Trajectories in random monads

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Abstract

Let us have a non-empty finite set S with $n > 1$ elements which we call points and a map $M : S \to S$. After V. I. Arnold, we call such pairs $(S,\ M)$ *monads*, but we consider *random monads* in which all the values of $M(\cdot)$ are random, independent and uniformly distributed in S. We fix some $\odot \in S$ and consider the infinite sequence $M^t(\odot),\ t = 0, 1, 2, \ldots$. A point is called *visited* if it coincides with at least one term of this sequence. A visited point is called *recurrent* if it appears in this sequence at least twice; if a visited point appears in this sequence only once, it is called *transient*. We denote by $Vis,\ Rec,\ Tra$ the numbers of visited, recurrent and transient points respectively and study their distributions. The distributions of $Vis,\ Rec,\ Tra$ are unimodal. The modes of Rec and Tra equal their minimal values, that is 1 and 0 respectively. The mode of Vis is approximated by $\sqrt{n}$, plus-minus a constant. The mathematical expectations: $\mathbb{E}(Vis)$ is approximated by $2\sqrt{\pi\, n/8}$ plus-minus a constant; $\mathbb{E}(Rec)$ and $\mathbb{E}(Tra)$ are approximated by $\sqrt{\pi\, n/8}$ plus-minus a constant. For the standard deviations $\sigma(Vis)$ and $\sigma(Rec) = \sigma(Tra)$ respectively we present the approximations

$$\sqrt{\frac{4-\pi}{2} \cdot n} \quad \text{and} \quad \sqrt{\frac{16-3\pi}{24} \cdot n}\ ,$$

from which they also deviate at most by a constant. We prove that when n tends to infinity, the correlations $\mathrm{Corr}(Rec, Tra)$ and $\mathrm{Corr}(Rec, Vis) = \mathrm{Corr}(Tra, Vis)$ converge to

$$\frac{8-3\pi}{16-3\pi} \quad \text{and} \quad \sqrt{\frac{12-3\pi}{16-3\pi}}.$$

Keywords: *Arnold's monads, random dynamical systems, incomplete gamma function, recurrent, transient.*

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Main statements

Let us have a non-empty finite set S with $n > 1$ elements, which we call *points*, and a map $M : S \to S$. Such a pair $(S,\ M)$ may be called a finite deterministic dynamical system, but we call it a *monad* after V. I. Arnold. The word *monad* had been used for many years in classical and modern studies when several years ago Vladimir Arnold [3, 4, 5] proposed the quoted above definition of it. Arnold emphasized simplicity of his definition and initialized a study of several concrete monads [6, 7, 8]. These studies have shown i.a. that however interesting and important may be concrete monads, many of them are not simple. On the other hand, we suggest a really simple background, against which it will be possible (as we expect) to recognize peculiarities of concrete deterministic monads. We suggest such a background in the form of *random monads* defined below. In this sense our approach is analogous to the well-known chaos or mean-field approximation widely used to obtail rough approximations of processes with local interaction.

Let us denote by $\#(\cdot)$ the cardinality of any finite set. In particular, we denote $n = \#(S)$, that is S has n elements. Given a monad (S, M) we may iterate M to obtain a sequence of maps M^t for all $t = 0,\ 1,\ 2, \dots$, where $M^0(s) = s$ for all $s \in S$ and $M^t = M(M^{t-1})$ for all natural t by definition. Given any initial $\odot$ and any M, we call by *trajectory* generated by $\odot$ and M the infinite sequence

$$M^0(\odot),\ M^1(\odot),\ M^2(\odot), \dots \tag{1}$$

Having $\odot \in S$ fixed, for each M we denote

$$\overline{M} = \{M^t(\odot),\quad t = 0,\ 1,\ 2, \dots\}. \tag{2}$$

The set of possible values of $\#(\overline{M})$ is $\{1,\ 2, \dots,\ n\}$. Notice also that $\#(\overline{M})$ is the minimal value of t such that

$$\{M^k(\odot)\ :\ k \in [0, \dots, t-1]\} = \overline{M}.$$

It is evident that the sequence $(M^t(\odot))$ is periodic starting at a certain value of t and therefore consists of the following two parts. A point is called *visited* if it coincides with at least one term of the sequence (1) . A visited point is called *recurrent* if it appears in this sequence at least twice; if a visited point appears in this sequence only once, it is called *transient*. We denote by Vis, Rec, Tra the numbers of visited, recurrent and transient points respectively. Notice that

$$\#(\overline{M}) = Vis = Rec + Tra.$$

We assume that the values $M(s)$ for all $s \in S$ are independently and uniformly distributed in S. Thus we have a distribution $\mathbb{P}$ defined as follows:

$$\forall\, w_1, \ldots,\, w_n \in S:\ \mathbb{P}\left(M(s_1) = w_1,\ \ldots,\ M(s_n) = w_n \right) = \left(\frac{1}{n}\right)^n.$$

Thus Vis, Rec and Tra are integer random variables, whose possible values are delimited by the inequalities

$$Rec \geq 1, \quad Tra \geq 0, \quad \text{and} \quad Vis = Rec + Tra \leq n.$$

The main task of this article is to study the distributions of Vis, Rec and Tra. As usual, $\mathbb{E}$ means mathematical expectation, Med means median, Var means variance, σ means standard deviation, Cov means covariance and Corr means correlation.

Theorem 1.

$$(i) \quad \begin{cases} \mathbb{P}(Rec = \rho) > \mathbb{P}(Rec = \rho + 1) & \text{for all } \rho \in \{1, \ldots, n-1\}; \\ \mathbb{P}(Tra = \tau) > \mathbb{P}(Tra = \tau + 1) & \text{for all } \tau \in \{0, \ldots, n-2\}. \end{cases}$$

Therefore for every natural n the probabilities of the events $Rec = \rho$ and $Tra = \tau$ strictly decrease as values of ρ and τ grow within the ranges of Rec and Tra. Therefore these distributions are unimodal with modes 1 and 0 (their smallest possible values) respectively.

(ii) As λ grows within the range of Vis, the probability of $Vis = \lambda$ first strictly grows, then perhaps stays the same at most once, then strictly decreases. Therefore the distribution of Vis is unimodal (having maximum at one or two neighbor values) and denoting its mode by $\mathrm{Mode}(Vis)$, we have

$$\sqrt{n} - 2 \ \leq\ \mathrm{Mode}(Vis) \ \leq\ \sqrt{n} + 1. \tag{3}$$

Now let us speak about mathematical expectation. Let us denote for all natural n

$$\phi(n) = \sqrt{\pi\, n/8}. \tag{4}$$

Theorem 2.

$$\begin{aligned} (i)\quad 2\phi(n)-2 &< \mathbb{E}(Vis) < 2\phi(n)+2;\\ (ii)\quad \phi(n)-1 &< \mathbb{E}(Rec) < \phi(n)+2;\\ (iii)\quad \phi(n)-2 &< \mathbb{E}(Tra) < \phi(n)+1. \end{aligned}$$

Theorem 2 is illustrated by figure 1.

