Stability in Lattice Systems with Local Prohibitions*

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1. Definitions and Examples

We consider integral lattices $\mathbf{Z}^d$, in which every point v has a corresponding finite set of states $X_v = X_0$, the same for all points of the lattice. The set $X = \prod X_v$ is called the *configuration* space, a configuration being an infinite vector $x = (x_v)$ of components $x_v \in X_v$, where $v \in \mathbf{Z}^d$. The set $X_V = \prod_{v \in V} X_v$ of configurations x_v on a set $V \subset \mathbf{Z}^d$ is defined similarly.

The characteristic function $\chi_A : X \to \{0; 1\}$ for a set $A \subset X$ is defined to be equal to 1 on the configurations contained in A and on these only. A set $V \subset \mathbf{Z}^d$ is called a *support* of $A \subset X$ if $\chi_A(x)$ depends only on x_V. A set $A \subset X$ is called *cylindrical* if it has a finite support. For any cylindrical A, the smallest support is the collection of all v such that χ_A depends essentially on x_v.

For any $v \in \mathbf{Z}^d$ we define the shift mapping $T_v : X \to X$ by the condition: $\forall w \in \mathbf{Z}^d : (T_v(x))_w = x_{w-v}$. An image $T_v(A)$ is called a *translate* of A. Call a configuration *periodic* if it has only a finite number of distinct translates.

Definition 1. Let d be a given dimension and X_0 a finite set. Also let E be a countable collection of cylindrical subsets of $X = X_0^{\mathbf{Z}^d}$, satisfying the following three conditions:

(a) If a set C belongs to E, then all the translates of C also belong to E.

(b) If every set is defined to be equivalent to its translates, then E will break up into only a finite number of equivalence classes.

(c) All elements of E are nonempty and distinct from X.

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Then we say that (E, X) is a homogeneous lattice system with local prohibitions, from now on referred to as a *system* for the sake of brevity. We call a configuration $x \in X$ *consistent* in the system (E, X) if x belongs to the intersection of all the elements of E.

For any set $V \subset \mathbf{Z}^d$ we also call a configuration $y_V \in X_V$ consistent in the system (E, X) if $\{x : x_V = y_V\}$ belongs to the intersection of all the elements of E whose supports are subsets of V.

The *diameter* of a system is defined to be the smallest number D such that any element of E is supported on a cube with edges of length D which are parallel to the axes of $\mathbf{Z}^d$. In the case of a one-dimensional system, D is the smallest number such that every element of E is supported on a segment $[v; v + D]$.

Remark 1. Let (E, X) be a system of diameter D. Let us construct another system $(E^0; X)$. All elements of E^0 are the translates of a set C^0, whose support is the cube V with edges of length D, among which d edges lie on the positive half-axes of $\mathbf{Z}^d$. The set C^0 is the intersection of all elements of E supported in V. It is easy to see that the set of consistent configurations of (E, X) is the same as that of (E^0, X).

Remark 2. All that has been said above can be restated in another language which we shall need only in the one-dimensional case for the systems (E^0, X) constructed in Remark 1. We call a sequence of n terms—elements of X_0—a *word* of length n in the alphabet X_0. Let us assign to every one-dimensional system (E^0, X) of diameter D a list of *prohibited* words. To do this we choose an element C^0 of E^0 with support $[0; D]$ and include in our list all words $x_0, \ldots, x_D$ such that

$$\{y \in X : y_0 = x_0, \ldots, y_D = x_D\} \not\subset C^0.$$

We say that a configuration y contains the word $x_1, \ldots, x_n$ if there exist $v \in \mathbf{Z}$ such that $y_{v+1} = x_1, \ldots, y_{v+n} = x_n$. A configuration will be consistent if it contains no prohibited words.

Remark 3. It is easy to state an algorithm which would determine, for any one-dimensional system, whether it contains a consistent configuration and whether such a configuration is unique. Construct a finite oriented graph g with $|X_0|^D$ vertices, each corresponding to a word of length D in the alphabet X_0. We construct an edge from a vertex $x_1, \ldots, x_D$ to a vertex $y_1, \ldots, y_D$ if $x_2 = y_1$, $x_3 = y_2, \ldots, x_D = y_{D-1}$, and, moreover, the word $x_1, \ldots, x_D, y_D$ is not prohibited. If the graph g has no oriented cycles, then the system contains no consistent configurations. If g has exactly one oriented cycle, consisting of one vertex of the form $a, a, \ldots, a$, then the system contains exactly one consistent configuration $\ldots, a, a, a, \ldots$. In any other case the system contains more than one consistent configuration.

The analogous questions for the case $d > 1$ are algorithmically unsolvable; see [2].

Consider a system (E, X). Let M denote the class of probability measures on X, or, more precisely, on the σ-algebra generated by the cylindrical sets. We define a subclass $M_\varepsilon \subset M$ for any parameter $\varepsilon \in [0; 1]$. A measure μ is contained in M_ε if the following inequality holds for any $k \in \mathbf{Z}_+$ and for any k distinct elements $C_1, \ldots, C_k$ of E:

$$\mu(C_1 \cup C_2 \cup \cdots \cup C_k) \geqslant 1 - \varepsilon^k.$$

For any configuration x we write $N(x)$ for the collection of configurations that differ from x only on a finite number of points. For a point v we define

$$N_v(x) = \{ y \in N(x) : y_v \neq x_v \}.$$

Definition 2. A consistent configuration is called *stable* in the system (E, X) if

$$\lim_{\varepsilon \to 0} \sup_{\substack{\mu \in M_\varepsilon \\ v \in \mathbf{Z}^d}} \mu(N_v(x)) = 0.$$

Remark 4. Since x is consistent, M_ε is nonempty because it contains the δ-measure concentrated on x. This is why the supremum makes sense. Furthermore,

$$\varepsilon_1 < \varepsilon_2 \Rightarrow M_{\varepsilon_1} \subset M_{\varepsilon_2},$$

so the supremum depends on ε in a monotonic way; therefore the limit always exists.

The purpose of the rest of this section is to explain why such constructions are considered. The starting point of the present paper is a particular case, which we describe in the following example.

Example 1. We denote the points of $\mathbf{Z}^d$ by (s, t), where $s \in \mathbf{Z}^{d-1}$, $t \in \mathbf{Z}$. Introduce the set $U = \{(s, t) : |s| \leqslant R;\ -m \leqslant t \leqslant -1\}$, where R and m are parameters. Let us choose a function $f : X_0^{|U|} \to X_0$ and form the collection E of the translates of the set $\{x : x_{0,0} = f(x_U)\}$.

