UNSTABLE MULTICOMPONENT SYSTEMS

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Systems are considered that consist of an infinite number of interconnected finite probabilistic automata operating in discrete time. Systems are called ergodic if they "forget" their initial state in the limit with respect to time. A sufficient condition for erogidicity is given. Two examples are cited of families of systems that are ergodic for positive parameter values and possess the opposite property for zero values.

§1. Introduction

We will consider Markov chains with a continual set of states, that describe the behavior of systems consisting of an infinite number of automata (regarding this topic, see, e.g., [1-4], where references can be found to some other studies). The main part of this study involves the examples in Sec. 3 of "unstable" systems whose behavior is markedly altered when arbitrarily weak random noise of a certain kind is superimposed. The propositions in Sec. 2 are of an ancillary nature. Proposition 1 helps to delineate the situation in which the examples become interesting. Propositions 2 and 3 are needed for the proofs pertaining to the examples. In this section we introduce the requisite concepts.

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Let X be an ensemble of mappings of countable set V onto finite set $\{0, 1, \ldots, n\}$. Elements of X will be called states and denoted by $x = (x_h)$, where $x_h \in \{0, 1, \ldots, n\}$, $h \in V$. States of the form "all x_h are equal to k" will be denoted by **k**. The quantities x_h will be called components of state x. Set X is rigged with a product (point-by-point convergence) topology. Let M be the ensemble of normalized measures on X, more precisely on the σ-algebra generated by cylindrical sets in X. A measure concentrated at $x \in X$ will be denoted by the same letter x. Mapping $P: M \to X$ that is continuous in the weak topology will be called an operator. The result of P acting on μ will be written as μP. We will consider only linear operators P, i.e., operators such that $(\lambda\mu + (1-\lambda)\nu)P = \lambda\mu P + (1-\lambda)\nu P$ for any measures $\mu, \nu \in M$, and numbers λ, $0 \leq \lambda \leq 1$. Linear and continuous operators P can be specified nearly by indicating how they act on point measures x. Each operator P corresponds to an array of probabilities p(C, x) that represent the value of measure xP, where $x \in X$, on cylindrical set $C \subset X$. For given C the quantity p(C, x) depends continuously on x.

An operator will be called deterministic if it carries all point measures into point measures. Deterministic operators will be denoted by D. They can be regarded as acting on X.

Operator P will be called ergodic if the weak limit $\lim_{t\to\infty} \mu P^t$ exists and is the same for all $\mu \in M$. Here we consider only those situations in which this limit is **n**, a measure concentrated in the state "all x_h are equal to n."

Let g be a one-to-one mapping of V onto V. We denote by g the mappings of X onto X and of M onto M, that it induces. The first of these is defined by the formula $\hat{g}x_h = x_{gh}$. We call g an automorphism of P if G commutes with P. We call P homogeneous if the group of automorphisms of P is transitive on V.

Operator P will be called local if for any finite $W \subset V$ there exists a finite $U(W) \subset V$ such that the projection of measure μP onto $X_W = \{(x_h), h \in W\}$ is expressed in terms of the projection of measure μ onto $X_{U(W)}$. We also introduce the notation $U^k(W) = U(U^{k-1}(W))$ where $U^0(W) = W$.

We introduce a partial ordering on X, taking $x \prec y$ if $x_h \leq y_h$ for all $h \in V$. Then we introduce an ordering on M in accordance with [2]; specifically, we call measureable set $C \subset X$ complete from above if $x \in C$ and $x \prec y$ imply $y \in C$. We will say that $\mu \prec \nu$, where $\mu, \nu \in M$, if $\mu(C) \leq \nu(C)$ for any C that is complete from above. We will say that $P \prec Q$ if $xP \prec xQ$ for any $x \in X$. Operator $P: M \to M$ will be called monotonic if $\mu \prec \nu$ implies that $\mu P \prec \nu P$.

In particular, deterministic operator D is monotonic if $x \prec y$ implies that $xD \prec yD$. If P is a monotonic operator, then the condition

$$\forall h \in V (\lim_{t\to\infty} 0P^t(x: x^h = n) = 1) \tag{1}$$

is equivalent to the fact that P is ergodic, and has measure **n** as a limiting measure.