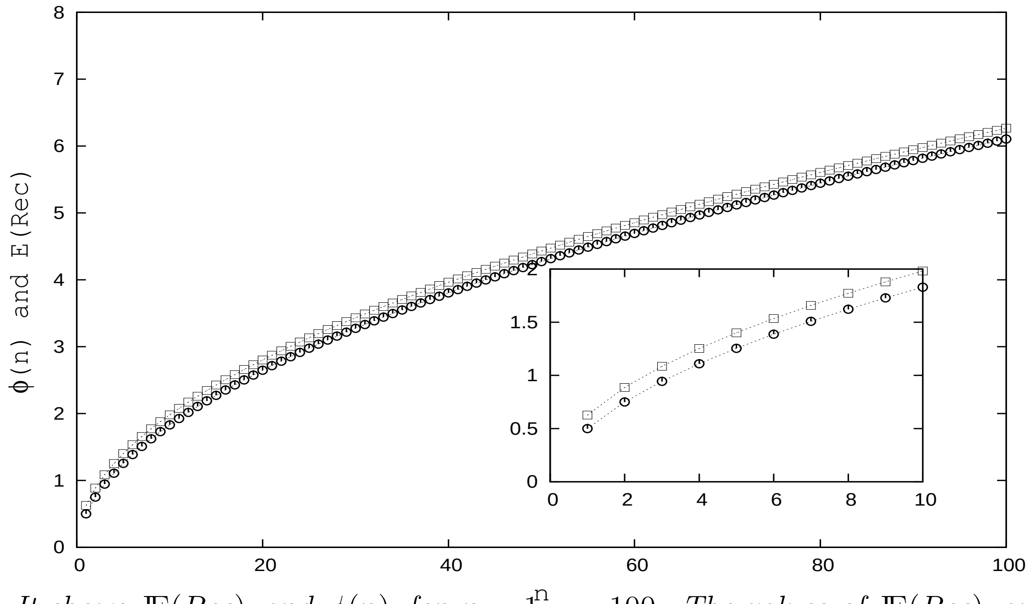

Figure 1: *It shows* $\mathbb{E}(Rec)$ *and* $\phi(n)$ *for* $n = 1, \ldots, 100$. *The values of* $\mathbb{E}(Rec)$ *are represented by squares, the values of* $\phi(n)$ *are represented by balls. This figure suggests that the difference* $\mathbb{E}(Rec) - \phi(n)$ *is always positive. Our calculations suggest that this difference decreases as* n *increases. If this is so, the greatest value of this difference is achieved when* $n = 1$ *and equals* $1 - \sqrt{\pi/8} \approx 0.373$.

Since $\mathbb{P}(Rec = \alpha) = \mathbb{P}(Tra = \alpha - 1)$ for all α and $Vis = Rec + Tra$, graphs of $\mathbb{E}(Vis)$ and $\mathbb{E}(Tra)$ are essentially the same, so we do not include them.

Theorem 3.

$$\begin{aligned} (i)\ \sqrt{\psi_1 \cdot n} - 5 < &\quad \sigma(Vis) &< \sqrt{\psi_1 \cdot n} + 3,\\ (ii)\ \sqrt{\psi_2 \cdot n} - 2 < &\ \sigma(Rec) = \sigma(Tra) &< \sqrt{\psi_2 \cdot n} + 2, \end{aligned}$$

where

$$\psi_1 = \frac{4-\pi}{2} \quad \text{and} \quad \psi_2 = \frac{16-3\pi}{24}.$$

Theorem 3 is illustrated by figures 2 and 3.

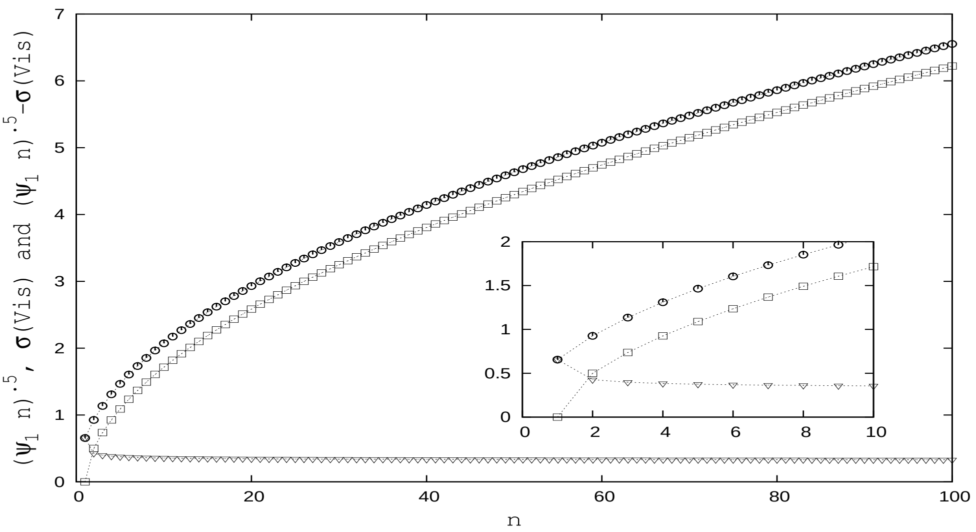

Figure 2: *This figure shows the values of $\sigma(Vis)$ represented by squares, values of $\sqrt{\psi_1 n}$ represented by balls and the difference between them represented by triangles. This figure suggests that the difference $\sqrt{\psi_1 n} - \sigma(Vis)$ is always positive. Our calculations suggest that this difference decreases as n increases. If this is so, its maximum is achieved at $n = 1$ and equals* $\sqrt{(4-\pi)/2} \approx 0.655$.

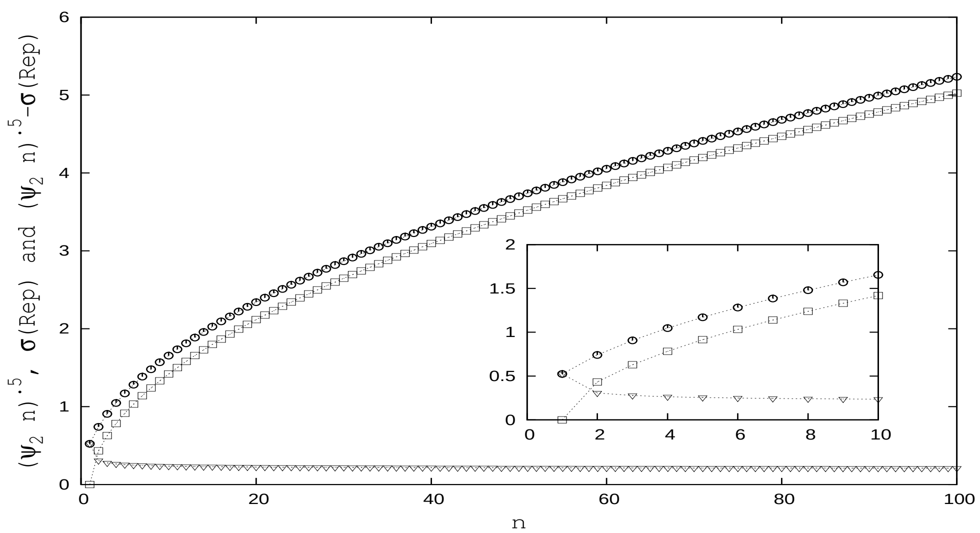

Figure 3: *This figure shows the values of $\sigma(Rec)$ represented by squares, values of $\sqrt{\phi_2 n}$ represented by balls and the difference between them represented by triangles. This figure suggests that the difference $\sqrt{\psi_2 n} - \sigma(Vis)$ is always positive. Our calculations suggest that this difference decreases as n increases. If this is so, its maximum is achieved at $n = 1$ and equals* $\sqrt{(16-3\pi)/24} \approx 0.523$.

