

Simple One-Dimensional Interaction Systems with Superexponential Relaxation Times¹

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Finite one-dimensional random processes with local interaction are presented which keep some information of a topological nature about their initial conditions during time, the logarithm of whose expectation grows asymptotically at least as M^3 , where M is the "size" of the set R_M of states of one component. Actually R_M is a circle of length M . At every moment of the discrete time every component turns into some kind of average of its neighbors, after which it makes a random step along this circle. All these steps are mutually independent and identically distributed. In the present version the absolute values of the steps never exceed a constant. The processes are uniform in space, time, and the set of states. This estimation contributes to our awareness of what kind of stable behavior one can expect from one-dimensional random processes with local interaction.

KEY WORDS: Random processes; one-dimensional local interaction; relaxation time; smoothing; Cramér-Edgeworth expansion; harnesses.

1. INTRODUCTION

It is well known that many one-dimensional systems lack those qualitative properties that systems whose dimension is greater than one may have and to which students of statistical physics pay most attention. For example, Lieb and Mattis wrote in the introduction to their still valuable collection,⁽¹⁾ "In one dimension bosons do not condense, electrons do not superconduct, ferromagnets do not magnetize, and liquids do not freeze" (p. vi). Another example: §152 of Landau and Lifshitz's famous monograph⁽⁹⁾ was called, "The impossibility of the existence of phases in one-dimensional systems" and an argument of a physical nature (which

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Lieb and Mattis justly call "heuristic") was presented in support of this impossibility.

However, for a long time most models which moved physicists to single out the one-dimensional case described equilibrium states. Non-equilibrium phenomena may be described using uniform random processes with local interaction, a class of which we describe here. One important characteristic of random processes is how long they can remember something about their initial conditions. Let us call this time their "relaxation time" (below we define it for our processes). Some infinite processes, even in the presence of random noise, can keep some knowledge about their initial condition forever; these are often called *nonergodic*, as opposed to *ergodic* ones which forget everything about their initial condition as $t \rightarrow \infty$. Nondegenerate finite processes cannot remember anything forever, but their relaxation times varies enormously depending on particulars of the interaction.

Examples of ergodic processes or processes with small relaxation times are easy to present: it is sufficient to make the interaction "strong enough" (see, for example, Chapters 3 and 4 of ref. 3, where other references can be found). On the other hand, for all $d > 1$, nondegenerate d -dimensional processes have been proposed which are nonergodic in the infinite case and have large relaxation times in the finite case (see, for example, ref. 13, where other references can be found). It is no wonder, however, that nonergodicity in the one-dimensional case has always presented special difficulties.

Much work has been done on *random cellular automata*, whose components have finite sets of states. The *positive rates conjecture*, which was proposed by several authors, claims that all nondegenerate one-dimensional random cellular automata are ergodic, that is, have a unique limit behavior (see, for example, Chapter 4, Section 3 of ref. 10, p. 115 of ref. 3, and ref. 6). However, the systems which mathematicians consider are much more general than those which arise from physical considerations, and may well contradict physical intuition. Now the positive rates conjecture seems to have been refuted: after some preliminary work,^(1,8) Péter Gács proposed a nonergodic, nondegenerate one-dimensional system.⁽⁴⁾ However, Gács's construction is very elaborate, which makes it difficult to apply in physics.

Several one-dimensional cellular automata which display some kind of stability were proposed in ref. 5. Now all of us seem to be (intuitively) sure that all the systems proposed in ref. 5 are ergodic (unless the noise is degenerate). However, their time of relaxation seems to grow unusually fast when ϵ , the probability of errors, tends to 0. For example, de Sa and Maes⁽²⁾ simulated one of these models (known as the "soldiers model") and

claimed that the relaxation time grows as an exponent of $1/\epsilon$. (Note, however, that it is difficult to make a reliable conclusion about the way this time grows from computer experiments alone, because of the very nature of the question.) The relaxation time may be expected to grow fast also for the "two-line voting" system described in ref. 14.

Considering all that has been said, it seems worthwhile to present (as we do) simple one-dimensional systems which display properties which may seem to need more than one dimension. We prove that our systems keep some information about their initial conditions for a very long time, which grows at least as an exponent of M^3 , when M , the "size" of the set of states of a single component, tends to infinity. A physicist might expect M^2 ; the unusual exponent of M^3 in our estimations is briefly commented upon in Note 4 below and will be discussed in more detail elsewhere.⁽¹⁵⁾ For technical reasons our components' sets of states are continuous, but computer experiments suggest that the same is true for systems with discrete sets of states of components (see Note 3).

Constructions presented here are not models of any particular physical phenomenon. However, they seem to be closer to physics than to computer science, since we avoid any sophisticated mechanism which the human mind might intentionally create to preserve information about the initial condition. The functioning of our systems involves only smoothing (which is quite imaginable in natural systems) and a symmetric random noise which adds a random increment to every component at every step of the discrete time. This is what we mean by calling our systems "simple" and in this sense the approach of this paper is different from refs. 1, 8, 5, and 4 because we ask what kind of stability may be expected of systems which lack any special mechanism of preservation except "smoothing."

This simplicity has a price, however. It is typical of the nonergodic or stable systems presented until now that their components are mostly in the "correct" states in spite of the random noise. In our systems, however, there is no such thing as a "correct" or "incorrect" state of a single component. What is remembered instead is a topological property of the whole system's state. This makes it more difficult to generalize our results to systems with an infinite set of components. Another property of our systems which is natural from the physical point of view but useless in computer science is uniformity in the set of states in addition to uniformity in space and time.