Let $m = \| m_{ab} \|$, where $0 \leq a \leq n$, $0 \leq b \leq n$ is a stochastic matrix of order (n + 1). We associate with matrix m and arbitrary $h \in V$ an operator $S_{m,h}: M \to M$ that carries coordinate x_h from each state a to state b with probabilities m_{ab}, all the remaining coordinates being unaltered. We denote by $S_m = \prod_{h \in V} S_{m,h}$ an operator that acts in this fashion on all coordinates independently. We denote by S the set of all S_m defined by matrices m that satisfy the pair of conditions

$$\begin{cases} \text{if} \quad b < a, & \text{then } m_{ab} = 0, \\ \text{if} \quad b = a < n, & \text{then } m_{ab} < 1. \end{cases}$$

The first of these conditions guarantees that $m_{nn} = 1$, and therefore $\mathbf{n}S_m = \mathbf{n}$ for any $S_m \in S$ In the proofs it will be convenient to employ operators S_m of a special form which we denote by S_ε where ε is a number, $0 \leq \varepsilon \leq 1$. We denote by S_ε and S_m where the matrix m is as follows:

$$m_{ab} = \begin{cases} 1, & \text{if} \quad a = b = n, \\ 1 - \varepsilon, & \text{if} \quad a = b < n, \\ \varepsilon, & \text{if} \quad a + 1 = b, \\ 0 & \text{otherwise.} \end{cases}$$

Operators S_ε are monotonic for all ε from 0 to 1. Obviously, for any $S_m \in S$ there exists an $\varepsilon > 0$ such that $S_\varepsilon \prec S_m$. From this and from [2] (Sec. 2) we have that if P is monotonic and $\mathbf{n} = \mathbf{n}P$, then the fact that PS_ε is ergodic for all $\varepsilon > 0$ implies that PS_m is ergodic for all $S_m \in S$. We will also require a representation of operators S which utilizes auxiliary variables ω_h equal to 0 or 1. All the ω_h are mutually independent, $\omega_h = 1$

with probability ε and $\omega_h = 0$ with probability $1-\varepsilon$. Then for any $x \in X$ the measure $x \prod_{h \in K} S_{\varepsilon,h}$ is defined as being induced by the measure on $\Omega = \{\omega\}$, $\omega = (\omega_h)$ under the mapping

$$y_h = \begin{cases} x_h, & \text{if} \quad h \bar{\in} K, \\ \min\{n, x_h + \omega_h\}, & \text{if} \quad h \in K. \end{cases}$$

In considering expressions of the form $0(PS_\varepsilon)^t$, we will assume that each use of S_ε corresponds to its own variables $\omega_h^1, \omega_h^2, \ldots, \omega_h^t$ that are mutually independent.

§2. Definitions and Propositions

We will consider superpositions of the form PS_m and will attempt to determine the P for which operators PS_m are ergodic for all $S_m \in S$. Proposition 1 is one elementary result of this kind. First let us consider Definition 1; we denote by $I_k(x)$ the ensemble of those h for which $x_h \geq k$.

Definition 1. We will call $x \in X$ an island if $I_1(x)$ is finite (i.e., if the number of its nonzero components is finite). Operator D will be called conservative if there exists an island x such that $I_n(xD^t)$ is nonempty for all $t = 0, 1, 2, \ldots$.

Proposition 1. Assume that operator D is monotonic and conservative. Then $\mathbf{n} = \mathbf{n}D$ and the operator DS_m is ergodic for all $S_m \in S$.

We will not prove Proposition 1, first, because it is easy, and second, because a more general proposition (Proposition 3) will be proved below.

We introduce the space of trajectories $\tilde{x} = x^{Z^+}$ whose elements are $\tilde{x} = (x^0, x^1, \ldots, x^t, \ldots)$, where all $x^t \in X$. We denote by $P_{P,\mu}$ the probability distribution on the space of trajectories that corresponds to a Markov process with initial distribution μ and transition operator P.

Definition 2. For any island $y \in X$ and any infinite sequence $(h_t) = (h_0, h_1, \ldots, h_t \ldots)$, where all $h_t \in V$, we write $\alpha(P, y, (h_t)) = \mathbf{P}_{P,y}\{\tilde{x} : x_{h_0}^0 = n,\ x_{h_1}^1 = n, \ldots, x_{h_t}^t = n, \ldots\}$. We also write $\alpha_P = \sup_{y, (h_t)} \alpha(P, y, (h_t))$, where the supremum is taken over all islands $y \in X$ and all (h_t).

Obviously, if deterministic operator D is conservative, then $\alpha_D = 1$, while if D is nonconservative, we have $\alpha_D = 0$. In the more general case, however, we cannot give an example of a homogeneous local P (V being infinite) for which α_P is not 0 or 1. The situation is partially clarified by the following.

Proposition 2. Assume that operator Q is such that for any island x, measure xQ is concentrated on a finite number of states, each of which is an island. Then α_Q is equal to 0 or 1.