This system can be interpreted as a system of finite automata placed at points $s \in \mathbf{Z}^{d-1}$ and working in discrete time $t \in \mathbf{Z}$. A configuration is consistent if the state of every automaton at any point in time is computed as a function f of its own state and the states of its neighbors within radius R during the m previous instants of time. A measure $\mu \in M_\varepsilon$ then arises if the automata computing the function $f(\cdot)$ err in a random fashion, and their errors happen seldom enough and are not too dependent on each

other and on the previous history. For instance, $\mu \in M_\varepsilon$ if every automaton at any point in time errs with probability ε independently of other errors. Stability of a configuration x means that the deviations from x caused by the small independent errors of the automata are not accumulated.

Theorems providing sufficient conditions for stability of configurations in the systems described in Example 1 are given in [8], [9]. The method of proving them is a variant of the well-known Peierls method [3], [5], [6]. The present paper is an attempt to carry the method of [8], [9] back to systems where the time axis is not singled out. The following example is an illustration of this sort of system.

Example 2. Let every point of $\mathbf{Z}^2$ possess two states, 0 and 1. The collection E consists of all translates of the following two sets:

$$\{x : x_{0,0} = x_{0,1}\}, \qquad \{x : x_{0,0} = x_{1,0}\}.$$

It is obvious that here we have exactly two consistent configurations: "all zeros" and "all ones." It follows from Propositions 5 and 6 that these configurations are stable. This system reminds one of the classical Ising model.

Unfortunately, at present, our Propositions 5 and 6 cannot be successfully applied to Gibbs distributions, since we cannot find sufficiently interesting measures in M_ε for the case $d > 1$. The only step in this direction is Proposition 4 which treats the case $d = 1$, but Propositions 5 and 6 are inapplicable precisely in this case. One would like to prove the analog of Proposition 4 for all dimensions.

2. Formulations

Our first proposition plays an auxiliary role.

Proposition 1. *Let (E, X) and (E', X) be two given systems. Let every $C \in E$ contain the intersection of a finite number of the elements of E' as a subset. Let every $C' \in E'$ contain the intersection of a finite number of the elements of E as a subset. Then the systems (E, X) and (E', X) have identical sets of consistent configurations and identical sets of stable configurations.*

Proposition 1 shows that in order to study a system (E, X), we can replace it by another system (E^0, X) with the same consistent and stable configurations, where all the elements of E^0 are translates of each other. The construction of the system (E^0, X) is described in Remark 1.

The following proposition, together with Remark 3, gives a rather complete description of one-dimensional systems.

Proposition 2. *A consistent configuration in a one-dimensional system is stable if and only if no other consistent configurations exist.*

The following proposition contrasts with Proposition 2.

Proposition 3. *Let $d \geqslant 2$ be a given dimension and X_0 a finite set containing at least two elements. Let a finite collection of configurations in the space $X = X_0^{\mathbf{Z}^d}$ be given, satisfying the following condition: if a configuration belongs to this collection, then all its translates also belong to it. Then there exists a system (E, X) in which all the configurations from our collection are stable and all others are inconsistent.*

Proposition 6, fundamental to this paper, will establish the stability of certain configurations. Obviously, the value of the statement that a configuration is stable depends on the wealth of the classes M_ε. Furthermore, to achieve the largest possible values of $\mu(N_\varepsilon(x))$, we should take measures μ concentrated on $N(x)$. Proposition 4 shows that in the case $d = 1$ such measures can be found among the Gibbs measures. Unfortunately, we must require periodicity of x.

Before we formulate Proposition 4 let us perform some constructions. Let (E, X) be a one-dimensional system of diameter D and y a consistent configuration in it. We are also given a segment $V \subset \mathbf{Z}$ and a function $q : X_0^{D+1} \to \mathbf{R}_+$, assuming only strictly positive values. We define a Gibbs measure μ on X, depending on y, V, $q(\cdot)$ as well as on parameters. The measure μ outside of V is concentrated on the configuration $y_{\mathbf{Z}\setminus V}$. The value of $\mu(x_V)$ for any configuration $x_V \in X_V$ is equal to a normalizing coefficient Ξ^{-1} multiplied by the product

$$\prod q(x_v, x_{v+1}, \ldots, x_{v+D}),$$

in which v runs over the values for which $[v; v + D]$ has at least one point in common with V. Here Ξ is the sum of such products over all x_V.

Proposition 4. *Let (E, X) be a system whose set of consistent configurations is nonempty. Then for any $\varepsilon > 0$ we can choose a function $q(\cdot)$ so that for any consistent periodic configuration y and any segment V, the Gibbs measure μ, described in the previous paragraph, belongs to M_ε.*

In order to formulate the next two propositions we must introduce some definitions. Let (E, X) be a d-dimensional system and y a configuration in it. We call a set $V \subset \mathbf{Z}^d$ a *guarantor* (of y) if for any point $v \in \mathbf{Z}^d$ there exists a finite family $\{C_1, \ldots, C_k\} \subset E$ such that

$$\{x : x_v \neq y_v, x_{v+V} = y_{v+V}\} \cap C_1 \cap \cdots \cap C_k = \emptyset.$$

Furthermore, let us immerse $\mathbf{Z}^d$ in $\mathbf{R}^d$ with the same origin 0. We identify the points of $\mathbf{R}^d$ with vectors. Call the unit vectors *directions*. The set of all directions, i.e., the unit sphere, is denoted by Ω. We call the set of vectors which have negative (nonpositive) scalar products with a given direction ω the open (closed) half-space opposite ω. A set $V \subset \mathbf{R}^d$ is called a guarantor if $V \cap \mathbf{Z}^d$ is a guarantor.

Proposition 5. *Let (E, X) be a system and y a periodic consistent configura-*

tion in it. Let there exist a finite set of directions $\omega_0, \omega_1, \ldots, \omega_n$ with the following two properties:

(a) *There exist positive $p_0, p_1, \ldots, p_n$ such that*

$$p_0\omega_0 + p_1\omega_1 + \cdots + p_n\omega_n = 0.$$

(b) *For any m from 1 through n the intersection of the two open half-spaces opposite ω_m and ω_0 is a guarantor of y.*

Then y is stable.

The next proposition applies to the case $d = 2$. In this case $\mathbf{R}^d = \mathbf{R}^2$ is a plane, Ω is a circle, and half-spaces are half-planes. We call a direction ω *coguarant* if the open half-plane opposite it is a guarantor.