Our main objects are systems whose elements have \mathbf{R}_M , a circle of length M , as the set of states. As a tool we consider systems with a linear interaction, the states of whose elements are real numbers. There is some analogy between the latter systems and those described in Chapter IX of ref. 10, since both are linear. Indeed, if we exclude noise, our linear systems become (rather trivial) discrete analog of the "smoothing processes"

described in Chapter IX of ref. 10. However, our main questions are quantitative rather than qualitative. On one hand the existence of a process, which is a major concern throughout ref. 10, is trivial in our case due to the discreteness of time. On the other hand, our main estimations (3) and (5) seems to have no analogs for systems examined in Chapter IX of ref. 10.

2. DEFINITIONS AND THEOREMS

Components are indexed by $0, \dots, L-1$, elements of the additive group of residues modulo $L \geq 2$. A state of the process is an L -tuple $a = (a_0, \dots, a_{L-1})$, where all a_s belong to the set \mathbf{R}_M , which is defined as follows. Choose a positive number M and define \mathbf{R}_M as the additive group of classes of equivalence if we declare two real numbers equivalent whenever their difference equals M , multiplied by an integer number. Our processes may be thought as linear operators P which act on the set of probability measures on the set \mathbf{R}_M^L of states, more exactly, on the σ -algebra generated by its cylinder subsets.

We use mutually independent "hidden" real random variables η'_s , $s = 0, \dots, L-1$ and $t = 1, 2, 3, \dots$. Every η'_s is distributed as a real random variable v , called *noise*. We assume that v is not a constant and that there is a constant v_{\max} such that

$$\text{Prob}(|v| > v_{\max}) = 0 \quad (1)$$

Choose a natural number N and different integer numbers $v_1, \dots, v_N \in \mathbf{Z}$ such that the differences $v_i - v_1$ generate \mathbf{Z} . Components $s + v_1, \dots, s + v_N$ are those that influence the component s at every time step. Also choose positive numbers w_1, \dots, w_N whose sum equals one (intensities of this influence). A *system* (v_i, w_i, v) is given by $v_1, \dots, v_N, w_1, \dots, w_N$, and the distribution of v . All the values which depend only on these parameters will be called constants. To every system there correspond *processes*, parametrized by L and M .

Let us define the transition function $F: \mathbf{R}_M^N \rightarrow \mathbf{R}_M$. (Note that the following definition is consistent.) Given a real number x , $\mu(x) \in \mathbf{R}_M$ denotes its class. Given a class $y \in \mathbf{R}_M$, its intersection with the segment $(-M/2, M/2]$ consists of one number, denoted $\mu_0^{-1}(y)$. For any $x_1, \dots, x_N \in \mathbf{R}_M$:

- If there is $q \in \mathbf{R}_M$ such that

$$\forall j = 1, \dots, N: \quad -M/4 < \mu_0^{-1}(x_j - q) < M/4$$

the value of the transition function equals

$$F(x_1, \dots, x_N) = \mu \left(\sum_{i=1}^N w_i \cdot \mu_0^{-1}(x_i - q) \right) + q$$

- Otherwise the value of $F(x_1, \dots, x_N)$ is undefined.

A process (a'_s) is the distribution of random variables $a'_s \in \mathbf{R}_M$ induced by the distribution of the hidden variables η'_s with the map defined in the following inductive way:

$$a'_s = \begin{cases} F(a'_{s+v_1}, \dots, a'_{s+v_N}) + \mu(\eta'_s) & \text{if } F(a'_{s+v_1}, \dots, a'_{s+v_N}) \text{ is defined} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

where a_s^0 are components of the initial condition.

We also define an integer-valued function $\text{rot}(\cdot)$ on \mathbf{R}_M^L as follows, where $a = (a_0, \dots, a_L)$:

$$\text{rot}(a) = \begin{cases} M^{-1} \cdot \sum_{s=0}^{L-1} \rho(a_s, a_{s+1}) & \text{if all the addends in this sum are defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Here the real function $\rho(\cdot)$ is defined by \mathbf{R}_M^2 as follows:

$$\rho(x, y) = \begin{cases} \mu_0^{-1}(y - x) & \text{if } \mu_0^{-1}(y - x) \neq M/2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Informally speaking, $\text{rot}(a)$ shows how many times we go around the continuous circle \mathbf{R}_M following the components of our states $a = (a_0, \dots, a_{L-1})$ as stepping stones as their index goes around the "discrete circle" $\{0, \dots, L-1\}$.

Consider a process (a'_s) with an initial condition a^0 for which $\text{rot}(a^0)$ is defined. Let a^t denote the state of the process at time t . The first time $t = t^*$ when $\text{rot}(a^t)$ is different from $\text{rot}(a^0)$ or undefined is called the *relaxation time* of this process. The following is our main theorem.

Theorem 1. For any system (v_i, w_i, ν) with $N > 1$ there are positive constants M_0 , C , and K such that for any initial condition a^0 of the form

$$a_s^0 = \mu(R \cdot s/L) \tag{2}$$

where R is an integer number, the expectation $E(t^*)$ of the relaxation time in all corresponding processes with $M \geq M_0$ is bounded from below by

$$L^{-1} \cdot \exp[C(M - K |R|)^3] \leq E(t^*) \tag{3}$$

Note that for states a^0 of the form (2) with $|R| < ML$ the function $\text{rot}(a^0)$ is defined and equals R . The following theorem gives the opposite estimate for a special case.