Theorem 4.

$$\left.\begin{array}{l} (i)\ \ \mathrm{Corr}(Rec, Tra) \text{ tends to } \dfrac{8-3\pi}{16-3\pi} \quad \text{and} \\[2ex] (ii)\ \mathrm{Corr}(Rec, Vis) = \mathrm{Corr}(Tra, Vis) \text{ tends to } \sqrt{\dfrac{12-3\pi}{16-3\pi}} \end{array}\right\} \text{ when } n \to \infty\,.$$

About behavior of medians of Vis, Rec and Tra, we have only proposition 7 and the following conjecture supported by figure 4.

Conjecture. There are positive constants C_1, C_2 and C such that for all n

$$\left|Med(Vis) - C_1\sqrt{n}\right| \le C \quad \text{and} \quad \left|Med(Rec) - C_2\sqrt{n}\right| \le C$$

where, according to our estimations, $C_1 \approx 1.2$ and $C_2 \approx 0.5$.

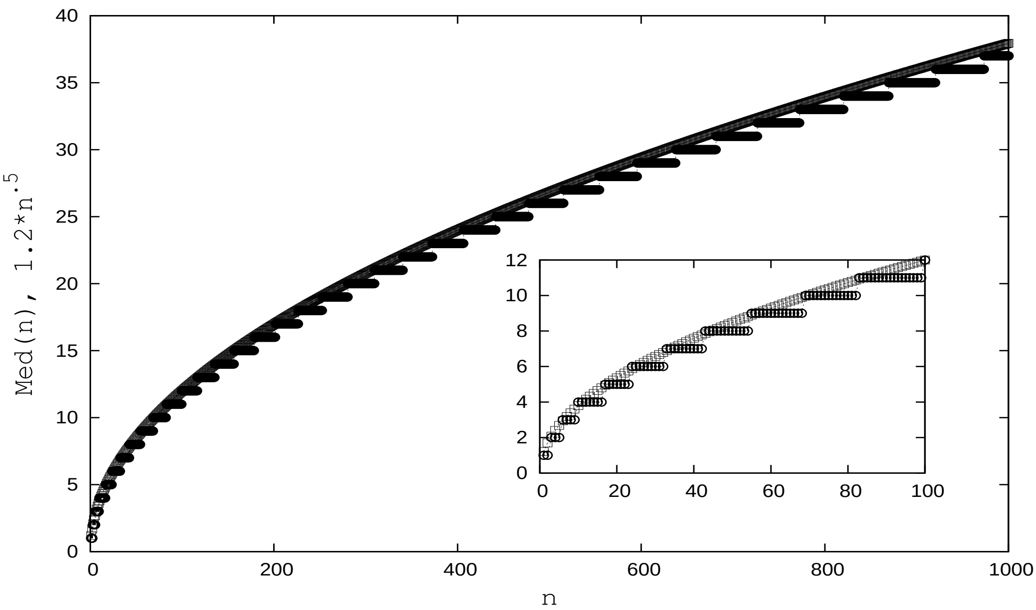

Figure 4: *This figure shows values of median of Vis for $n = 1, \ldots, 100$ evaluated by the expression at proposition 7, item (i) and the values of $1.2\sqrt{n}$.*

Thus we have stated all our main results as theorems; it remains to prove them. The use of these results, as we see it, is that they describe the most "neutral" behavior of an "average" monad. Thus we provide a background to compare with all those concrete deterministic monads which researchers may choose to study in the present and future. In fact we have exact expressions for all those quantities, which we estimate in our theorems 2, 3 and 4. These expressions are given in propositions 3, 4, 5 and 6.

Proofs

Proposition 1. For all integer numbers $\rho \geq 1, \quad \tau \geq 0$ and λ such that $\lambda = \tau + \rho \leq n$

$$
\begin{array}{llcl}
(i) & \mathbb{P}(Vis = \lambda) & = & \dfrac{\lambda \cdot (n-1)!}{n^{\lambda} \cdot (n-\lambda)!} \ \text{ for } \ \lambda = 1, \ldots, n; \\[2ex]
(ii) & \mathbb{P}(Rec = \rho,\ Tra = \tau) & = & \dfrac{(n-1)!}{n^{\lambda} \cdot (n-\lambda)!} \ \text{ for } \ \rho = 1, \ldots, n,\ \tau = 0, \ldots, n-1; \\[2ex]
(iii) & \mathbb{P}(Rec = \rho) & = & \displaystyle\sum_{\lambda=\rho}^{n} \ \frac{(n-1)!}{n^{\lambda} \cdot (n-\lambda)!} \ \text{ for } \ \rho = 1, \ldots, n; \\[2ex]
(iv) & \mathbb{P}(Tra = \tau) & = & \displaystyle\sum_{\lambda=\tau+1}^{n} \frac{(n-1)!}{n^{\lambda} \cdot (n-\lambda)!} \ \text{ for } \ \tau = 0, \ldots, n-1; \\[2ex]
(v) & \mathbb{P}(Rec = \alpha) & = & \mathbb{P}(Tra = \alpha - 1) \ \text{ for } \ \alpha = 1, \ldots, n. \\[2ex]
(vi) & \mathrm{Mode}(Rec) & = & \mathrm{Mode}(Tra) + 1; \\[2ex]
(vii) & \mathbb{E}(Rec) & = & \mathbb{E}(Tra) + 1; \\[2ex]
(viii) & \mathbb{E}(Vis) & = & 2\,\mathbb{E}(Rec) - 1 = 2\,\mathbb{E}(Tra) + 1; \\[2ex]
(ix) & \mathrm{Var}(Tra) & = & \mathrm{Var}(Rec), \quad \sigma(Tra) = \sigma(Rec); \\[2ex]
(x) & \mathrm{Cov}(Rec, Vis) & = & \mathrm{Cov}(Tra, Vis) = \mathrm{Var}(Tra) + \mathrm{Cov}(Rec, Tra).
\end{array}
$$

Proof. For any condition C let us denote by $\{M \,|\, C\}$ the event " C is satisfied". For every integer $k \in [1,\ n-1]$ we denote by D_k the event "all the terms of the sequence $M^i(\odot)$, where i runs from 0 to k, are different from each other". Notice that for every $k \in [1,\ n-1]$ the probability of D_k equals

$$
\mathbb{P}(D_k) \ = \ \prod_{i=1}^{k} \ \frac{n-i}{n} \ = \ \frac{(n-1)!}{n^k \cdot (n-k-1)!}. \tag{5}
$$

Also notice that

$$\{M \,|\, Vis = \lambda\} = D_{\lambda-1} \setminus D_\lambda \text{ and } D_{\lambda-1} \supset D_\lambda \text{ for all integer } \lambda \in [1,\ n].$$

Therefore

$$\mathbb{P}(Vis = \lambda) = \mathbb{P}(D_{\lambda-1}) - \mathbb{P}(D_\lambda).$$

Hence, using (5) , we get item (i) .