Proposition 6. *If, in a two-dimensional system (E, X), the set of all coguarant directions of a given periodic consistent configuration y contains an arc which constitutes more than a half of the circle Ω, then y is stable.*

3. Proofs

Proof of Proposition 1. It is evident that sets of consistent configurations are identical. Suppose that a configuration y is stable in (E, X). Let us prove that it is stable in (E', X). For this purpose it suffices to find, for any $\varepsilon > 0$, a $\delta > 0$ such that $M'_\delta \subset M_\varepsilon$, where M'_δ corresponds to (E', X). From homogeneity considerations we can choose a k such that every $C \subset E$ contains the intersection of no more than k distinct elements of E' as a subset. Since all elements of E are distinct from X, we can choose a number l such that every $C' \in E'$ is a subset of no more than l distinct elements of E. Let $\delta = (\varepsilon/k)^l$. Suppose that $\mu \in M'_\delta$. Let us prove that $\mu \in M_\varepsilon$. Take any $C_1, \ldots, C_m \in E$. Due to our choice of k, we can write:

$$C'_{1,1} \cap \cdots \cap C'_{1,k} \subset C_1,$$
$$C'_{m,1} \cap \cdots \cap C'_{m,k} \subset C_m,$$

where all the sets $C'_{1,1}, \ldots, C'_{m,k}$ are elements of E'. We may have added arbitrary elements of E' to every intersection so that each intersection consists of exactly k terms. Clearly,

$$\begin{aligned}\mu(C_1 \cup \cdots \cup C_m) &\\ &\geqslant \mu\big[(C'_{1,1} \cap \cdots \cap C'_{1,k}) \cup \cdots \cup (C'_{m,1} \cap \cdots \cap C'_{m,k})\big]\\ &= \mu\big[(C'_{1,1} \cup \cdots \cup C'_{m,1}) \cap \cdots \cap (C'_{1,k} \cup \cdots \cup C'_{m,k})\big].\end{aligned}$$

Here the third expression is obtained from the second by interchanging the order of intersections and unions. It contains k^m pairs of parentheses. The sets inside the first pair are taken from different sets $C_1, \ldots, C_m$. Hence there are at least m/l distinct sets among them. Therefore,

$$\mu(C'_{1,1} \cup \cdots \cup C'_{m,1}) \geqslant 1 - \delta^{m/l}.$$

The same reasoning applies to every pair of parentheses. Thus:

$$\mu(C_1 \cup \cdots \cup C_m) \geqslant 1 - k^m \cdot \delta^{m/l} = 1 - \varepsilon^m.$$

Proof of Proposition 2. Proposition 1 allows us to assume that all the elements of E are the translates of a single set C^0, whose support is the segment $[0; D]$. In the course of the proof of Proposition 2, only $[0; D]$ is called the support of C^0 and only $[v; v + D]$ for any v is called the support of $T_v(C^0)$.

I. Suppose that a one-dimensional system (E, X) contains (at least) two distinct consistent configurations y and z. Let us prove that y is unstable. We can assume that $y_0 \neq z_0$; in fact, for any $\varepsilon > 0$ we can find a measure $\mu \in M_\varepsilon$ such that $\mu(N_0(y)) = 1$. We shall construct the measure μ in the form

$$\mu = \frac{1}{n}(\delta_1 + \delta_2 + \cdots + \delta_n),$$

where $\delta_1, \ldots, \delta_n$ are δ-measures concentrated on configurations $y_1, \ldots, y_n$ which will now be constructed. Choose an integer $n > \varepsilon^{-2D}$ and define y_k for all k from 1 through n in the following way:

$$y_k(v) = \begin{cases} z(v) & \text{if } |v| \leqslant k(D+2), \\ y(v) & \text{if } |v| > k(D+2). \end{cases}$$

The measure μ is defined. Clearly, $\mu(N_0(y)) = 1$. Let us prove that $\mu \in M_\varepsilon$. Denote by E_k the collection of the elements of E which do not contain y_k. Notice that if $C \in E_k$, i.e., $y_k \notin C$, then the support of C either contains the points $k \cdot (D + 2)$ and $k \cdot (D + 2) + 1$ or the points $-k \cdot (D + 2)$ and $-[k \cdot (D + 2) + 1]$. It follows that the sets E_k are disjoint and that each of them contains no more than $2D$ elements. Now suppose that $C_1, \ldots, C_m$ are distinct elements of E. Let us prove that

$$\mu(C_1 \cup \cdots \cup C_m) \geqslant 1 - \varepsilon^m.$$

Consider the following two cases:

(a) All of $C_1, \ldots, C_m$ belong to a single E_k. Then $\mu(C_1 \cup \cdots \cup C_m) = 1 - 1/n$. On the other hand, $l \leqslant 2D$, and hence the inequality in question holds.

(b) Not all of $C_1, \ldots, C_m$ belong to one E_k. Then every E_k fails to contain at least one of $C_1, \ldots, C_m$. In this case $\mu(C_1 \cup \cdots \cup C_m) = 1$.

II. Suppose that a one-dimensional system (E, X) contains exactly one consistent configuration y. Let us prove that y is stable. Notice that all the components of y are the same. Let $y = \ldots, a, a, a, \ldots$.

We call a word $z_1, \ldots, z_n$ in an alphabet X_0 consistent if the configuration $z_{[1:n]} = (z_1, \ldots, z_n)$ is consistent.

Lemma. *We set $M = |X_0|^D + D$. Any consistent word has all its component letters—except perhaps M on the left end and M on the right—equal to a.*

Proof. Suppose otherwise. Discarding extra letters, we obtain the consistent word:

$$z_{-M}, \ldots, z_0, \ldots, z_M,$$

where $z_0 \neq a$. Clearly, a word of length M in the alphabet X_0 has at least one combination of D consecutive letters repeated twice. Let $z_{k+1}, \ldots, z_{k+D}$ and $z_{l+1}, \ldots, z_{l+D}$, where $k < l$, be two identical combinations in the sequence $z_1, \ldots, z_M$. In the word $z_{-M}, \ldots, z_0, \ldots, z_M$ remove the letters with indices larger than $l + D$ and add on the right of it an infinite number of periods, each coinciding with $z_{k+D+1}, \ldots, z_{l+D}$. We carry out a similar operation on the left end. Thus we obtain a consistent sequence which is infinite in both directions, hence a configuration containing a component different from a, which contradicts our hypothesis.