Theorem 2. Consider the system with $N=2$, $v_1=0$, $v_2=1$, $w_1=w_2=1/2$, and

$$v = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases} \quad (4)$$

There are positive constants K and C such that for any initial condition a^0 the expectation of the relaxation time in all corresponding processes with $L > KM$ is bounded from above by

$$E(t^*) \leq \exp(CM^3) \quad (5)$$

3. PROOF OF THEOREM 1

The processes described above will be called *finite* processes now. They are finite in two respects: (a) the number of components is finite and equals L ; (b) the set of states is a circle. Our main tool are *infinite* processes whose set of components is \mathbf{Z} and whose set of states in \mathbf{R} . However, the number L still serves as a parameter in their definition. Given a system (v_i, w_i, ν) and a number L , an infinite process (b'_s) is a distribution of real random variables b'_s , where $s \in \mathbf{Z}$, which is induced by the distribution of the same hidden variables η'_s which we used in finite processes, where $s \in \{0, \dots, L-1\}$, with the map defined in the following inductive way:

$$b'_s = \sum_{i=1}^N w_i \cdot b'_{s+v_i} + \eta'_{r(s)}$$

for all $s \in \mathbf{Z}$, $t=1, 2, 3, \dots$, where $r(s)$ is the residue when s is divided by L and $b'_s{}^0$ are components of the initial condition. Given an infinite process (b'_s) , we denote $\Delta b'_s = b'_{s+1} - b'_s$.

The following proposition shows which properties of infinite processes underlie Theorem 1.

Proposition 1. For any system (v_i, w_i, ν) with $N > 1$ there is a positive constant C such that for all L , all $D > 0$, and all corresponding infinite processes (b'_s) with the initial condition $b'_s{}^0 \equiv 0$ the expectation $E(t^0)$ of the first time t^0 when $\sup_s \Delta b'_s > D$ is bounded from below by $L^{-1} \cdot \exp(CD^3)$.

Proof. Note that the noise in the infinite processes, as we define them, is space-periodic. If the initial condition is also periodic, that is,

$b_{s+L}^0 \equiv b_s^0$ (which is certainly true if $b_s^0 \equiv 0$), then the process is also periodic, that is, $b_{s+L}^t \equiv b_s^t$ for all s, t . Note also that in the infinite process every variable b_s^t and every difference $\Delta b_s^t = b_{s+1}^t - b_s^t$ is a linear combination of some hidden variables. Let us write these formulas for b_0^t and Δb_0^t :

$$b_0^t = \sum_{n=0}^{t-1} \sum_s p_s^n \cdot \eta_s^{t-n} \tag{6}$$

$$\Delta b_0^t = \sum_{n=0}^{t-1} \sum_s \Delta p_s^n \cdot \eta_s^{t-n}, \quad \text{where } \Delta p_s^n = p_{s+1}^n - p_s^n$$

Let us prove that

$$\max_s |\Delta p_s^n| = O(n^{-1}) \quad \text{and} \quad \sum_s (\Delta p_s^n)^2 = O(n^{-3/2}) \tag{7}$$

Proof of the First Statement in (7). Given v_1, \dots, v_N and nonnegative w_1, \dots, w_N whose sum equals 1, we can define a real random variable V :

$$V = \begin{cases} v_1 & \text{with probability } w_1 \\ \dots & \dots \\ v_N & \text{with probability } w_N \end{cases} \tag{8}$$

We can prove by induction that for all s, n the value of p_s^n equals the probability that $V_n \equiv s$ modulo L , where V_n is a sum of n independent random variables, every one of which is distributed as V . (This observation is analogous to the duality between smoothing processes and potlatch processes, which is described, e.g., in Chapter IX of ref. 10.) Thus

$$p_s^n = \sum_{m=-\infty}^{\infty} q_{s+mL}^n \tag{9}$$

where $q_s^n = \text{Prob}(V_n = s)$. The Cramér-Edgeworth expansion for convolutions of identical lattice distributions (for example, refer to Theorem 13 in ref. 12, Chapter VII, p. 205) gives us for any $s, n, k \geq 1$

$$q_s^n = Q_k(x(s, m)) + o(n^{-k/2}) \tag{10}$$

where

$$x(s, m) = \frac{s + m \cdot L - n \cdot \mu}{\sigma \sqrt{n}}, \quad Q_k(x) = \frac{\exp(-x^2/2)}{\sigma (2\pi n)^{1/2}} \left(1 + \sum_{j=1}^k P_j(x) n^{-j/2} \right) \tag{11}$$

μ and σ are the expectation and standard deviation of V , and every $P_j(x)$ is a polynomial whose degree is j and whose coefficients are determined by the first j moments of the distribution of V .