Now let us prove item (ii) . Notice that

$$\{M \,|\, Rec = \rho,\ Tra = \tau\} \ = \ D_{\lambda-1} \ \cap \ \{M \,|\, M^{\lambda+1}(\odot) = M^{\tau}(\odot)\} \tag{6}$$

for all integer $\lambda,\ \rho,\ \tau$ such that

$$\rho \geq 1, \quad \tau \geq 0, \quad \lambda = \rho + \tau \leq n.$$

Since the events intersected in the right side of (6) are independent from each other, the probability of the intersection is a product of their probabilities. The probability of the latter one is $1/n$. Therefore

$$\mathbb{P}(Tra = \tau,\ Rec = \rho) \ = \ \mathbb{P}(D_{\lambda-1}) \cdot \frac{1}{n}.$$

Substituting here (5) , we get item (ii) . The remaining items follow from this one. *Proposition 1 is proved.*

Now let us prove theorem 1. Proof of item (i) follows from items (iii) and (iv) of proposition 1. Let us prove item (ii). Due to item (i) of proposition 1,

$$\frac{\mathbb{P}(Vis = \lambda + 1)}{\mathbb{P}(Vis = \lambda)} = \frac{(\lambda + 1) \cdot (n - \lambda)}{\lambda \cdot n}. \tag{7}$$

This expression is less than one iff $\lambda^2 + \lambda > n$. Let us denote by λ_0 the minimal integer value of λ for which the expression (7) is less than one. Notice that $\mathrm{Mode}(Vis)$ equals either λ_0 or $\lambda_0 - 1/2$. Thus

$$\lambda_0 - 1/2 \leq \mathrm{Mode}(Vis) \leq \lambda_0.$$

On the other hand let us denote by λ_1 that real value of λ at which the expression (7) equals one, that is the real positive root of the equation $\lambda^2+\lambda=n$. It is evident that

$$\lambda_1 \le \lambda_0 \le \lambda_1 + 1$$

and

$$\sqrt{n}-1 \le \lambda_1 \le \sqrt{n}.$$

The estimation (3) follows from these inequalities. *Theorem 1 is proved.*

Throughout this article we assume that summing over an empty set results in zero. Thus whenever $b<a$,

$$\sum_{k=a}^{b} \text{anything} \quad = 0. \tag{8}$$

Lemma 1. For all natural n

$$e^n\left(\frac{1}{2}-\sqrt{\frac{9}{2\,\pi\,n}}\right) \;<\; \sum_{k=0}^{n-3}\frac{n^k}{k!} \;<\; \frac{e^n}{2}. \tag{9}$$

Proof. First let us prove the right inequality in (9) . It may be rewritten as

$$e^{-n}\sum_{k=0}^{n-3}\frac{n^k}{k!} \;=\; \sum_{k=0}^{n-3}\mathbb{P}(X=k) \;<\; 1/2,$$

where the random variable X has Poisson distribution with parameter n, that is

$$\mathbb{P}(X=k)=\frac{e^{-n}\cdot n^k}{k!} \quad \text{for all} \quad k=0,\ 1,\ 2,\ldots$$

For this case it was shown in [2] that

$$\sum_{k=0}^{n}\mathbb{P}(X=k)\ge 1/2 \quad \text{and} \quad \sum_{k=0}^{n-1}\mathbb{P}(X=k)<1/2.$$

Therefore

$$\sum_{k=0}^{n-3}\mathbb{P}(X=k) < \sum_{k=0}^{n-1}\mathbb{P}(X=k) \;<\; 1/2.$$

Now let us prove the left inequality in (9) . Evidently,

$$e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}\ge\frac{1}{2} \iff e^{-n}\sum_{k=0}^{n-3}\frac{n^k}{k!}\ge\frac{1}{2}-e^{-n}\sum_{k=n-2}^{n}\frac{n^k}{k!}.$$

Using (16) , we get

$$\sum_{k=n-2}^{n} \frac{n^k}{k!} = \frac{3n^n - n^{n-1}}{n!} \le \frac{e^n \cdot (3n^n - n^{n-1})}{\sqrt{2\pi n} \cdot n^n} \le \frac{3 \cdot e^n}{\sqrt{2\pi n}}.$$

So

$$e^{-n} \sum_{k=n-2}^{n} \frac{n^k}{k!} \ \le \ \frac{3}{\sqrt{2\pi n}}.$$

Therefore

$$e^{-n} \sum_{k=0}^{n-3} \frac{n^k}{k!} \ = \ \frac{1}{2} - e^{-n} \sum_{k=n-2}^{n} \frac{n^k}{k!} \ \ge \ \frac{1}{2} - \frac{3}{\sqrt{2\pi n}}.$$

Lemma 1 is proved.

We shall use the upper incomplete gamma function or $\Gamma(z,\ a)$

$$\Gamma(z,\ a) = \int_a^\infty t^{z-1} e^{-t} dt.$$

We use $\Gamma(z,\ a)$ only for integer values of $z \ge 1$ and $a \ge 0$. In this case all its properties, which we need, can be found in [1]. In particular

$$\Gamma(z,\ a) = (z-1)! \cdot e^{-a} \cdot \sum_{k=0}^{z-1} \frac{a^k}{k!}. \tag{10}$$

Based on (10) , it is easy to show for all integer $j \ge 0$ and $n \ge 1$ that

$$\Gamma(j+1,\ n) = j \cdot \Gamma(j,\ n) + e^{-n}\, n^j, \tag{11}$$

$$\Gamma(j,\ n) = \frac{\Gamma(j+2,\ n) - e^{-n} \cdot n^j \cdot (n+j+1)}{(j+1) \cdot j} \tag{12}$$

and

$$\sum_{k=0}^{j} \frac{n^k}{k!} \ = \ \frac{e^n \cdot \Gamma(j+1,\ n)}{j!}. \tag{13}$$

For all natural n we denote

$$\Phi(n) = \frac{n!}{2\, n^n} \sum_{k=0}^{n-1} \frac{n^k}{k!}. \tag{14}$$

Using $\Gamma(z,a)$, we may rewrite (14) as follows:

$$\Phi(n) = \frac{e^n \ \cdot \Gamma(n,\ n)}{2\, n^{n-1}}.$$

Proposition 2.

$$\phi(n) - 3/2 < \Phi(n) - 1 + \frac{1}{2n} < \phi(n), \tag{15}$$

where $\phi(n)$ was defined in (4) and $\Phi(n)$ was defined in (14) .

We shall use the following form of Stirling's approximation [9]: for all natural n

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e^{\lambda_n} \quad \text{with} \quad 0 \le \lambda_n \le \frac{1}{12n}. \tag{16}$$

Now let us prove the left inequality in (15) . For $n = 1$ and $n = 2$ we check it numerically. Now let $n \ge 3$. Then, using the left inequality in (9) , we get

$$\Phi(n) - 1 + \frac{1}{2n} = \frac{n!}{2n^n}\sum_{k=0}^{n-3}\frac{n^k}{k!} \ge$$

$$\frac{n!}{2n^n} \cdot e^n \cdot \left(\frac{1}{2} - \frac{3}{\sqrt{2\pi n}}\right) \ge$$

$$\phi(n) - 3/2.$$

Now let us prove the right inequality in (15) . For $n = 1$ and $n = 2$ we check it numerically. For $n \ge 3$ we use the right inequality of (9) to obtain

$$\Phi(n) - 1 + \frac{1}{2n} < \phi(n) \Longleftrightarrow \sum_{k=0}^{n-3}\frac{n^k}{k!} < \frac{e^n}{2}.$$

Of course

$$\Phi(n) - 1 + \frac{1}{2n} < \phi(n) \Longleftrightarrow \sum_{k=0}^{n-3}\frac{n^k}{k!} < \frac{2n^n}{n!}\phi(n).$$

But

$$\frac{2n^n}{n!}\phi(n) \le \frac{e^n}{\sqrt{2\pi n}}\sqrt{\frac{\pi n}{2}} = \frac{e^n}{2}.$$

Proposition 2 is proved.