Now consider a measure $\mu \in M_e$ and let us estimate $\mu(N_0(y)) \leqslant \mu(\{x : x = \alpha\})$ from above. We expand the last expression in the form

$$\sum \mu(\{x : x_{[-M;M]} = (z_{-M}, \ldots, z_0, \ldots, z_M)\}),$$

where the sum is taken over all $(z_{-M}, \ldots, z_0, \ldots, z_M)$ in which $z_0 \neq a$. It follows from the lemma that all such words are not consistent; hence every summand is at most ε. Thus,

$$\mu(\{x : x_0 \neq a\}) \leqslant |X_0|^{2M+1} \cdot \varepsilon,$$

which approaches 0 when ε does.

Proof of Proposition 3. Let $y^1, \ldots, y^n$ be the given collection. We denote by (v, ρ) the set $\{w \in \mathbf{Z}^d : |w - v| \leqslant \rho\}$, and call it a *ball* of radius $\rho > 0$ with center $v \in \mathbf{Z}^d$.

Define E to be the collection of the translates of a set C_ρ determined by the following condition: a configuration y belongs to C_ρ if the restriction of y to the ball $(0, \rho)$ coincides with at least one of the restrictions of $y^1, \ldots, y^n$ to this ball. The only requirement on ρ is that it is to be sufficiently large. We now describe how to choose ρ.

We call a shift T_v *proper* to a configuration y if $y = T_v(y)$, and we denote by $G(y)$ the group of proper shifts of y. Set $G = \bigcap_{i=1}^n G(y^i)$. It follows from the hypothesis of our Proposition that the quotient group $\mathbf{Z}^d / G(y^i)$ is finite for all i from 1 through n; hence $\mathbf{Z}^d / G$ is finite as well. Choose a representative $w_1, \ldots, w_m$ from every coset in $\mathbf{Z}^d / G$. Also choose a basis $u_1, \ldots, u_d$ of the d-dimensional group G. We set

$$\rho = \max\{|w_1|, \ldots, |w_m|, |u_1|, \ldots, |u_d|\} + 1$$

and proceed to prove Proposition 3.

I. Suppose y is consistent. Then $y_v = y_{v+u_j}$ for any $v \in \mathbf{Z}^d$ and any j from 1 through d. It follows that $G \subset G(y)$. Let y coincide with y^i on the

ball $(0, \rho)$. Then y coincides with y^i on the points $w_1, \ldots, w_m$. This and $G \subset G(y)$ imply that $y = y^i$.

II. The stability of $y^1, \ldots, y^n$ can easily be proved by the contour method.

Proof of Proposition 4. Proposition 1 allows us to assume that all the elements of E are the translates of a set whose support is $[0; D]$ and which is defined by a list of prohibited words of length $D + 1$.

I. Graphs. We consider finite oriented graphs which for any pair of vertices a and b (including $a = b$) have at most one edge directed from a to b and denoted by $a \to b$, if it exists. *A path* of length m going from a_0 to a_m is a sequence of vertices and edges which can be written as $a_0 \to a_1 \to \cdots \to a_m$. A path is called a *cycle* if its first and last vertices coincide and these two vertices are identified.

Our basic graph G has a set of vertices X_0^D, and a set of words of length D in the alphabet X_0. An edge goes from a vertex $(x_1, \ldots, x_D)$ to a vertex $(y_1, \ldots, y_D)$ of $x_{k+1} = y_k$ for all k from 1 through $D - 1$. Every word $(x_0, \ldots, x_D)$ corresponds in a one-to-one relationship to the edge going from $(x_0, \ldots, x_{D-1})$ to $(x_1, \ldots, x_D)$. We call an edge prohibited if it corresponds to a prohibited word.

A sequence that is infinite in both directions is written $\cdots \to a_{v-1} \to a_v \to a_{v+1} \to \cdots$ and called an infinite path. Every configuration $x = (x_v)$ corresponds in a one-to-one relationship to the infinite path $\cdots \to (x_{v+1}, \ldots, x_{v+D}) \to \cdots$ in the graph G. A configuration is consistent if its corresponding path has only unprohibited edges. *Cyclic* vertices and edges of G are those through which an infinite path, corresponding to a periodic consistent configuration, can pass. It is equivalent to define a vertex or an edge as cyclic if a cycle of G which contains only unprohibited edges passes through it. Define two cyclic vertices to be equivalent if both of them belong to a cycle containing only unprohibited edges, and call the resulting equivalence classes *pools*. The cyclic edges connecting the edges of a pool will also be considered to belong to this pool.

II. The choice of the function $q(\cdot)$. The function $q(\cdot)$ must be defined on the set X_0^D or, equivalently, on the set of the edges of G. We look for $q(\cdot)$ in the form

$$q(x_0, \ldots, x_D) = p(x_0, \ldots, x_D) \cdot \tau^{r(x_0, \ldots, x_D)}.$$

The functions $p(\cdot)$ and $r(\cdot)$, which we shall choose, depend only on (E, X); and $p(\cdot)$ assumes only positive values, whereas $r(\cdot)$ assumes only integral non-negative values. The number $r(x_0, \ldots, x_D)$ is called the *degree* of the edge $(x_0, \ldots, x_D)$. The degree of a path is the sum of the degrees of its edges, counted as many times as they are included in the path. We also assign to every vertex a of G the degree $r(a)$, equal to the minimum of the

degrees of the paths starting on cyclic vertices and terminating at a. The parameter $\tau \geqslant 0$ depends only on (E, X) and ε, is positive when $\varepsilon > 0$, and approaches 0 when ε does. The precise dependence of τ on ε will be indicated later.

Assign to our function $q(\cdot)$ a square non-negative matrix Q_τ, whose rows and columns are indexed by the elements of X_0^D. The element of Q_τ at the intersection of the ath row and bth column is $q(a \to b)$ if the edge $a \to b$ is in G, and 0 otherwise. Clearly, Q_τ is irreducible for $\tau > 0$. Therefore, by Perron's theorem, Q_τ has a positive eigenvalue of maximal absolute value, denoted by λ_τ. We denote the corresponding eigenvector, which we normalize so that its components add to 1, by v^τ.

Now we shall require certain conditions to hold for $p(\cdot)$ and $r(\cdot)$, and then show that we can choose $p(\cdot)$ and $r(\cdot)$ so that they satisfy these conditions.

1. $r(\cdot)$ is zero on all cyclic edges and integral positive on all noncyclic edges.