Since our interaction has a finite range, the sum in (9) actually is always finite and contains $O(n)$ terms. [In fact it is sufficient to add its terms only from $m = -rn$ to $m = rn$, where $r = \max(|v_1|, \dots, |v_N|)$.] Therefore

$$4p_s^n = \sum_{m=-O(n)}^{O(n)} [Q_k(x(s+1, m)) - Q_k(x(s, m))] + \sum_{m=-O(n)}^{O(n)} o(n^{-k/2}) \quad (12)$$

Let $k = 4$. Then the summing of $o(n^{-k/2})$ results in $O(n^{-1})$. It remains to prove that the first sum in the right side of (12) is also $O(n^{-1})$. For any real function f denote

$$\Sigma(f) = \sup_{d, p \neq 0} \left| \sum_{m=-\infty}^{\infty} f(d + m \cdot p) \right|$$

Note that for any f and any positive constant C

$$\Sigma(C \cdot f) = C \cdot \Sigma(f) \quad \text{and} \quad \Sigma(f(C \cdot x)) = \Sigma(f(x))$$

Lemma. If a real function f is differentiable, $|f'(x)| \leq 1$, and $|f(x)| \leq 1/(1+x^2)$ for all x , and $\int_{-\infty}^{\infty} f(x) dx = 0$, then $\Sigma(f) \leq 8$.

Proof. Without loss of generality we assume that $p > 0$. If $p \geq 1$,

$$\left| \sum_{m=-\infty}^{\infty} f(d + mp) \right| \leq 2 \sum_{n=0}^{\infty} \frac{1}{1+n^2} < 8$$

Now let $0 < p \leq 1$. Then

$$\left| \sum_{m=-\infty}^{\infty} f(d + mp) \right| \leq \sum_{|d+mp| \geq 1/p} |f(d + mp)| + \left| \sum_{|d+mp| < 1/p} f(d + mp) \right| \quad (13)$$

The first sum in the right side of (13) does not exceed

$$2 \sum_{n=0}^{\infty} \frac{1}{1+(np+1/p)^2} \leq \frac{2}{p} \int_{1/p}^{\infty} \frac{dx}{1+x^2} \leq \frac{2}{p} \int_{1/p}^{\infty} \frac{dx}{x^2} = 2$$

Let us estimate the second sum in the right side of (13). Let m_1 and m_2 denote the smallest and the largest integer values of m for which

$|d + mp| < 1/p$. Then the second term (without the absolute value sign) can be interpreted as p times the rectangle rule approximation of the integral

$$\int_{d+m_1 p}^{d+(m_2+1)p} f(x) dx$$

The difference between this finite integral and the infinite integral (which equals zero) does not exceed

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$$

Since $|f'| \leq 1$, the error of the rectangle rule approximation, multiplied by p , does not exceed $\frac{1}{2}(m_2 - m_1 + 1)p^2 \leq 2$. Thus $\Sigma(f) \leq 2 + \pi + 2 < 8$ in this case also. The lemma is proved. ■

Let us use the properties of $\Sigma(\cdot)$ to estimate the first sum in the right side in (12). Let us expand its summand into a Taylor polynomial of degree 3:

$$\begin{aligned} Q_4(x(s+1, m)) - Q_4(x(s, m)) \\ = \frac{Q_4'(x(s, m))}{\sigma\sqrt{n}} + \frac{Q_4''(x(s, m))}{2\sigma^2 n} + \frac{Q_4'''(x(s, m))}{6\sigma^3 n\sqrt{n}} + O(n^{-2}) \end{aligned} \quad (14)$$

Note that for every i the function $n^{i/2}Q_4^{(i)}(n^{i/2}x)$, multiplied by a suitable positive constant, satisfies all the conditions of the lemma. Hence the sum of the three first terms in (14) is $O(n^{-1})$. Summing the last term also gives $O(n^{-1})$, as before. Thus the first statement in (7) is proved.

The Second Statement in (7). This follows from the first one and from

$$\sum_s |\Delta p_s^n| = O(n^{-1/2}) \quad (15)$$

Let us prove (15). Again we may neglect the remainder term in (10). Without it the right side of (10) certainly has a finite variation, because its derivative changes its sign only a finite number of times. This assures (15). Thus both statements in (7) are proved. ■

For any real random variable ξ let us define its moment generation function (MGF) as

$$\phi(z | \xi) = \int_{-\infty}^{\infty} \exp(zx) dF(x) \quad (16)$$

where $F(x) = \text{Prob}(\xi \leq x)$. The MGF $\phi(z | \xi)$ is a real function of a real argument z , the distribution of ξ serving as a parameter. We use the MGF only when the absolute value of ξ never exceeds a constant; in these case the integral (16) certainly converges. Let us use the MGF for $\xi = \Delta b'_s$ to estimate for any $D > 0$ and $z > 0$

$$\begin{aligned} \text{Prob}(\Delta b'_s > D) &= \int_D^\infty dF(x) \leq \int_{-\infty}^\infty \exp[z(x - D)] dF(x) \\ &= \exp(-zD) \phi(z | \Delta b'_s) \end{aligned}$$

Note that $\phi(z | c \cdot \xi) = \phi(c \cdot z | \xi)$ for any random variable ξ and any numbers z and c and that the MGF of a sum of several independent random variables equals the product of their MGFs. Also remember that all η'_s are distributed as ν . All this allows us to rewrite the last expression as

$$\exp(-zD) \prod_{n=0}^t \prod_s \phi(z | \Delta p_s^n \cdot \eta'_s{}^{-n}) = \exp(-zD) \prod_{n=0}^t \prod_s \phi(\Delta p_s^n \cdot z | \nu)$$

Choosing some integer v between 1 and t , we can rewrite this expression as

$$\exp(-zD) \cdot \prod_{n=0}^{v-1} \prod_s \phi(\Delta p_s^n \cdot z | \nu) \cdot \prod_{n=v}^t \prod_s \phi(\Delta p_s^n \cdot z | \nu) \quad (17)$$

Let A be a positive constant, whose value will be chosen later. Since we are interested in the asymptotics as $D \rightarrow \infty$, we may assume that $AD^2 \geq 1$. Take

$$z = AD^2 \quad \text{and} \quad v = \begin{cases} [AD^2] & \text{if } 1 \leq [AD^2] \leq t \\ t & \text{if } t \leq [AD^2] \end{cases}$$

Let us estimate each product in (17).