Here it is not required that Q be homogeneous, local, and monotonic. Our proof will be indirect: assume that $0 < \alpha_Q < 1$; we arrive at a contradiction. We take an island x and sequence (h_t) such that $\alpha(Q, x, (h_t)) > 0$. The condition implies that for any natural T, measure xQ^T is concentrated on a finite number of islands. Let

$$xQ^T = \sum_{k=1}^{K} \pi_k y_k,$$

where all the y_k are measures concentrated on islands y_k, all the $\pi_k > 0$, $\sum_{k=1}^{K} \pi_k = 1$. Then

$$\alpha(Q, x, (h_t)) = \mathbf{P}_{Q,x}\{\tilde{x}: \text{all the } x_{h_t}^t = n,\ t \in \mathbf{Z}^+\} = \sum_{k=1}^{K} \mathbf{P}_{Q,x}\{\tilde{x}: x^T = y_k, \text{ all the } x_{h_t}^t = n,\ t \in \mathbf{Z}^+\} =$$

$$= \sum_{k=0}^{K} \mathbf{P}_{Q,x}\{\tilde{x}: x^T = y_k, \text{ all the } x_{h_t}^t = n,\ 0 \leq t \leq T\} \alpha(Q, y_k, (h_T, h_{T+1}, \ldots)) \leq$$

$$\leq \alpha_Q \sum_{k=0}^{K} \mathbf{P}_{Q,x}\{\tilde{x}: x^T = y_k, \text{ all the } x_{h_t}^t = n,\ 0 \leq t \leq T\} = \alpha_Q \mathbf{P}_{Q,x}\{\tilde{x}: \text{all the } x_{h_t}^t = n,\ 0 \leq t \leq T\}.$$

But this is impossible since

$$\lim_{T \to \infty} \mathbf{P}_{Q,x}\{\tilde{x}: \text{all the } x_{h_t}^t = n, 0 \leq t \leq T\} = \alpha(Q, x, (h_t)).$$

Proposition 2 has thus been proved.

The following proposition reveals the significance of the case $\alpha_P = 1$ for the aspects of ergodicity under consideration here.

Proposition 3. Assume that P is a homogeneous local monotonic operator that can be written in the form $P = P'S_\varepsilon$, and $\alpha_P = 1$. Then P is ergodic and has limiting measure **n**.

Proof. We introduce an arbitrary $\delta > 0$. It suffices to construct a number T_δ such that for $T \geq T_\delta$ we have $0(PS_\varepsilon)^T\{x : x_0=n\}\geqslant 1-\delta$, where the zero subscript denotes an arbitrary element of V. First we take an island y and sequence (h_t) such that $\alpha(P, y, (h_t))\geqslant 1-(\delta/2)$. For each natural t we fix an automorphism g_t such that $g_t(h_t) = 0$. Then we will obviously have $\mathbf{P}_{P,0}(x_0^T=n|x^\tau>g_{t-\tau}(y))\geqslant 1-(\delta/2)$, where $1 \leq \tau \leq T$. This inequality also remains valid if we add any constraint on $x^1, \ldots, x^{T-1}$ to the condition, in particular the constraint that τ is the first instant t at which $x^t > g_{T-t}(y)$. This array of inequalities with this constraint on τ, varying from 1 to T, yields $\mathbf{P}_{P,0}(x_0^T=n|\exists\tau : 1\leqslant\tau\leqslant T,\ x^\tau>g_{T-\tau}(y))\geqslant 1-(\delta/2)$. Let us now bound from below the probability of the condition in this formula. We use the representation of operator S_ε in terms of the variables ω_h^t. We note that for $\tau \geq n$ the condition

$$\omega_h^t=1 \quad \text{for all} \quad h\in U^{\tau-t}(I_1(g_{T-\tau}(y))),\ \tau-n+1\leqslant t\leqslant\tau, \tag{2}$$

ensures that $x^\tau > g_{T-\tau}(y)$. We consider τ values that are multiples of n. Conditions (2) are mutually independent for such τ, and hence the probability that $x^\tau > g_{T-\tau}(y)$ for at least one $\tau \leq T$ (more precisely, even only for τ that are multiples of n) is not less than 1 minus the following expression:

$$\left(1-\varepsilon^{\sum_{k=0}^{n-1}|U^k(I_1(g_{T-\tau}(y)))|}\right)^{[T/n]}\sim \mathrm{const}^T,$$

where the base of the exponent is less than 1, and therefore the entire expression tends to 0 as $T \to \infty$. Taking T_δ to be such that for $T \geq T_\delta$ this expression is less than $\delta/2$, we obtain what was required.