This implies that if $\tau = 0$, then $q(\cdot)$ is zero on all the noncyclic edges, and $Q_0 = Q_{\tau=0}$ factors into the submatrices corresponding to the pools. To every pool B there corresponds an irreducible non-negative matrix Q_B. Denote the corresponding maximal eigenvalue by λ_B and the corresponding eigenvector, normalized so that its components add to 1, by v^B. It is clear that as we vary the values of $p(\cdot)$ over all positive numbers on the edges of B, we can make λ_B assume any given positive value. All that we ask of $p(\cdot)$ on the cyclic edges is that λ_B be the same for all pools. On noncyclic edges we merely require $p(\cdot)$ to be positive. Without striving for generality, we impose the following two conditions on $p(\cdot)$:

2. The values of $p(\cdot)$ on all cyclic edges are positive and such that all λ_B are equal to 1.
3. The values of $p(\cdot)$ on all noncyclic edges are equal to one.

We also require the following statements to hold for the function $r(\cdot)$ and positive integers R:

4. If the first and the last vertices of a path are cyclic, then the degree of the path is either 0 or at least R.
5. Construct the graph γ whose vertices are in one-to-one correspondence with the pools and are denoted by the same symbols. We construct an edge from a vertex B_1 to a vertex B_2 of γ if the graph G contains a path of degree R going from the pool B_1 to the pool B_2. For any two vertices B_1 and B_2, a path from B_1 to B_2 exists in γ.

Let us prove that we can choose $r(\cdot)$ and R so that conditions 4 and 5 hold. For this purpose we first prove that G contains a Hamiltonian cycle,

i.e., a cycle containing all the vertices of G, once each. It follows from the corollary to Theorem 2 of Chapter 10 in [1] that G contains a factor, i.e., a system of cycles such that every vertex has exactly one cycle passing through it just once. If more than one such cycle exists, their number can easily be decreased by merging two suitable cycles into one.

Let us choose a Hamiltonian cycle in G and cut it into pieces at all cyclic vertices. Every piece is a path containing at least one edge, and every piece has only its first and last vertices cyclic. If a piece contains a cyclic edge, then the piece consists entirely of this edge. Otherwise all the edges of the piece are noncyclic, and we assign them integral positive degrees such that their sum—the degree of the piece—is R. Thus, we chose $p(\cdot)$, $r(\cdot)$, and R so that all five of our conditions hold. We shall use only these conditions.

III. Estimates. In the next part of the proof we use the following two conventions.

(a) Distinct positive entities depending only on the system (E, X), the functions $p(\cdot)$ and $r(\cdot)$, and the number R are called constants and denoted by "const."

(b) τ is assumed to be positive and less than a sufficiently small constant.

Lemma 1. $\lambda_\tau \leqslant 1 + \text{const} \cdot \tau^R$.

Proof. It follows from formula (40) of [4] that if λ is the largest positive eigenvalue of an irreducible non-negative matrix M and v is a positive vector of the appropriate dimension, then

$$\lambda \leqslant \max_i \frac{(Mv)_i}{v_i}.$$

To make use of this formula we construct the vector v by the following method. Identify the restriction v_B of v on a pool B with the eigenvector v^B. Now let a be a noncyclic vertex. Let $\Pi_{a,b}$ denote the collection of all paths of degree $r(a)$ which terminate on a, start at a noncyclic vertex b, and contain only noncyclic edges. Denote by Π_a the union of $\Pi_{a,b}$ over all cyclic b. Clearly, Π_a is always finite and nonempty. We write $a < b$ if $\Pi_{a,b}$ is nonempty. Set

$$v_a = (2\tau)^{r(a)} \cdot \sum_{b\,:\,a<b} v_b \cdot |\Pi_{a,b}|.$$

This defines the vector v. Let us compare the components of v and $Q_\tau v$.

1. If a is noncyclic, then $(Q_\tau v)_a < v_a$. In fact, $(Q_\tau v)_a$ is a polynomial in τ whose term of lowest degree has degree $r(a)$ and is equal to $\sum v_h \cdot q(h \to a)$, where h runs over all possible penultimate vertices of the paths belonging to Π_a. We can transform this expression using the inequality

$r(h) \leqslant r(a) - 1$:

$$\sum_h v_h \cdot q(h \to a)$$

$$\leqslant \tau^{r(a)} \cdot \sum_h 2^{r(h)} \cdot \sum_{b\,:\,h<b} v_b \cdot |\Pi_{h,b}| \leqslant \tau^{r(a)} \cdot 2^{r(a)-1} \cdot \sum_h \sum_{b\,:\,h<b} v_b \cdot |\Pi_{h,b}|$$

$$\leqslant \frac{1}{2}(2\tau)^{r(a)} \cdot \sum_{b\,:\,a<b} v_b \cdot |\Pi_{a,b}| = \frac{1}{2} v_a ,$$

and the inequality follows.

2. It is easy to prove that if a is cyclic, then

$$(Q_\tau v)_a \leqslant v_a + \text{const} \cdot \tau^R,$$

which implies Lemma 1.

Lemma 2. *For all vertices a,*

$$\text{const} \cdot \tau^{r(a)} \leqslant v_a^\tau \leqslant \text{const} \cdot \tau^{r(a)}.$$

Proof

1. *The upper bound.* This is evident when a is cyclic. Let a be noncyclic. Consider the inequality

$$v_a^\tau \geqslant (Q_\tau^{r(a)} v^\tau)_a ,$$

which is obvious since $\lambda_\tau \geqslant 1$. The right-hand side equals the sum, over all paths of length $r(a)$ which terminate at a, of the values of $q(\cdot)$ on the edges of the path multiplied by v_B^τ, where b is the starting point of the path. The degree of each of these paths is at least $r(a)$. Therefore v_a^τ is bounded above by a polynomial in τ whose term of lowest degree has degree $r(a)$; the upper bound follows.

2. *The lower bound.* We first consider cyclic vertices. In this case it suffices to show that

$$v_a^\tau \geqslant \text{const} \cdot v_B^\tau$$

for all cyclic vertices a and all vertices b. The inequality is obvious when both a and b belong to the same pool. We now prove it when a and b belong to different pools. Since our function $r(\cdot)$ satisfies condition 5, the inequality is implied by the following proposition. Suppose that there exists a path of degree R from pool B to pool A. Then there exist two vertices $b \in B$, $a \in A$ such that $v_a^\tau \geqslant \text{const} \cdot v_b^\tau$. Let us prove this statement. Write $N = |X_0|^D - 1$. Apply the matrix Q_0^N to the vector v^τ. The matrix Q_A^N acts on the components of $(v^\tau)_A$ in the pool A. It follows from the formula at the beginning of the proof of Lemma 1 and the fact that $\lambda_A = 1$ that at least one component will not decrease in the course of this action. Let this be the

ath component. Thus,

$$(Q_0^N v^\tau)_a \geqslant v_a^\tau .$$

Subtract this inequality from the following equation:

$$(Q_\tau^N v^\tau)_a = \lambda_\tau^N v_a^\tau .$$

The difference of the left-hand sides is the increase of the ath component resulting from τ's not being zero. We choose N large enough so that there exists a path of length N and degree R which terminates at a and starts at a vertex $b \in B$. Then

$$(Q_\tau^n v^\tau)_a - (Q_0^N v^\tau)_a \geqslant p_0^N \cdot \tau^R \cdot v_B^\tau ,$$

where $p_0 = \min\{1, \min p(\cdot)\}$. Hence,

$$v_a^\tau(\lambda_\tau^N - 1) \geqslant p_0^N \cdot \tau^R \cdot v_B^\tau .$$

This and Lemma 1 imply our claim.