First Product. From (1) there is a constant E such that $\phi(x | \nu) \leq \exp(E|x|)$. Hence

$$\begin{aligned} \prod_{n=0}^{v-1} \prod_s \phi(AD^2 \cdot \Delta p_s^n | \nu) &\leq \prod_{n=0}^{v-1} \prod_s \exp(AD^2 E \cdot |\Delta p_s^n|) \\ &= \exp\left(AD^2 E \cdot \sum_{n=0}^{v-1} \sum_s |\Delta p_s^n|\right) \\ &\leq \exp\left(AD^2 E \cdot \sum_{n=0}^{v-1} O(n^{-1/2})\right) \\ &\leq \exp(AD^2 E \cdot v^{1/2}) \\ &\leq \exp\{AD^2 E \cdot O(A^{1/2}D)\} \\ &= \exp\{O(A^{3/2})\} \end{aligned} \quad (18)$$

Second Product. Due to (7) for $n \geq v$

$$\begin{aligned} |AD^2 \Delta p_s^n| &\leq AD^2 \cdot \max_s |\Delta p_s^n| \leq AD^2 \cdot O(n^{-1}) \\ &\leq AD^2 \cdot O(v^{-1}) \leq AD^2 \cdot O\left(\frac{1}{AD^2}\right) = O(1) \end{aligned}$$

If we add a constant to the noise of a process, the distribution of its relaxation time does not change. Based on this, we may assume without loss of generality that $E(v) = 0$. (This turns our processes into *harnesses*, introduced in ref. 7.) Hence from (1) there is a positive constant E such that $\phi(x | v) \leq \exp(E \cdot x^2)$. This assures

$$\begin{aligned} \prod_{n=v}^i \prod_s \phi(AD^2 \cdot \Delta p_s^n | v) &\leq \prod_{n=v}^i \prod_s \exp\{A^2 D^4 E \cdot (\Delta p_s^n)^2\} \\ &= \exp\left(A^2 D^4 E \sum_{n=v}^i \sum_s (\Delta p_s^n)^2\right) \\ &\leq \exp\left(A^2 D^4 E \sum_{n=v}^i O(n^{-3/2})\right) \\ &\leq \exp\{A^2 D^4 \cdot O(v^{-1/2})\} \\ &= \exp\{O(A^{3/2} D^3)\} \end{aligned} \tag{19}$$

Together (17)–(19) give

$$\text{Prob}(\Delta b'_s > D) \leq \exp\{-AD^3 + O(A^{3/2} D^3)\}$$

Now we can choose A so small that the last expression will not exceed $\exp(-CD^3)$, where C is a positive constant. We have proved that $\text{Prob}(\Delta t'_s > D) \leq \exp(-CD^3)$ uniformly in s and t . Summing this over s and t , we get for any T

$$\text{Prob}(t^0 \leq T) = \text{Prob}(\exists s, t \leq T: \max_s \Delta b'_s > D) \leq TL \exp(-CD^3)$$

whence Proposition 1 follows. ■

For any finite process (a'_s) with an initial condition a^0 for which $\text{rot}(a^0)$ is defined, let us choose an infinite process (b'_s) which we shall call *coupled* with a'_s . As before, (b'_s) is a distribution of real random variables b'_s , where $s \in \mathbb{Z}$, induced by the same hidden variables η'_s , where $s \in \{0, \dots, L-1\}$, with the map

$$b'_s = \sum_{l=1}^N w_l \cdot b'_{s+v_l} + \eta'_{r(s)}, \quad \text{where } s \in \mathbb{Z}, \quad t \in \{1, 2, 3, \dots\}$$

Here components of the initial condition b_s^0 for $s \in \mathbf{Z}$ equal

$$b_s^0 = a_{r(s)}^0 + q(s) \cdot M \cdot \text{rot}(a^0) \quad (20)$$

where $q(s)$ and $r(s)$ are the quotient and the residue when $s \in \mathbf{Z}$ is divided by L .

Proposition 2. Take any finite process (a'_s) and let (b'_s) be the coupled infinite process. Denote $\Delta v = \max(v_1, \dots, v_N) - \min(v_1, \dots, v_N)$. While

$$\max_s |\Delta b'_s| < \frac{M - 2v_{\max}}{2(\Delta v + 1)} \quad (21)$$

$\text{rot}(a')$ remains defined and equals its initial value $\text{rot}(a^0)$.