Proposition 3.

$$(i) \qquad \mathbb{E}(Vis) = 2\Phi(n);$$

$$(ii) \qquad \mathbb{E}(Rec) = \Phi(n) + \frac{1}{2};$$

$$(iii) \qquad \mathbb{E}(Tra) = \Phi(n) - \frac{1}{2}.$$

Proof. Let us first prove that

$$\mathbb{E}(Rec) = \Phi(n) + \frac{1}{2}. \tag{17}$$

For $n = 1$ and $n = 2$ we check this by numerical computation. Now let $n \geq 3$. From item (iii) of proposition 1

$$\mathbb{E}(Rec) = \sum_{\rho=1}^{n} \rho \cdot \sum_{\lambda=\rho}^{n} \frac{(n-1)!}{n^{\lambda} \cdot (n-\lambda)!} =$$

$$(n-1)! \sum_{\lambda=1}^{n} \sum_{\rho=1}^{\lambda} \frac{\rho}{n^{\lambda}(n-\lambda)!} =$$

$$\frac{(n-1)!}{2} \cdot \sum_{\lambda=1}^{n} \frac{\lambda^2 + \lambda}{n^{\lambda} \cdot (n-\lambda)!}. \tag{18}$$

Substituting $k = n - \lambda$ into (18) , we get

$$\frac{1}{(n-1)!} \cdot \mathbb{E}(Rec) =$$

$$\frac{1}{2n^n} \left((n^2 + n) \sum_{k=0}^{n-1} \frac{n^k}{k!} \; - \; (2n+1) \sum_{k=0}^{n-1} \frac{k\, n^k}{k!} \; + \; \sum_{k=0}^{n-1} \frac{k^2\, n^k}{k!} \right). \tag{19}$$

To transform (19) further, let us note that

$$\sum_{k=0}^{n-1} \frac{k\, n^k}{k!} = n \sum_{k=1}^{n-1} \frac{n^{k-1}}{(k-1)!} = n \sum_{k=0}^{n-2} \frac{n^k}{k!} \tag{20}$$

and

$$\sum_{k=0}^{n-1} \frac{k^2\, n^k}{k!} = n^2 \sum_{k=0}^{n-3} \frac{n^k}{k!} + n \sum_{k=0}^{n-2} \frac{n^k}{k!}. \tag{21}$$

Thus (19) turns into

$$\begin{aligned}
\frac{1}{(n-1)!}\cdot \mathbb{E}(Rec) &= \frac{1}{2n^n}\left((n^2+n)\sum_{k=0}^{n-1}\frac{n^k}{k!} - 2n^2\sum_{k=0}^{n-2}\frac{n^k}{k!} + n^2\sum_{k=0}^{n-3}\frac{n^k}{k!}\right)\\
&= \frac{1}{2n^n}\left(n\sum_{k=0}^{n-3}\frac{n^k}{k!} + (n-n^2)\frac{n^{n-2}}{(n-2)!} + (n^2+n)\frac{n^{n-1}}{(n-1)!}\right)\\
&= \frac{1}{2n^n}\left(n\sum_{k=0}^{n-1}\frac{n^k}{k!} + \frac{n^n}{(n-1)!}\right)
\end{aligned}$$

Hence, using (13) , we get

$$\frac{1}{(n-1)!}\cdot \mathbb{E}(Rec) = \frac{1}{2n^n}\left(n\left(\frac{e^n\cdot\Gamma(n,\ n)}{(n-1)!}\right) + \frac{n^n}{(n-1)!}\right).$$

Thus (17) is proved. *Now, due to (17) and items (vii) and (viii) of proposition 1, proposition 3 is proved.*

Lemma 2.

$$\mathbb{E}(Rec^2) = \frac{\Phi(n)}{3} + \frac{4n+1}{6}.$$

Proof. For $n \le 3$ we prove this by calculation. Now let $n \ge 4$. Then

$$\mathbb{E}(Rec^2) = \sum_{\rho=1}^{n}\rho^2\cdot\sum_{\lambda=\rho}^{n}\frac{(n-1)!}{n^\lambda\cdot(n-\lambda)!} = \frac{(n-1)!}{6}\sum_{\lambda=1}^{n}\frac{2\lambda^3+3\lambda^2+\lambda}{n^\lambda\cdot(n-\lambda)!}. \tag{22}$$

Substituting $k = n-\lambda$ into (22) , we get

$$\frac{6\,n^n}{(n-1)!}\cdot\mathbb{E}(Rec^2) \;=\; (2n^3+3n^2+n)\sum_{k=0}^{n-1}\frac{n^k}{k!}$$

$$-(6n^2+6n+1)\sum_{k=0}^{n-1}\frac{k\cdot n^k}{k!} + (6n+3)\sum_{k=0}^{n-1}\frac{k^2\cdot n^k}{k!} - 2\sum_{k=0}^{n-1}\frac{k^3\cdot n^k}{k!}.$$

Notice that

$$\sum_{k=0}^{n-1}\frac{k^3 n^k}{k!} = n\left(\sum_{k=0}^{n-2}\frac{n^k}{k!} + 3n\sum_{k=0}^{n-3}\frac{n^k}{k!} + n^2\sum_{k=0}^{n-4}\frac{n^k}{k!}\right), \tag{23}$$

Using (20) , (21) and (23) , we get

$$\frac{6\,n^n}{(n-1)!}\cdot \mathbb{E}(Rec^2) \;=\; (2n^3+3n^2+n)\sum_{k=0}^{n-1}\frac{n^k}{k!} - (6n^3+6n^2+n)\sum_{k=0}^{n-2}\frac{n^k}{k!}$$

$$+(6n+3)\left(n^2\sum_{k=0}^{n-3}\frac{n^k}{k!} + n\sum_{k=0}^{n-2}\frac{n^k}{k!}\right) - 2n\left(\sum_{k=0}^{n-2}\frac{n^k}{k!} + 3n\sum_{k=0}^{n-3}\frac{n^k}{k!} + n^2\sum_{k=0}^{n-4}\frac{n^k}{k!}\right).$$

We may rewrite this as

$$\frac{6\,n^n}{(n-1)!}\cdot \mathbb{E}(Rec^2) = n\sum_{k=0}^{n-4}\frac{n^k}{k!} + \frac{2n^{n-2}}{(n-1)!}\cdot(2n^3+2n^2-2n+1). \tag{24}$$

Thus, using (13) , we get

$$\sum_{k=0}^{n-4}\frac{n^k}{k!} = \frac{e^n\cdot\Gamma(n,\ n)}{(n-1)!} - \sum_{k=n-3}^{n-1}\frac{n^k}{k!}.$$