Now from the upper bound for noncyclic vertices one can easily prove that $v_a^\tau \geqslant \text{const} \cdot v_b^\tau$ also holds when a is cyclic and b is not. Then, once the estimate from below for cyclic vertices is proved, it can also be proved for noncyclic vertices.

Lemma 3. $\text{const} \cdot \lambda_\tau^{|V|} \leqslant \Xi \leqslant \text{const} \cdot \lambda_\tau^{|V|}$.

Proof. The upper bound is obvious. Let us establish the lower bound. If $|V| \leqslant \text{const} \cdot (\tau^R)^{-1}$, then the lower bound is also obvious, since in that case $\lambda_\tau^{|V|} \leqslant \text{const}$ by Lemma 1, while Ξ is no less than its summand corresponding to the configuration y and equal to 1.

Let $V = [0; N]$. Then Ξ is equal to the component with the index $(y_{N+1}, \ldots, y_{N+D})$ of the vector $Q^{N+D+1}v$, where v has the component with the index $(y_{-D}, \ldots, y_{-1})$ equal to 1 and all other components equal to zero. We use this fact in the case when $N \geqslant N_0 \cdot (\tau^R)^{-1}$, where N_0 is a constant to be chosen in the course of the proof. In this case the lower bound is obtained as follows:

Suppose that the vector v has one component $v_a = 1$, where a is a cyclic vertex, and all other components zero. Then there exists a constant N_0 such that if $N = N_0 \cdot (\tau^R)^{-1}$, then the following inequality holds:

$$Q_\tau^N v \geqslant \text{const} \cdot v^\tau .$$

Let us prove this claim. Let $|A|$ be the number of vertices in the pool A which contains a. Then

$$(Q_\tau^{|A|} v)_A \geqslant \text{const} \cdot v .$$

Suppose that a path of length l and degree R exists which leads from the

pool A to the vertex b of another pool B. Then

$$(Q^{|A|+l}v)_b \geqslant \text{const} \cdot \tau^R;$$

hence,

$$(Q^{|A|+l+|B|}v)_B \geqslant \text{const} \cdot \tau^R \cdot v^B.$$

Induction on positive integral k yields

$$(Q^{k(|A|+l+|B|)}v)_B \geqslant \text{const} \cdot k \cdot \tau^R \cdot v^B.$$

Setting $k = [(\tau^R)^{-1}]$ we obtain $(Q^N v)_B \geqslant \text{const} \cdot v^B$ for $N \geqslant \text{const} \cdot (\tau^R)^{-1}$. After const $\cdot (\tau^R)^{-1}$ additional steps, we will have the components bounded below by a constant in the third pool, to which a path of degree R leads out of B; etc. By condition 5 we will reach every pool in this fashion. Therefore, for $C =$ const large enough, in $C \cdot (\tau^R)^{-1}$ steps we will obtain a vector all of whose cyclic components are no less than a constant. After a constant number of additional steps we find that the value of every noncyclic component a is not less than const $\cdot \tau^{r(a)}$, and this, by Lemma 2, is sufficient.

Now let us prove Proposition 4. From the sum for which Ξ stands, choose the terms corresponding to those configurations x_V which have prohibited words on k fixed segments of length $D + 1$. The sum of such terms is easily estimated from above by

$$(\text{const} \cdot \tau)^k \cdot \lambda_\tau^{|V|}.$$

We divide this expression by the estimate for Ξ from Lemma 3 and obtain the required estimate for τ as ε times a sufficiently small constant.

The proof of Proposition 5 is similar to the proof of Theorem 1 in [9] and we omit it.

Proof of Proposition 6. The first five lemmas do not require the hypothesis of Proposition 6.

Lemma 1. *A set containing a guarantor is also a guarantor.*

Lemma 2. *Every guarantor contains a finite guarantor.*

Lemma 3. *Let A' be a subset of guarantor A. Suppose every element a of A' is in some correspondence with the guarantor B_a. Then the set*

$$(A \setminus A') \cup \bigcup_{a \in A'} (a + B_a)$$

is also a guarantor.

The proofs of the first three lemmas are obvious.

A nonempty intersection of two distinct closed half-planes, with 0 not included, is called a *angle*. An *r-section* of an angle α is any bounded subset $\sigma \subset \alpha$ which satisfies the following condition: if $v_0, v_1, v_2, \ldots$ is a sequence

of points with $v_0 = 0$ such that all its other points belong to $\alpha \backslash \sigma$ and $|v_{i+1} - v_i| \leqslant r$ for all i, then $\{v_i\}$ is bounded.

Lemma 4. *If an angle α is a guarantor, then there exists a number $r(\alpha) > 0$ such that any $r(\alpha)$-section of α is also a guarantor.*

Proof. Let the angle α be a guarantor. By Lemma 2 there exists a finite guarantor $A \subset \alpha$. Set $r(\alpha)$ equal to the maximal distance from the points of A to 0. Let σ be an $r(\alpha)$-section of α. Let us prove that σ is a guarantor. Denote by ρ the set of points which can belong to sequences $v_0, v_1, v_2, \ldots$. With $v_0 = 0$, all the other terms belong to $\alpha \backslash \sigma$ and $|v_{i+1} - v_i| \leqslant r(\alpha)$ for all i. From what was said above, ρ is bounded; hence, the collection of the elements of E whose supports intersect ρ is also bounded. Suppose that a configuration x belongs to all such elements of E and that $x_0 \neq y_0$. Let us find a point $v \in \sigma$ such that $x_v \neq y_v$.