Proof. Let t be less than or equal to the last time when (21) holds. Then $\text{rot}(a')$ is defined, because for all s

$$|\Delta b'_s| < \frac{M - 2v_{\max}}{2(\Delta v + 1)} < \frac{M}{2}$$

It remains to prove that $\text{rot}(a') = \text{rot}(a'^{-1})$. For any $a, b, c \in \mathbf{R}_M$

$$\begin{cases} \text{if } \rho(a, b) \text{ and } \rho(b, c) \text{ are defined and } |\rho(a, b)| + |\rho(b, c)| < M/2, \\ \text{then } \rho(a, c) \text{ is also defined and equals } \rho(a, b) + \rho(b, c) \end{cases} \quad (22)$$

Therefore for any $a, b, c, d \in \mathbf{R}_M$, if

$$|\rho(a, b)| + |\rho(b, c)| < M/2 \quad \text{and} \quad |\rho(a, d)| + |\rho(d, c)| < M/2$$

then

$$\rho(a, b) + \rho(b, c) = \rho(a, c) = \rho(a, d) + \rho(d, c) \quad (23)$$

Note also that for all s

$$\max(b'_{s+v_1}{}^{-1}, \dots, b'_{s+v_N}{}^{-1}) - \min(b'_{s+v_1}{}^{-1}, \dots, b'_{s+v_N}{}^{-1}) < \frac{(M - 2v_{\max}) \Delta v}{2(\Delta v + 1)}$$

whence for all s

$$|\rho(a'_{s+v_1}{}^{-1}, a'_s)| < \frac{(M - 2v_{\max}) \Delta v}{2(\Delta v + 1)} + v_{\max} \quad (24)$$

Thus for any s both

$$|\rho(a'_{s+v_1}{}^{-1}, a'_s)| + |\rho(a'_s, a'_{s+1})| \quad \text{and} \quad |\rho(a'_{s+v_1}{}^{-1}, a'_{s+1+v_1}{}^{-1})| + |\rho(a'_{s+1+v_1}{}^{-1}, a'_{s+1})|$$

are less than the sum of the right sides of (21) and (4), which equals $M/2$. This allows us to apply (23) to obtain

$$\rho(a'_{s+v_1-1}, a'_s) + \rho(a'_s, a'_{s+1}) = \rho(a'_{s+v_1-1}, a'_{s+1+v_1-1}) + \rho(a'_{s+1+v_1-1}, a'_{s+1})$$

Summing this over $s=0, \dots, L-1$ gives $\text{rot}(a'^{-1}) = \text{rot}(a')$. ■

Now to prove Theorem 1. Proposition 2 reduces our task to estimating the time while (21) holds in the infinite process coupled with the finite process in question. The initial condition b^0 of this infinite process is $b'_s = Rs/L$. [The result of application of (20) to (2).] Proposition 1 can be easily extended to this initial condition, because it adds only a constant R/L to $\Delta b'_s$. Thus we obtain the estimation

$$L^{-1} \exp \left\{ C_1 \left(D - \left| \frac{R}{L} \right| \right)^3 \right\} \leq \mathbf{E}(T)$$

Substitution here of the right side of (21) as D proves Theorem 1. ■

4. PROOF OF THEOREM 2

Proposition 3. There are positive constants C_0, C_1 , and K such that in every finite process (a'_s) which corresponds to the system in Theorem 2 with $L > KM$ and with any initial condition a^0 for which $\text{rot}(a^0)$ is defined, the value of $\text{rot}(a^t)$ changes or becomes undefined in the time span $[0, C_1 M^2]$ with a probability which is not less than $\exp(-C_0 M^3)$.

Proof. As before, consider the coupled infinite process (b'_s) . Now

$$\Delta b'_0 = I_t + \sum_{n=0}^{t-1} \sum_s \Delta p'_s \cdot \eta_s^{t-n}$$

where I_t is the contribution from the initial condition. Note that for the system in Theorem 2 the random variable V [defined in (8)] has expectation $\mu = 1/2$, and that is why the following definition is useful. Given some $C_1, C_2 > 0$ (whose values will be chosen later), let us denote $T = [C_1 M^2]$ and classify all pairs (s, n) , where $0 \leq n < T$, into two classes, "relevant" and "irrelevant," according to the following rule:

$$(s, n) \text{ is } \begin{cases} \text{relevant} & \text{if } |s + n/2| \leq C_2 M \\ \text{irrelevant} & \text{otherwise} \end{cases}$$

Given two functions f and g , let $f \asymp g$ mean that $f = O(g)$ and $g = O(f)$. Using the expansion (9)–(11), we can prove that $\sum_s |\Delta p'_s| \asymp n^{-1/2}$ [which

is a stronger version of (15)]. Based on this, we can choose K and C_2 so large that the sum of $|\Delta p_s^n|$ over all irrelevant (s, n) will be less than half of the sum of $|\Delta p_s^n|$ over all relevant (s, n) .

Now let us classify all relevant pairs (s, n) into two classes: "left," for which Δp_s^n is positive, and "right," for which Δp_s^n is negative. Now denote by E the following event:

$$E = \begin{cases} \text{if } I_t \geq 0, \text{ then every left variable equals } -1 \\ \quad \text{and every right variable equals } 1 \\ \text{otherwise every left variable equals } 1 \\ \quad \text{and every right variable equals } -1 \end{cases}$$

We can choose C_1 so large that, event E assumed, the contribution of hidden variables η_s^{T-n} , where pairs (s, n) are relevant, to $\Delta b'_0$ will exceed M and therefore $|\Delta b'_0|$ will certainly exceed $M/2$. Thus we can choose our constants $C_1, C_2,$ and K in such a way that E will assure $|\Delta b'_0| > M/2$.