Finally, we use (24) . *Lemma 2 is proved.*

Proposition 4.

$$\mathrm{Var}(Tra) = \mathrm{Var}(Rec) = -(\Phi(n))^2 - \frac{2\Phi(n)}{3} + \frac{8n-1}{12}.$$

Proof of Proposition 4. From item (ix) of proposition 1, $\mathrm{Var}(Tra) = \mathrm{Var}(Rec)$. Using this with the well-known identity $\mathrm{Var}(Rec) = \mathbb{E}(Rec^2) - \mathbb{E}(Rec)^2$ and using (11) and lemma 2, we complete our *proof of proposition 4.*

Lemma 3.

$$\mathbb{E}(Tra\cdot Rec) = -\frac{\Phi(n)}{3} + \frac{2n-1}{6}.$$

Proof. For $n \le 4$ we prove this by numerical computation. Now let $n \ge 5$. Then proposition 1 gives us

$$\frac{\mathbb{E}(Tra\cdot Rec)}{(n-1)!} = \sum_{\tau=0}^{n-1}\sum_{\rho=1}^{n-\tau}\frac{\tau\cdot\rho}{(n-\lambda)!\cdot n^\lambda} =$$

$$\sum_{k=1}^{n-1}\left(\frac{1}{(n-(k+1))!\cdot n^{k+1}}\cdot\sum_{l=1}^{k} l(k-l+1)\right).$$

But

$$\sum_{l=1}^{k} l(k-l+1) = (k+1)\sum_{l=1}^{k} l - \sum_{l=1}^{k} l^2 =$$

$$\frac{k(k+1)^2}{2} - \frac{k(k+1)(2k+1)}{6} = \frac{k^3+3k^2+2k}{6}.$$

Therefore

$$\frac{6\,\mathbb{E}(Tra \cdot Rec)}{(n-1)!} = \sum_{k=1}^{n-1} \frac{k^3+3k^2+2k}{(n-(k+1))! \cdot n^{k+1}}.$$

Taking $j = n-(k+1)$, we get

$$\frac{6\,\mathbb{E}(Tra \cdot Rec)}{(n-1)!} = \sum_{j=0}^{n-2} \frac{(n-j-1)((n-j-1)^2+3(n-j-1)+2)}{j! \cdot n^{n-j}}.$$

So

$$\frac{6n^n\,\mathbb{E}(Tra \cdot Rec)}{(n-1)!} =$$

$$(n^3-n)\sum_{j=0}^{n-2}\frac{n^j}{j!} + (-3n^2+1)\sum_{j=0}^{n-2}\frac{jn^j}{j!} + 3n\sum_{j=0}^{n-2}\frac{j^2n^j}{j!} - \sum_{j=0}^{n-2}\frac{j^3n^j}{j!}.$$

From (20) , (21) and (23) , we get

$$\frac{6n^n\,\mathbb{E}(Tra \cdot Rec)}{(n-1)!} = -n\sum_{j=0}^{n-5}\frac{n^j}{j!} + \frac{n^{n-3}}{(n-2)!}\cdot(2n^3-3n^2+7n-6). \tag{25}$$

Using (12) , we get

$$\Gamma(n-4,\ n) = \frac{\Gamma(n-2,\ n) - e^{-n}\,n^{n-4}\,(2n-3)}{(n-3)(n-4)}.$$

Hence, using (13) , we have

$$\sum_{j=0}^{n-5}\frac{n^j}{j!} = \frac{e^n\cdot\Gamma(n-4,\ n)}{(n-5)!} = \frac{e^n\cdot\Gamma(n-2,\ n) - n^{n-4}\,(2n-3)}{(n-3)!}.$$

So, returning to (25)

$$\frac{6n^n\,\mathbb{E}(Tra\cdot Rec)}{(n-1)!}=$$

$$-n\left(\frac{e^n\cdot\Gamma(n-2,\ n)-n^{n-4}\,(2n-3)}{(n-3)!}\right)+\frac{n^{n-3}}{(n-2)!}\,(2n^3-3n^2+7n-6)=$$

$$-n\left(\frac{e^n\cdot\Gamma(n-2,\ n)}{(n-3)!}\right)-\frac{n^{n-1}}{(n-2)!}(1-2n).$$

Thus, using (12) again, *lemma 3 is proved.*

Lemma 4.

$$\mathrm{Cov}(Tra,\ Rec)=-(\Phi(n))^2-\frac{\Phi(n)}{3}+\frac{4n+1}{12}.$$

Proof. The well-known formula

$$\mathrm{Cov}(Tra,\ Rec)=\mathbb{E}(Tra\cdot Rec)-\mathbb{E}(Tra)\cdot\mathbb{E}(Rec)$$

allows us to use proposition 3 and lemma 3. *Lemma 4 is proved.*

Proposition 5.

$$\mathrm{Var}(Vis)=-4(\Phi(n))^2-2\Phi(n)+2n.$$

Proof. From item (ix) of proposition 1, $\mathrm{Var}(Tra)=\mathrm{Var}(Rec)$. Therefore

$$\mathrm{Var}(Vis)=2\left(\mathrm{Var}(Tra)+\mathrm{Cov}(Tra,Rec)\right).$$

Hence, using proposition 4 and lemma 4. *Proposition 5 is proved.*

Proof of theorem 2. From items (vii) and $(viii)$ of proposition 1 we have

$$\mathbb{E}(Tra)=\mathbb{E}(Rec)-1\quad\text{and}\quad\mathbb{E}(Vis)=2\,\mathbb{E}(Rec)-1.$$

Let us denote $k=\Phi(n)$ and use the function

$$f(k)=\mathbb{E}(Rec)=k+\frac{1}{2}.$$

Of course $f(k)$ is a linear function of k and it increases as k increases. But from proposition 2

$$\phi(n)-1<\Phi(n)<\phi(n)+1 \Longrightarrow f\left(\phi(n)-1\right)<f\left(\Phi(n)\right)<f\left(\phi(n)+1\right).$$

Therefore,

$$f\left(\phi(n)-1\right)<\mathbb{E}(Rec)<f\left(\phi(n)+1\right).$$

Thus

$$f\left(\phi(n)-1\right)-1<\mathbb{E}(Rec)-1<f\left(\phi(n)+1\right)-1$$

and

$$2f\left(\phi(n)-1\right)-1<2\mathbb{E}(Rec)-1<2f\left(\phi(n)+1\right)-1.$$

Theorem 2 is proved.