We construct points v_n inductively until the point to be defined as v has been constructed. Set $v_0 = 0$. Assume that we have constructed a point $v_n \in \rho$ for which $x_{v_n} \neq y_{v_n}$. Since A is a guarantor and since x belongs to all the elements of E whose supports intersect ρ, we can find a point $a \in v_n + A$ such that $x_a \neq y_a$. If $a \in \rho$, then take $v_{n+1} = a$. If a does not belong to ρ, then we stop the construction and take $v = a$. Evidently, in the latter case, $a = v \in \sigma$, as required. The construction must terminate by the definition of a section.

Lemma 5. *Let the angles α and β be guarantors and subsets of another angle; also assume that $\alpha \cap \beta$ is nonempty. Then any angle which properly contains $\alpha \cap \beta$ is a guarantor.*

Proof. Since $\gamma \backslash (\alpha \cap \beta)$ is nonempty, it intersects either α or β; let it intersect α. Denote by σ the intersection of α with the annulus centered at 0 with outer radius $R + r(\alpha)$ and inner radius R, where R is chosen sufficiently large. By Lemma 4, σ is a guarantor. To every point $v \in \sigma \backslash \gamma$ we assign a set B_v by the following rule: B_v is the intersection of β with $\gamma - v$ and a disk with center at 0 and sufficiently large radius ρ. It is easy to see that we can first choose R sufficiently large and then ρ sufficiently large so that B_v is an $r(\beta)$-section of β and hence is a guarantor by Lemma 4. Then, by Lemma 3, the set

$$(\sigma \cap \gamma) \cup \bigcup_{v \in \sigma \backslash \gamma} (v + B_v)$$

is a guarantor; therefore, the angle γ that contains it is also a guarantor.

We call an arc $\alpha \subset \Omega$ *coguarant* if the intersection of all open half-planes which are opposite a point of α is a guarantor. We can restate Lemma 5 in the following way.

Lemma 5'. *If two coguarant arcs intersect, and their union is not larger than half the circle, then any open arc which is properly contained in their union is also coguarant.*

From now we assume the hypothesis of Proposition 6. The following lemma is stated in two equivalent formulations.

Lemma 6. *There exist two angles α and β, which are guarantors and subsets of another angle, such that the intersection of α and β is empty.*

Lemma 6^{I}. *There exist two coguarant arcs whose union is an arc larger than half the circle.*

It suffices to prove Lemma 6^{I}. By Lemma 2 any coguarant direction belongs to at least one coguarant arc; hence, the union of all coguarant arcs contains a closed half-circle. It follows by a well-known lemma on finite coverings that there exists a finite collection of coguarant arcs whose union contains a closed half-circle. If there are two of these arcs, then our Lemma is proved. Let the number of such arcs be $M > 2$. We now prove the following statement by induction on the parameter m, decreasing from M to 2: "There exist m coguarant arcs whose union is an arc larger than half the circle Ω." The basis of our induction for $m = M$ is the statement proved above, and the inductive step reduces to Lemma 5^{I}.

We add an equivalent restatement of Lemmas 6 and 6^{I}.

Lemma 6^{II}. *In $\mathbf{R}^2$ there exist oblique coordinates s, t with the origin at 0, and finite guarantors A and B, such that $s < 0$, $t > 0$ at all points of A and $s > 0$, $t > 0$ at all points of B.*

Lemma 6^{II} is the only reason for proving the previous lemmas. We now fix the coordinates s and t and the sets A and B for which Lemma 6^{II} holds. Quantities which are uniquely defined by this convention are called constants.

To every point $v \in \mathbf{Z}^2$ we assign the following set:

$$D_v = \{x \in X : x_v \neq y_v \text{ and } (x_{v+A} = y_{v+A} \text{ or } x_{v+B} = y_{v+B})\}.$$

We assign to every finite set $V \subset \mathbf{Z}^2$ the set D_V which is the intersection of the sets D_v over all $v \in V$. The following lemma is the only one which uses the periodicity of y.

Lemma 7. *There exists a finite collection $\{C_1, \ldots, C_m\} \subset E$ such that for any $v \in \mathbf{Z}^2$ the following intersection is empty:*

$$D_v \cap T_v(C_1) \cap \cdots \cap T_v(C_m) = \emptyset.$$

The proof follows easily from the fact that A and B are guarantors.

Besides our given system (E, X), we construct the system (E', X). To obtain E' we adjoin to E the set $C' = C_1 \cap \cdots \cap C_m$ and all its translates, where $C_1, \ldots, C_m$ are the sets described in Lemma 7. By Proposition 1 the systems (E, X) and (E', X) have identical sets of stable configurations; hence, it suffices to establish the stability of y in (E', X). The system (E', X) is convenient because if $\mu \in M_\varepsilon$, then $\mu(D_V) \leqslant \varepsilon^{|V|}$ for any finite $V \in \mathbf{Z}^2$.

We now prove Proposition 6 by an argument similar to the proof of nonergodicity of Stavskaya's problem in [7]. Our s axis is analogous to the line on which the automata in [7] are positioned, our t axis corresponds to the time axis with the reversed sign, our configuration y corresponds to the state "all automata are 0 all the time," and the relation $x \in D_v$ is analogous to the relation "spontaneous excitation in the realization of x occurs at the point v." In order to obtain an upper bound for the measure of $N_0(y)$ we cover it by a countable system of the sets D_V for certain special V which we now construct.

By a (piecewise linear) *curve* we shall mean a finite sequence $P = \{P_1, \ldots, P_n\}$, whose terms will be called *knots*. Every knot is of the form $P_k = (v_k; \Pi_k)$, where $v_k \in \mathbf{Z}^2$, $\Pi_k \in \{0; 1\}$. We say that the kth knot $P_k = (v_k; \Pi_k)$ is located at the point v_k and has the label Π_k. We call the knot $P_k = (v_k; \Pi_k)$ *labelled* if $\Pi_k = 1$. We denote by $M(P)$ the set of points at which the labelled knots P are located. We call the vector difference $v_{k+1} - v_k$ an *edge* of our curve, $1 \leqslant k \leqslant n-1$.

Call a curve $P = (P_1, \ldots, P_n)$ *admissible* if it satisfies the following five conditions:

1. The first and the last knots are at 0.
2. The curve contains at least one labelled knot.
3. All the labelled knots of the curve are at distinct points.
4. Each edge $v_{k+1} - v_k$, where $1 \leqslant k \leqslant n-1$, belongs to $A \cup (B - A) \cup (-B)$.

