_k. Then the sets E_k form a partition of the set $\{1, 2, 3\}$. Let $E'_k = \{1, 2, 3\} \smallsetminus E_k$. Then v is the j-th pole of H_k if and only if $j \in E'_k$. Let $e_k(j) = 1$ if $j \in E_k$ and 0 otherwise. Then $j \in E'_k$ if and only if $1 - e_k(j) = 1$. The number of k's such that v is the j-th pole of H_k is therefore

$$\sum_k (1 - e_k(j)) = p - \sum_k e_k(j) = p - 1.$$

This number does not depend on j, which is what we had to prove.   □

3.2. **Building an explanation tree.** Let $v = (a, b, u, t + 1)$ with $\xi'(v) = 1$. If $v \notin \textit{Noise}\,'$ then there is an u' such that $\xi'(w) = 1$ for at least two members w of the set

$$\{\,(a, b, u', t), (a + 1, b, u', t), (a, b + 1, u', t)\,\}.$$

Let us define the set $\textit{Excuse}\,(v)$ as such a pair of elements w. We define $\textit{Excuse}\,(v)$ to be empty in all other cases. By Lemma 3.1, we can turn $\textit{Excuse}\,(v)$ into a spanned set, $(\textit{Excuse}\,(v), v_1, v_2, v_3)$ with span 1. Let us denote $\textit{Excuse}\,_i(v) = v_i$.

Lemma 3.3 (Excuse Size Lemma). *If $\mathbf{V} = (V, v_1, v_2, v_3)$ is a spanned set and v_i are not in $\textit{Noise}\,'$ then*

$$\sum_i L_i(\textit{Excuse}\,_i(v_i)) = \textit{Span}\,(\mathbf{V}) + 1.$$

Proof. Let T be the triangle of points u satisfying the inequalities $L_1(u) \leq 0$ $L_2(u) \leq 0$, $L_3(u) \leq 1$. Then T has size 1 and $v + T$ always contains the set $\textit{Excuse}\,(v)$. Since the chosen poles turn $\textit{Excuse}\,(v)$ into a spanned set of size 1 the function L_i achieves its maximum in $T + v$ on $\textit{Excuse}\,_i(v)$. We have

$$L_i(\textit{Excuse}\,_i(v)) = \max_{u \in T+v} L_i(u) = \max_{u \in T} L_i(u) + L_i(v)$$

Hence

$$\sum_i L_i(\textit{Excuse}\,_i(v_i)) = \sum_i \max_{u \in T} L_i(u) + \sum_i L_i(v_i)$$
$$= \textit{Size}\,(T) + \textit{Span}\,(\mathbf{V}) = 1 + \textit{Span}\,(\mathbf{V}).$$

□

For every b with $\xi'(b) = 1$ we define V (the set of points that participated in "misleading" b) the sets $\textit{Arrows}$ and $\textit{Forks}$ of undirected edges on V as follows. $V = V_n$, $\textit{Arrows} = A_n$, $\textit{Forks} = F_n$, where the sets V_i, A_i, F_i $(i = 0, \dots, n)$ and the number n are defined as follows. Let $V_0 = \{b\}$, $A_0 = F_0 = \emptyset$. Suppose that V_i, A_i, F_i is defined. If for all $v \in V_i$ with $\xi'(v) = 1$ we have $\textit{Excuse}\,(v) \subset V_i$ then $n = i$. Else let v be such that $\xi'(v) = 1$ and $\textit{Excuse}\,(v) \not\subset V_i$. Let $\textit{Excuse}\,(v) = \{a, b\}$. Then

$$V_{i+1} = V_i \cup \{a, b\},$$
$$A_{i+1} = A_i \cup \{\{v, a\}, \{v, b\}\},$$
$$F_{i+1} = F_i \cup \{\{a, b\}\}.$$

For an arbitrary node u of the *investigation graph* defined above, G_u is a subgraph of $(V, \textit{Arrows}\,)$ induced by

$$\{\,v \in V : \textit{Time}\,(v) \leq \textit{Time}\,(u)\,\}.$$

Notice that G_u includes points v with $Time\,(v) = Time\,(u)$. The set $History\,(u)$ is the connected component of G_u containing u. We will use the notation $Time\,(K)$ for histories K without ambiguity. For a history K we define the graph $G_K = (V_K, E_K)$ as follows:

- $V_K = \{\,\text{history } R \subset K : Time\,(R) = Time\,(K) - 1\,\}$.
- $\{R, S\} \in E_K$ iff $\{R, S\} \subset V_K$ and for some $\{v, w\} \in Forks$ we have $v \in R$ and $w \in S$ and $Time\,(v) = Time\,(w) = Time\,(K) - 1$.