§3. Examples

Paper [2] proved an assertion (with a constraint that can be readily circumvented) that is part of the converse to Proposition 1. Let us formulate it in our terms. Assume that D is a homogeneous univariate local monotonic operator $D : X \to X$, where $X = \{0, 1\}^Z$ (the automata have two states 0 and 1) and $\mathbf{1}D = \mathbf{1}$. Then if D is nonconservative, the operators are nonergodic for sufficiently small $\varepsilon > 0$. Paper [3] proved a more general assertion (for some multivariate operators). We can assume that an analogous assertion holds for all multivariate operators, and if this is the case it is valid for all D with a commutative transitive group of automorphisms (but only under the condition that the automata have two states).

In this section we will set up two examples of homogeneous local monotonic D that show that, in general form, the converse of Proposition 1 is not valid. In the first example the automata have free states 0, 1, and 2 and are connected in a univariate chain. In the second example the automata have two states, 0 and 1, but are connected in such a way that the group of automorphisms of D that is transitive on V is noncommutative. In both examples D possesses the following properties: a) it is nonconservative, b) operators DS_m are ergodic for all $S_m \in S$ and have limiting measure **n**.

The outline of the proof of ergodicity of DS_m is similar in both cases. We set up a monotonic operator Q such that $Q \prec DS_\varepsilon$ for specified $\varepsilon > 0$, which converts any island into a measure concentrated on a finite number of islands. Then we prove that $\alpha_Q > 0$. From this we have $\alpha_Q = 1$, by Proposition 2, and hence $\alpha_{DS_\varepsilon} = 1$ out of consideration of monotonicity. Therefore, operators DS_ε are ergodic for all $\varepsilon > 0$, by Proposition 3. But hence DS_m are also ergodic for all $S_m \in S$, since any S_m is minorized by some S_ε, $\varepsilon > 0$.

Operator Q is equal to $Q = DS_\varepsilon'$ in both examples, where S'_ε is given by the following equation:

$$xS_\varepsilon'=x\prod_{h\in U(I_1(x))} S_{\varepsilon,h},$$

the $S_{\varepsilon,h}$ being defined in Sec. 1, while $I_1(x)$ is the set of those $h \in V$ where $x_h \neq 0$.

In other words, S_ε' acts on an arbitrary $x \in X$ like S_ε in the neighborhood of nonzero components x and like an identity operator outside this neighborhood.

Example 1.* As stated above, here we have $X = \{0, 1, 2\}^Z = \{x\}$, where $x = (x_h)$, $h \in Z$, $x_h \in \{0; 1; 2\}$.

*This example was suggested by [4].

Each $(xD)_h$ (the h-th component of state xD) can be expressed in terms of x_{h-1}, x_h, x_{k+1} as follows:

$$(xD)_h = \begin{cases} 0, & \text{if} \quad x_{h-1} = 0,\ x_h = x_{h+1} = 1, \\ 1, & \text{if} \quad x_{h-1} = x_h = 2,\ x_{h+1} = 1, \\ 2, & \text{if} \quad x_{h-1} + x_h = x_{h+1} = 2, \\ & \text{the integer closest to} \quad \frac{1}{3}(x_{h-1} + x_h + x_{h+1}) \text{ otherwise.} \end{cases} \tag{3}$$

We will show that D is nonconservative. It suffices to consider how D acts on islands of the form ...000222...222000.... As a result of the second row on the right side of (3), the action of D causes the 2's in this island to change successively (right to left) to 1's, so that we obtain an island of the form ...000111...111000.... Moreover, because of the first row in (3) the action of D causes the 1's in this island to change successively (left to right) to 0's, so that any island is eventually transformed to the "all zero" state.

We will not prove that DS_m is ergodic, because the proof is analogous to that of the same property for Example 2.

Example 2. Here set V consists of all pairs of the form (i, j), where $i \in Z$, $j \in \{1, -1\}$, while $X = \{0, 1\}^V = \{x\}$, where $x = (x_h)$, $h \in V, x_h \in \{0, 1\}$. Operator $D: X \to X$ is specified by the condition $(xD)_{(i,j)} = (x_{(i,j)} \vee x_{(i,-j)}) \wedge x_{(i-j,j)}$. In other words there will be a 1 at point (i, j) at the next instant if and only if at point (i − j, i) there was a 1 at the preceding instant and there was a 1 at least one of the points (i, j), (i, −j).

First we will prove that D is a nonconservative operator. It suffices to consider how D acts on islands of the form

...000111...111000...
...000111...111000... .