Now let us classify all real numbers x into three groups:

$$x \text{ is } \begin{cases} \text{subcritical} & \text{if } |x| < M/2 - 2 \\ \text{critical} & \text{if } M/2 - 2 \leq |x| < M/2 \\ \text{supercritical} & \text{if } M/2 \leq |x| \end{cases}$$

Denote $y'_s = (b'_s + b'_{s+1})/2$ and $\Delta y'_s = y'_{s+1} - y'_s$. If $t + 1$ is the first time when at least one among $\Delta b'_{s+1}$ is supercritical, then

$$\begin{cases} \text{none among } \Delta b'_s \text{ and } \Delta y'_s \text{ is supercritical} \\ \text{but at least one among } \Delta y'_s \text{ is critical} \end{cases} \tag{25}$$

Of course, $\text{Prob}(E) \geq \exp(-C_0 M^3)$ for any choice of our constants with a suitable $C_0 = \text{const} > 0$. Now let E_t denote the event (25) for any given t . We have proved that the union of E_t over $t < T = C_1 M^2$ has probability which is not less than $\exp(-C_0 M^3)$.

To prove Proposition 3, it is sufficient to prove that, E_t assumed, $\text{rot}(a^{t+1})$ is different from $\text{rot}(a^t)$ or undefined with a probability which is not less than a positive constant. Let us assume E_t and denote $x'_s = \mu(y'_s)$. Since none of the $\Delta b'_s$ are supercritical,

$$y'_s - b'_s = b'_{s+1} - y'_s < M/4$$

Since $|\rho(\mu(x), \mu(y))| \leq |y - x|$ for any real x, y ,

$$|\rho(a'_s, x'_s)| < M/4 \quad \text{and} \quad |\rho(x'_s, a'_{s+1})| < M/4$$

Hence and from (22)

$$\rho(a'_s, a'_{s+1}) = \rho(a'_s, x'_s) + \rho(x'_s, a'_{s+1}),$$

$$\rho(x'_{s-1}, x'_s) = \rho(x'_{s-1}, a'_s) + \rho(a'_s, x'_s)$$

Summing this over s gives $\text{rot}(x') = \text{rot}(a')$. Now let us consider two cases.

Case 1. At least one among $\Delta y'_s$ is subcritical. Then there is s such that $\Delta y'_s$ is critical, but $\Delta y'_{s+1}$ is subcritical (or vice versa, which is analogous). In this case let us consider two new events:

$$E^+: \eta'_{s-1} = -1 \quad \text{and} \quad \eta'_{s+1} = 1$$

$$E^-: \eta'_{s-1} = -1 \quad \text{and} \quad \eta'_{s+1} = -1$$

Both events are independent of E_s and have probability $1/4$. Values of $\text{rot}(a'^{+1})$ (if defined) are different from each other in these two cases for any particular prehistory and values of all η'_u for $u \neq s-1, s$. Therefore in this case the conditional probability (given E_s) that $\text{rot}(a'^{+1})$ is different from $\text{rot}(a')$ or undefined is not less than $1/4$.

Case 2. All $\Delta y'_s$ are critical. Suppose that not all of them have one and the same sign and come to a contradiction. We can find s such that $\Delta y'_s$ is positive and $\Delta y'_{s+1}$ is negative (or vice versa, which is analogous). Then $y'_s \geq M/2 - 2$ and $y'_{s+1} \leq -(M/2 - 2)$, which means that $(b'_s + b'_{s+1})/2 \geq M/2 - 2$ and $(b'_{s+1} + b'_{s+2})/2 \leq -(M/2 - 2)$, whence $b'_s - b'_{s+2} \geq 2(M - 4)$. But, according to our choice of t , none of $\Delta b'_s$ and $\Delta b'_{s+1}$ is supercritical. This provides a contradiction if $M \geq 8$. Thus all $\Delta y'_s$ have one and the same sign. In this case $\text{rot}(a'^{+1})$ is defined and equals $\text{rot}(x') = \text{rot}(a')$ only if all η'_{s+1} have one and the same value for all s . The probability of this events is 2^{1-L} , which is less than or equal to $1/2$ for $L \geq 2$.

Proposition 3 is proved. ■

Thus, starting from any initial condition, after at most $\tau = C_1 M^2$ time steps, $\text{rot}(a')$ becomes different from $\text{rot}(a^0)$ or undefined with a probability which is not less than $\varepsilon = \exp(-C_0 M^3)$. Therefore $\text{Prob}(t^* \geq n\tau) \leq (1 - \varepsilon)^n$, whence

$$E(t^*) = \sum_{k=1}^{\infty} \text{Prob}(t^* \geq k) \leq \tau \sum_{n=1}^{\infty} (1 - \varepsilon)^n = \frac{\tau}{\varepsilon} = C_1 M^2 \exp(C_0 M^3)$$

which does not exceed $\exp(CM^3)$ with a suitable constant C . Theorem 2 is proved. ■

5. FINAL NOTES

Note 1. Theorem 1, which is true whenever $N \geq 2$, contrasts with the degenerate case $N=1$ (in which components essentially do not interact). For simplicity let us assume that $a_s^0 = 0$ and that the noise ν is distributed as in Theorem 2. In this case there is a constant M_0 such that for all $M > M_0$

$$E(t^*) \asymp \frac{M^2}{\ln L} \quad (26)$$

where t^* is the relaxation time, as before.

Let us prove (26). We may assume that $L > L_0 = \text{const}$. Since $N=1$, we may assume that $\nu_1 = 0$. As before, let (b'_s) be the coupled infinite process. For every $s \in \{0, \dots, L-1\}$, as t grows, b'_s performs a random walk, every step of which is distributed as (4) independently from all the other walks. Therefore every $\Delta b'_s$ is a sum of t independent, identically distributed random variables. Every one of these variables has zero expectation, because it is distributed as a difference of two independent random variables, every one of which is distributed as ν . For any $p \in (0, 1)$ let $t(p)$ be the first time t when $\text{Prob}(|\Delta b'_0| \geq M/2) \geq p$. From the central limit theorem $t(p) \asymp M^2 / (-\ln p)$. Now we estimate $E(t^*)$.