Lemma 3.4. G_K *is connected.*

Proof. In the graph of arrows the history K is a connected component. The sub-histories in G_K are connected with each other only through pairs of arrows going trough time $Time\,(K)$. The tails of each such pair of arrows are connected by a fork. $\qquad\square$

A *spanned history* is a spanned set that is a history in which all spanning points v_i have the maximum value of $Time$. It is easy to see that a spanned history goes through several instants of time if and only if none of its spanning points is in $Noise\,'$.

The explanation tree will be built from an intermediate object that is a tree whose nodes are of two kinds: spanned histories, and *intermediate nodes*. The edges of the tree are original edges of the graph adjacent to the intermediate nodes or the poles of these spanned histories. Such a tree will be called a *partial explanation tree*. The *span* of the explanation tree will be the sum of the spans of its nodes (intermediate nodes have span 0) and the sizes of its forks (horizontal edges: their sizes are all 1).

We start with a node $u \notin Noise\,'$ such that $\xi'[u] = 1$. Then $(History\,(u), u, u, u)$ is itself a spanned history, forming a one-node partial explanation tree. Now we apply repeatedly an operation called *refinement* to the tree.

Let $\mathbf{K} = (K, v_1, v_2, v_3)$ be a spanned history in the tree going through several instants of time. Then v_i are not in $Noise\,'$. Consider the graph $G_K = (V_K, E_K)$ defined above. Let $\mathcal{M} = V_K \cup E_K$, i.e. the set of all histories in V_K and all edges in G_K connecting them, taken as two-element sets. Let L be the union of these sets, $\mathbf{L} = (L, u_1, u_2, u_2)$ where $u_i = Excuse_i(v_i)$. Lemma 3.4 implies that the set $\mathcal{M}$ is connected by intersection. Let us apply Lemma 3.2 to $\mathbf{L}$ and $\mathcal{M}$. We find a set $\mathbf{M}_1, \ldots, \mathbf{M}_n$ of spanned sets such that

$$\sum_i Span\,(\mathbf{M}_i) = Span\,(\mathbf{L}) = \sum_i L_i(u_i).$$

It follows from Lemma 3.3 that the latter sum is $Span\,(\mathbf{K}) + 1$. Also, the u_i are among the poles of these sets. Some of these sets are disjoint spanned histories, others are forks connecting them, adjacent to their poles. Consider these forks again as edges and the spanned histories as nodes. By the minimality property of the lemma, they form a tree. Now the refinement operation deletes all non-pole nodes of the spanned set $\mathbf{K}$ in the old tree and adds the tree just built. For each of the 1,2 or 3 nodes that were poles in $\mathbf{K}$ it chooses a pole v_i in it and adds the arrow from v_i to u_i to connect the new tree to the old one. Notice that the operation increased the span by at least 1 and the number of arrows by at most 3.

When the refinement operation cannot be applied any longer then all nodes of the tree belong to $Noise\,'$.

Proof of Lemma 2.1. What is left to prove is the estimate on the number of edges. Let us contract each arrow (u, v) of the explanation tree one-by-one into its bottom point v. The edges of the resulting tree are the forks. All the intermediate nodes will be contracted into the remaining unrefinable one-node spanned histories that are actually elements of *Noise* $'$. If k is the number of these nodes then $k - 1$ is the number of forks. The span of the explanation tree just constructed is the sum of sizes of the forks, i.e. $k - 1$. The number of arrows is at most $3(k - 1)$ since each introduction of 3 arrows was accpanied by an increase of the span by 1. The total number of edges of the explanation tree is thus at most $4(k - 1)$. $\qquad\qquad\blacksquare$

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