Lemma 5. Let us denote

$$f(k)=-k^2-\frac{2k}{3}+\frac{8n-1}{12}, \tag{26}$$

$$g(k)=-4k^2-2k+2n. \tag{27}$$

Then

$$(i)\ f\big(\phi(n)\big)-2\phi(n)-\frac{5}{3}<\ \mathrm{Var}(Rec)=\mathrm{Var}(Tra)\ <f\big(\phi(n)\big)+2\phi(n)-\frac{1}{3},$$

$$(ii)\ g\big(\phi(n)\big)-8\phi(n)-6<\qquad \mathrm{Var}(Vis)\qquad <g\big(\phi(n)\big)+8\phi(n)-2.$$

Proof. First let us prove item (i). From item (ix) of proposition 1 we know that $Var(Tra)=Var(Rec)$. Then, denoting $k=\Phi(n)$ and $f(k)=\mathrm{Var}(Tra)$, we have

$$f(k)=\mathrm{Var}(Tra)=\mathrm{Var}(Rec)=-k^2-\frac{2k}{3}+\frac{8n-1}{12}.$$

This expression is a quadratic function of k with a negative second derivative. So it is positive when k is between the roots

$$\alpha_1=\frac{-2-\sqrt{24n+1}}{6}\ \text{ and }\ \alpha_2=\frac{-2+\sqrt{24n+1}}{6}.$$

Notice also that $f(k)$ decreases for $k > (\alpha_1 + \alpha_2)/2 = -1/3$. Using proposition 2 for $n > 0$, we get

$$-\frac{1}{3} < \phi(n) - 1 < \Phi(n) < \phi(n) + 1.$$

Thus

$$f\left(\phi(n) + 1\right) < f\left(\Phi(n)\right) < f\left(\phi(n) - 1\right).$$

It is easy to evaluate

$$\begin{aligned} f\left(\phi(n) + 1\right) &= f(\phi(n)) - \left(2\phi(n) + \tfrac{5}{3}\right) \\ f\left(\phi(n) - 1\right) &= f(\phi(n)) - \left(2\phi(n) - \tfrac{1}{3}\right). \end{aligned}$$

Thus item (i) is proved. The proof of item (ii) is analogous. *Lemma 5 is proved.*

Proof of Theorem 3. First let us prove the right side of (i). Using lemma 5, we have

$$\begin{aligned} Var(Vis) &< \frac{4-\pi}{2} \cdot n + 6\sqrt{\frac{\pi}{8}} \cdot \sqrt{n} + 1 \\ &< \frac{4-\pi}{2} \cdot n + 6\sqrt{\frac{4-\pi}{2}} \cdot \sqrt{n} + 9 \\ &= \left(\sqrt{\frac{4-\pi}{2} \cdot n} + 3\right)^2 \end{aligned}$$

because

$$6\sqrt{\frac{\pi}{8}} < 6\sqrt{\frac{4-\pi}{2}} \text{ and } 1 < 9.$$

Therefore

$$\sigma(Vis) = \sqrt{\mathrm{Var}(Vis)} < \left| \sqrt{\frac{4-\pi}{2} \cdot n} + 3 \right| = \sqrt{\frac{4-\pi}{2} \cdot n} + 3.$$

Hence follows the right side of item (i) of theorem 3 for all n.

Now to prove the right side of (ii). Using lemma 5 again, we have

$$\begin{aligned} Var(Rec) = Var(Tra) &< \frac{16-3\pi}{24} \cdot n + \frac{4}{3}\sqrt{\frac{\pi}{8}} \cdot \sqrt{n} + \frac{4}{3} \\ &< \frac{16-3\pi}{24} \cdot n + 4\sqrt{\frac{16-3\pi}{24}} \cdot \sqrt{n} + 4 \\ &= \left(\sqrt{\frac{16-3\pi}{24} \cdot n} + 2\right)^2 \end{aligned}$$

because

$$\frac{4}{3}\sqrt{\frac{\pi}{8}} < 4\sqrt{\frac{16-3\pi}{24}} \text{ and } \frac{4}{3} < 4.$$

Therefore

$$\sigma(Rec) = \sigma(Tra) = \sqrt{\mathrm{Var}(Rec)} < \left| \sqrt{\frac{16-3\pi}{24} \cdot n} + 2 \right| = \sqrt{\frac{16-3\pi}{24} \cdot n} + 2.$$

Hence follows the right side of item (ii) of theorem 3 for all n.

Now to prove the left side of (i). Using lemma 5 again, we get

$$\begin{aligned} Var(Vis) &> \frac{4-\pi}{2} \cdot n - 10\sqrt{\frac{\pi}{8}} \cdot \sqrt{n} - 7 \\ &> \frac{4-\pi}{2} \cdot n - 10\sqrt{\frac{4-\pi}{2}} \cdot \sqrt{n} + 25 \\ &= \left(\sqrt{\frac{4-\pi}{2} \cdot n} - 5\right)^2 \quad \text{for } n \geq 50 \end{aligned}$$

because

$$-10\sqrt{\frac{\pi}{8}} \cdot \sqrt{n} - 7 > -10\sqrt{\frac{4-\pi}{2}} \cdot \sqrt{n} + 25 \text{ for } n \geq 50.$$

Therefore

$$\sigma(Vis) = \sqrt{Var(Vis)} > \left| \sqrt{\frac{4-\pi}{2} \cdot n} - 5 \right| = \sqrt{\frac{4-\pi}{2} \cdot n} - 5$$

for $n \geq 60$ because

$$\sqrt{\frac{4-\pi}{2} \cdot n} - 5 > 0 \text{ for } n \geq 60.$$

Hence follows the left side of item (i) of theorem 3 for all n because we have checked it numerically for $n < 60$ (See figure 2).

Now to prove the left side of (ii). Using lemma 5 again, we get

$$\begin{aligned} Var(Rec) = Var(Tra) &> \frac{16-3\pi}{24}\cdot n - \frac{8}{3}\sqrt{\frac{\pi}{8}}\cdot\sqrt{n} - 2 \\ &> \frac{16-3\pi}{24}\cdot n - 4\sqrt{\frac{16-3\pi}{24}}\cdot\sqrt{n} + 4 \\ &= \left(\sqrt{\frac{16-3\pi}{24}\cdot n} - 2\right)^2 \end{aligned}$$

because

$$-\frac{8}{3}\sqrt{\frac{\pi}{8}}\cdot\sqrt{n} - 2 \;>\; -4\sqrt{\frac{16-3\pi}{24}}\cdot\sqrt{n} + 4 \quad\text{for}\quad n \geq 225.$$

Therefore

$$\sigma(Rec) = \sigma(Tra) = \sqrt{Var(Rec)} > \left|\,\sqrt{\frac{16-3\pi}{24}\cdot n} - 2\,\right| = \sqrt{\frac{16-3\pi}{24}\cdot n} - 2$$

for $n \geq 15$ because

$$\sqrt{\frac{16-3\pi}{24}\cdot n} - 2 > 0 \quad\text{for}\quad n \geq 15.$$

Hence follows the left side of item (ii) of theorem 3 for all n because for $n < 225$ we have checked it numerically (see figure 3). *Theorem 3 is proved.*

Proposition 6.

$$(i)\ \ \mathrm{Corr}(Rec, Tra) = \frac{Var(Vis)}{2Var(Tra)} - 1;$$

$$(ii)\ \mathrm{Corr}(Rec, Vis) = \mathrm{Corr}(Tra, Vis) = \frac{\sigma(Vis)}{2\sigma(Rec)}.$$

Proof. It is easy to prove that

$$Var(Vis) = 2(Var(Tra) + \mathrm{Cov}(Tra, Rec)). \tag{28}$$

It is well-known that

$$\mathrm{Corr}(Rec, Tra) = \frac{\mathrm{Cov}(Rec, Tra)}{\sigma(Rec)\cdot\sigma(Tra)}.$$