Estimation from Below. While

$$\sup_s |\Delta b'_s| < M/2 \quad (27)$$

$\text{rot}(a^t)$ certainly remains defined and equal to $\text{rot}(a^0)$. So it is sufficient to estimate the expectation of the time while (27) holds. For simplicity we may assume that L is even. Then for any t all the variables $\Delta b'_s$ may be separated into two groups, those with odd s and those with even s , all the variables of each group being mutually independent. Note that, if $p = L^{-2}$, then

$$t(p) = t(L^{-2}) \asymp \frac{M^2}{\ln L} \quad (28)$$

For every s the probability that the absolute value of $\Delta b'_s$ reaches $M/2$ in time (28) does not exceed L^{-2} ; therefore the probability that the absolute value of at least one of these variables for even s reaches $M/2$ does not exceed $1 - (1 - L^{-2})^{L/2}$, which tends to zero when $L \rightarrow \infty$. The same is true for odd values of s , whence the probability that the absolute value of some $\Delta b'_s$ for t given by (28) reaches $M/2$ also tends to zero. Thus for large enough L this probability is less than $1/2$, whence $E(t^*)$ is not less than half of (28), which gives the desired estimation of $E(t^*)$ from below.

Estimation from Above. Arguing as in Theorem 2, one can show that, starting from any initial condition, after every $\text{const} \cdot M^2/\ln L$ steps, the probability that $\text{rot}(a^t)$ changes its value or becomes undefined exceeds a positive constant, whence the upper estimation for $E(t^*)$ can be deduced.

Note 2. It seems natural to consider a nonperiodic analog of our infinite processes, which is a distribution of real random variables b'_s , where $s \in \mathbb{Z}$ and $t = 0, 1, 2, \dots$, induced by independent "hidden" variables η'_s , where $s \in \mathbb{Z}$ and $t = 1, 2, \dots$, all of which are distributed as a nonconstant noise v , which satisfies (1), with the map

$$b'_s = \sum_{i=1}^N w_i \cdot b'^{t-1}_{s+v_i} + \eta'_s \tag{29}$$

for all $t > 0$, where b^0_s are components of the initial condition. Normally more basic objects should be examined first. So it looks like an omission that systems like (29) have not been given enough special consideration (ref. 7 stands out as a valuable study of such systems). Being linear and therefore certainly simpler than our processes, the processes (29) display a property which anticipates our results: On one hand, for any initial condition and any constants C_1 and C_2 the probability $\text{Prob}(C_1 \leq b'_s \leq C_2)$ tends to zero as $t \rightarrow \infty$. But, on the other hand (and this is our point) the distribution of differences $\Delta b'_s$ tends to one and the same probability distribution as $t \rightarrow \infty$ when we start from any initial condition b^0 for which $|b^0_s| \leq \text{const}$. (Only ref. 7 anticipated this.) To prove this, let us write an analog of (6) for the present case:

$$\Delta b^t_0 = \sum_{n=0}^{t-1} \sum_s \Delta p^n_s \cdot \eta'^{t-n}_s + \sum_s \Delta p^t_s \cdot b^0_s$$

If $|b^0_s| \leq \text{const}$, then due to (15) (which is true in the present case also), the contribution of the initial condition tends to zero as $t \rightarrow \infty$. We discuss these systems elsewhere.⁽¹⁵⁾

Note 3. A superexponential relaxation time seems to take place also for some systems with a discrete set of states. This is suggested by our computer simulations, where the set of states of a single component was the set \mathbb{Z}_M of residues modulo M , where M was a positive integer number and the noise v was the same as (4). When the transition function $F(a'_{s+v_1}, \dots, a'_{s+v_N})$ was not defined, a'_s was made equal to all the elements of \mathbb{Z}_M with equal probabilities independent of all the prehistory. We took $N=3$, $v_1 = -1$, $v_2 = 0$, $v_3 = 1$, and the transition function (acting on the set of residues

modulo M) was the finite-valued version of one the following infinite-valued transition functions:

- $(a'_{s-1}^{-1} + a'_s{}^{-1} + a'_{s+1}{}^{-1})/3$, rounded to the nearest integer.
- The median of the same three arguments $a'_{s-1}{}^{-1}$, $a'_s{}^{-1}$, $a'_{s+1}{}^{-1}$. (A median of three numbers is the middle one if they are sorted in increasing order.)

Note 4. Let us discuss the unusual non-Gaussian exponent $\exp(\text{const} \cdot D^3)$ in Proposition 1 in an informal manner. A physicist would define an energy of a state b of our infinite system as $\sum_s (\Delta b_s)^2$ and, assuming Gibbs distribution, would expect that the probability that $\Delta b_s > D$ would be of the order $\exp(-D^2)$, like a Gaussian one. However, in our case, first, energy, as defined, is not conserved in our systems, and, second, due to condition (1), a large value of Δb_s cannot emerge in one time step; it has to be accumulated in about D^2 steps, and meanwhile energy both decreases as “dissipates” at a distance about D , so that the total energy spent is about D^3 , in accordance with our estimation. We discuss further elsewhere.⁽¹⁵⁾

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