We will assume that the i axis is directed to the right, while the j axis is directed upward. It is easy to see that D causes the 1's in this island to change successively to 0's, beginning with the left end in the upper series and with the right end in the lower one. As a result, the island changes over to the "all zero" state over a time equal to the length of the 1's files.

Now we will show that operators DS_m, where $S_m \in S$, are ergodic. Operator Q was described at the beginning of this section. We will show that $\alpha_Q > 0$. We assert that as y, (h_t), such that $\alpha(Q, y, (h_t)) > 0$, we can take $h_t = (0, 1)$ for all t and $y = (y_h)$, where

$$y_{(i,j)} = \begin{cases} 1, & \text{if} \quad -1 \leqslant i \leqslant 1, \\ 0 & \text{otherwise}. \end{cases}$$

We will prove this. We will formulate four conditions regarding the ensemble of variables ω_h^t, where $h \in V$, $t \in Z^+$, that control the action of S'_ε. Satisfaction of these four conditions is sufficient (but not necessary) for all variables $x^t_{(0,1)}$, as functions of the ensemble (ω_h^t), to be equal to 1. Each of these conditions refers to its own group of variables, and therefore the probability that all four conditions will be satisfied is equal to the product of the four probabilities that they will be satisfied separately. The first condition refers to variables $\omega^t_{(i,1)}$, where $i > 1$. First we will describe it informally. We note that if $x^t_{(i,1)} = 1$ for $a \le i \le b$, then $x^{t+1}_{(i,1)} = 1$ for $a + 1 \le i \le b$. In other words, the file of 1's on the upper line can be shortened only from the left under the action of D; its right end stays as is. If we should have $\omega^{t+1}_{(b+1,1)} = 1$, then the file of 1's lengthens by 1 on the right. We introduce the sequence of quantities k_t, $t \in Z^+$ functions of the ensemble $\omega_{(i,1)}$, where $i > 1$. The k_t are the coordinates i of the right ends of the files of 1's obtained in the upper line at instants t. The definition of k_t is inductive: $k_0 = 1$, while for $t > 0$ $k_t = k_{t-1} + \omega^t_{(k_{t-1}+1,1)}$.

It is clear that the sequence k_t is the random-walk trajectory of a point that shifts to the right over one time cycle with probability ε and remains as is with probability $1 - \varepsilon$. As we know, the condition $k_t \ge 1/2\varepsilon t$ for all $t \in Z^+$ will hold with positive probability. This is our first condition. The second condition refers to variables $\omega^t_{(i,-1)}$, where $i > 0$. First we will describe it informally. Assume that $x^t_{(i,1)} = 1$ for $a \le i \le b$. Then $x^{t+1}_{(i,-1)} = x^t_{(i+1,-1)}$ for $a \le i \le b$. In other words, if there is a 1's file on the upper line, under the action of D the configuration on the lower line under this file shifts 1 to the left over each time step. If the operator S'_ε also acts, the 0's in this configuration are replaced by 1's with probability ε. Therefore there are fewer and fewer 0's as we move to the left. The second condition ensures that no zero will appear at the point (0, −1). Its exact formulation is as follows: for any $t \ge 1$ at least one of the quantities $\omega^{t-l}_{(l,-1)}$, where $0 \le l \le \varepsilon t/(2 + \varepsilon)$, is equal to 1.

Here the limit $\varepsilon t/(2+\varepsilon)$ of variation of l is chosen in such a way that all pairs $(l, t-l)$ lie to the left of the the line $i = 1/2\varepsilon t$, i.e., in the region in which, according to the first condition, the upper line is filled with 1's. Once a 1 has arisen at point $(l, -1, t-l)$, therefore, it shifts by 1 to the left over each time cycle.

We will show that the probability that the second condition holds is positive. Indeed, it is not less than $\prod_{t=1}^{\infty}(1-(1-\varepsilon)^{[\varepsilon t/(2+\varepsilon)]})$. This expression is greater than zero, since the series $\sum_{t=1}^{\infty}(1-\varepsilon)^{[\varepsilon t/(2+\varepsilon)]}$ converges.

The first and second conditions together guarantee that our island is "inaccessible" from the right. The third and fourth conditions have a similar function for the left side. The third and fourth conditions can be obtained from the first and second by making the substitution $(i, j) \to (-i, -j)$.

It is clear from the above that the probability that all four conditions are satisfied is positive. A fortiori, the probability of the condition that stems from them, namely, "$x^t_{(0,1)}=1$ for all $t \in Z^+$," is also positive, QED.

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