Before we state the requirements, observe that the sets A, $B - A$, and $-B$ are disjoint. This allows us to assign to every edge $v_{k+1} - v_k$ of an admissible curve a type d_k defined in the following way:

$$d_k = \begin{cases} 1 & \text{if } v_{k+1} - v_k \in A; \\ 2 & \text{if } v_{k+1} - v_k \in B - A; \\ 3 & \text{if } v_{k+1} - v_k \in -B. \end{cases}$$

Now we state the last requirement for an admissible curve:

5. If ($d_{k-1} = 3$ and $d_k = 1$) or ($d_{k-1} \neq 3$ and $d_k \neq 1$), then the knot P_k is labelled.

Lemma 8. *The set $N_0(y)$ is covered by the sets $D_{M(P)}$, where P runs over all admissible curves.*

Proof. Take an arbitrary configuration $x \in N_0(y)$. Corresponding to it, we construct an admissible curve $P(x)$ such that $x \in D_{M(P(x))}$. For convenience we assign to every curve P the set of points

$$\delta(P) = \{v \in M(P) : x \notin D_v\}.$$

Clearly, $x \in D_{M(P)}$ if and only if $\delta(P)$ is empty.

We construct a sequence of curves $P_0(x), P_1(x), \ldots$. At each step, after constructing the next $P_k(x)$ we prove that $P_k(x)$ is admissible and that the knots of $P_k(x)$ are located only at the points v where $x_v \neq y_v$. Then, using this fact, we transform $P_k(x)$ into the next curve $P_{k+1}(x)$. After a finite number of such steps we obtain the curve to be defined as $P(x)$. We omit the proofs, since they are obvious, and the inessential details of the construction.

First we describe an auxiliary operation called "discarding a loop." Suppose that a curve $P = (P_1, \ldots, P_m)$ does not satisfy requirement 3 for an admissible curve. Choose indices k and l such that $1 \leqslant k < l \leqslant n$ and $v_k = v_l$, $\Pi_k = \Pi_l = 1$. Discard from our curve the terms with indices i, where $k < i \leqslant l$, i.e., form the curve $P' = (P'_1, \ldots, P'_{n+k-l})$ in which

$$P'_j = \begin{cases} P_j & \text{if} \quad j \leqslant k; \\ P_{j+l-k} & \text{if} \quad j > k. \end{cases}$$

Clearly, starting from a curve which satisfies requirements 1, 2, 4, and 5 of the admissible curve, after discarding a loop sufficiently many times we obtain an admissible curve.

We now describe the inductive construction. As the first step we take the curve $P_0(x)$ which consists of one labelled knot positioned at 0.

Now assume that we have the curve $P_k(x) = (P_1, \ldots, P_n)$. If $x \in D_{M(P_k(x))}$, then the construction terminates and $P_k(x)$ is defined to be $P(x)$. Suppose that $x \notin D_{M(P_k(x))}$, i.e., $\delta(P_k(x))$ is nonempty. Take any point which is an element of $\delta(P_k(x))$. Let the labelled knot at this point be $P_i = (v_i, \Pi_i)$. By the inductive hypothesis $x_{v_i} \neq y_{v_i}$. Then $x_{v_i+A} \neq y_{v_i+A}$ *and* $x_{v_i+B} \neq y_{v_i+B}$, *i.e., there exist points* $w_A \in v_i + A$ *and* $w_B \in v_i + B$ *such that* $x_{w_A} \neq y_{w_A}$ *and* $x_{w_B} \neq y_{w_B}$. *Form the sequence* $P' = (P'_1, \ldots, P_{n+3})$ *by the following rule*:

$$P'_j = \begin{cases} P_j & \text{if} \quad j < i; \\ (v_i; 0) & \text{if} \quad j = i; \\ (w_A; 1) & \text{if} \quad j = i+1; \\ (w_B; 1) & \text{if} \quad j = i+2; \\ (v_i; 0) & \text{if} \quad j = i+3; \\ P_{j-3} & \text{if} \quad j > i+3. \end{cases}$$

Clearly the curve P' satisfies requirements 1, 2, 4, and 5 of an admissible curve. After discarding a loop sufficiently many times we obtain an admissible curve which we define to be $P_{k+1}(x)$.

Let us prove that the inductive process of curve transformations will necessarily terminate, i.e., that the set $\delta(P_k(x))$ will be empty at some step. We assign the following parameter to every curve $P = P_k(x)$:

$$\Sigma(P) = \sum_{v_i \in \delta(P)} 3^{-l(v_i)}.$$

This parameter will decrease strictly at each step of our construction. On the other hand, it can assume only a finite number of distinct values, since all the labelled knots of $P_k(x)$ are located at distinct elements of the finite set $\{v : x_v \neq y_v\}$.

Thus, $\mu(N_0(y)) \leqslant \sum \mu(D_{M(P)})$, where the sum is taken over all admissible curves P. It remains to estimate the sum. Denote by $|P|$ the number of terms of a curve ($|P| = n$, if $P = (P_1, \ldots, P_n)$).

Lemma 9. *There exists a constant $F > 0$ such that $|M(P)| \geqslant F \cdot |P|$ for any admissible curve P.*

Proof. It follows from Lemma 6[II] that $s < 0$ at all the points of $A \cup (-B)$. Denote by $R > 0$ the maximum of s over all points of $B - A$. Set $F = \frac{1}{3} r(R + r)$. Since the first and last knots of the curve coincide, the number of edges of type 2 is at least $(|P| - 1)r/(R + r)$. Next, it is easy to see that there exists at least one labelled knot between any two edges of the curve of type 2. Hence, the number of labelled knots is at least $[(|P|) \cdot r/(R + r)] - 1$. This expression is not less than $F|P|$ for $|P| \geqslant 3(R + r)/r$.

If $|P| < 3(R + r)/r$, then, since by requirement 2 the curve contains at least one labelled knot, the number of labelled knots is still no less than $F \cdot |P|$.

Now, using Lemma 9, we can obtain an upper bound for the sum of $\mu(D_{M(P)})$ over all admissible P for $\mu \in M_\varepsilon$. Observe that the number of admissible curves with n knots does not exceed H^n, where H is a constant. Therefore,

$$\sum \mu(D_{M(P)}) \leqslant \sum \varepsilon^{|M(P)|} \leqslant \sum \varepsilon^{F \cdot |P|} \leqslant \sum_{n=1}^{\infty} H^n \cdot \varepsilon^{Fn},$$

where the first three sums are taken over all admissible P.

This series converges for sufficiently small $\varepsilon > 0$, and its sum approaches 0 as $\varepsilon \to 0$. This fact and Lemma 8 yield an upper bound for $\mu(N_0(y))$ which guarantees the stability of y.

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