Hence, using (28) and after that item (ix) of proposition 1, we get

$$\mathrm{Corr}(Rec, Tra) = \frac{Var(Vis)}{2\sigma(Rec)\sigma(Tra)} - \frac{Var(Tra)}{\sigma(Rec)\sigma(Tra)} = \frac{Var(Vis)}{2Var(Tra)} - 1.$$

Item (i) is proved. Now let us prove item (ii). From item (x) of proposition 1,

$$\mathrm{Cov}(Tra, Vis) = \mathrm{Cov}(Rec, Vis) = Var(Rec) + \mathrm{Cov}(Rec, Tra).$$

Then, using first this and after that (28), we get $\mathrm{Corr}(Tra, Vis) = \mathrm{Corr}(Rec, Vis)$. Therefore

$$\begin{aligned}\mathrm{Corr}(Rec, Vis) = &\ \frac{\sigma(Rec)}{\sigma(Vis)} + \frac{\mathrm{Cov}(Rec, Tra)}{\sigma(Rec)\sigma(Vis)}\\ = &\ \frac{\sigma(Rec)}{\sigma(Vis)} + \frac{1}{\sigma(Rec)\sigma(Vis)}\left(\frac{Var(Vis)}{2} - Var(Tra)\right)\end{aligned}$$

Proposition 6 is proved.

Proof of Theorem 4. Let ψ_1 and ψ_2 be the same as in theorem 3. First we shall prove item (i). Using item (i) of proposition 6 and theorem 3, we get

$$\frac{(\sqrt{\psi_1 n} - 5)^2}{2(\sqrt{\psi_2 n} + 2)^2} - 1 \le \mathrm{Corr}(Rec, Tra) = \frac{Var(Vis)}{2Var(Rec)} - 1 \le \frac{(\sqrt{\psi_1 n} + 3)^2}{2(\sqrt{\psi_2 n} - 2)^2} - 1.$$

Going to the limit, we get

$$\lim_{n\to\infty} \mathrm{Corr}(Rec, Tra) = \frac{\psi_1}{2\psi_2} - 1 = \frac{8 - 3\pi}{16 - 3\pi}.$$

Item (i) is proved. Now let us prove item (ii). Using item (ii) of proposition 6 and Theorem 3, we get

$$\frac{\sqrt{\psi_1 n} - 5}{2(\sqrt{\psi_2 n} + 2)} \le \mathrm{Corr}(Rec, Vis) = \mathrm{Corr}(Tra, Vis) = \frac{\sigma(Vis)}{2\sigma(Rec)} \le \frac{\sqrt{\psi_1 n} + 3}{2(\sqrt{\psi_2 n} - 2)}.$$

Going to the limit, we get

$$\lim_{n\to\infty} \mathrm{Corr}(Rec, Vis) = \lim_{n\to\infty} \mathrm{Corr}(Tra, Vis) = \frac{1}{2}\sqrt{\frac{\psi_1}{\psi_2}} = \sqrt{\frac{12 - 3\pi}{16 - 3\pi}}.$$

Theorem 4 is proved.

Proposition 7. Let us denote by Med(Vis), Med(Rec) and Med(Tra) the medians of Vis, Rec and Tra respectively. Then, of course, Med(Tra) = Med(Rec)-1. In addition,

(i) Med(Vis) is the minimal integer which satisfies the condiction

$$n^{k+1}(n-(k+1))! \geq 2n!.$$

(ii) Med(Rec) is the minimal integer which satisfies the condiction

$$\frac{(n-1)!}{(n-k)!}\left(\frac{k\cdot e^n\cdot\Gamma(n-k+1,\ n)}{n^n}-n^{1-k}\right)\geq -1/2.$$

Proof. We start with item (i). First we shall prove that

$$\mathbb{P}(Vis \leq k) = 1-\frac{n!}{n^{k+1}(n-(k+1))!}.$$

For $n-\lambda=j$

$$\mathbb{P}(Vis\leq k)=\sum_{\lambda=1}^{k}\frac{\lambda(n-1)!}{n^{\lambda}(n-\lambda)!}=\frac{(n-1)!}{n^n}\left(n\sum_{j=n-k}^{n-1}\frac{1}{n^{-j}j!}-\sum_{j=n-k}^{n-1}\frac{1}{n^{-j}(j-1)!}\right).$$

Now we take $l=j-1$. Then

$$\begin{aligned}\mathbb{P}(Vis\leq k) &= \frac{(n-1)!}{n^n}\left(n\sum_{j=n-k}^{n-1}\frac{1}{n^{-j}j!}-n\sum_{j=n-k-1}^{n-2}\frac{1}{n^{-l}l!}\right)\\ &= \frac{(n-1)!}{n^n}\left(\frac{n}{n^{1-n}(n-1)!}-\frac{n}{n^{-n+k+1}(n-(k+1))!}\right)\\ &= 1-\frac{(n-1)!}{n^k(n-(k+1))!}.\end{aligned}$$

If $k=$ Med (Vis), then

$$\mathbb{P}(Vis\leq k)\geq\frac{1}{2}\Leftrightarrow -\frac{n!}{n^{k+1}(n-(k+1))!}\geq-\frac{1}{2}.$$

Now let us prove item (ii). We shall show that

$$\mathbb{P}(Rec\leq k) = 1+\frac{(n-1)!}{(n-k)!}\left(\frac{k\cdot e^n\cdot\Gamma(n-k+1,\ n)}{n^n}-n^{1-k}\right). \tag{29}$$

By item (iii) of proposition 1,

$$\begin{aligned}
\mathbb{P}(Rec \le k) &= \sum_{i=1}^{k}\sum_{\lambda=i}^{n}\frac{(n-1)!}{n^{\lambda}(n-\lambda)!} = \frac{(n-1)!}{n^n}\sum_{i=1}^{k}\sum_{j=0}^{n-i}\frac{n^j}{j!} \\
&= \frac{(n-1)!}{n^n}\left(k\sum_{j=0}^{n-k}\frac{n^j}{j!} + \sum_{j=1}^{k-1}\frac{n^{n-j}}{(n-j)!}\right) \\
&= \frac{(n-1)!}{n^n}\left(k\sum_{j=0}^{n-k}\frac{n^j}{j!} + \sum_{l=n-k+1}^{n-1}\frac{(n-l)n^l}{l!}\right) \\
&= \frac{(n-1)!}{n^n}\left(k\sum_{j=0}^{n-k}\frac{n^j}{j!} + n\sum_{j=n-k+1}^{n-1}\frac{n^j}{j!} - \sum_{j=n-k}^{n-2}\frac{n^{j+1}}{j!}\right) \\
&= \frac{(n-1)!}{n^n}\left(k\sum_{j=0}^{n-k}\frac{n^j}{j!} + n\left(\frac{n^{n-1}}{(n-1)!} - \frac{n^{n-k}}{(n-k)!}\right)\right)
\end{aligned}$$

Using (13) , we obtain (29). Therefore

$$\mathbb{P}(Rec \le k) \ge \frac{1}{2} \Leftrightarrow 1 + \frac{(n-1)!}{(n-k)!}\left(\frac{k\cdot e^n\cdot\Gamma(n-k+1,n)}{n^n} - n^{1-k}\right) \ge \frac{1}{2}$$

Proposition 7 is proved.

Acknowledgements

A. Toom thanks CNPq, which supported this work, process PQ/301266/2007-7.

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