NONERGODIC MULTIDIMENSIONAL SYSTEMS OF AUTOMATA

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Identical stochastic automata having a finite number of states are positioned at all points of a d-dimensional integer-valued space. At any instant of discrete time each automaton can go to any one of its states with never-vanishing probabilities depending on its own states and those of a finite number of its "neighbors" at the preceding instant. A system of this type is synthesized which is capable of "remembering" its initial state for an infinitely long time when the system commences operation in one of n distinct states of the type "all automata are in state k," where $1 \le k \le n$.

§1. Introduction

We propose to investigate a Markov chain having a continuum of states and describing the behavior of an infinite system of stochastic automata. All automata are identical. They are presumed to be situated at the nodes of the d-dimensional integer-valued lattice Z^d, where they are enumerated by index vectors $i \in Z^d$ (i is any d-dimensional integer-valued vector). The state of each automaton assumes a finite set of values $M_m = \{0, 1, \dots, m\}$. The time t is discrete. The state x_i^t of the i-th automaton at time t depends probabilistically on the states of r designated automata, known as its "neighbors," at time t−1, i.e., $x_i^t = b$ with probability $\varphi^b(a_1, \dots, a_r)$ if

$$x_{i+v_1}^{t-1}=a_1,\dots,x_{i+v_r}^{t-1}=a_r \text{ where } a_1,\dots,a_r,\ b\in M_m,\quad \sum_b \varphi^b=1.$$

The set of vectors $V_1, \dots, V_r$ and the function φ^b is the same for all automata. The set of indices $i + V_1, \dots, i + V_r$ is denoted by U(i). If the states of all automata at time t−1 are given, the states of the automata at time t are independent stochastic variables.

The foregoing description of the operation of automata determines a linear operator P_φ in measure space on the set $X_m = M_m^{Z^d}$, where the σ-algebra is generated by cylindrical sets. Let $a \in X_m$, i.e., $a = (a_i)$, $a_i \in M_m$, $i \in Z^d$. We denote by δ_a the measure on X_m concentrated at a. The operator P_φ maps δ_a into a measure $\delta_a P_\varphi$ in which all automaton states x_i, $i \in Z^d$ are independent and $x_i = b$ with probability $\varphi^b(a_{i+V_1}, \dots, a_{i+V_r})$. The outcome of an application of P_φ to any measure μ is defined as

$$\left\{\begin{aligned}&\mu P_\varphi\{x: x_i=b_i \quad \text{for all} \quad i\in I\}=\\ &=\sum_{\{a_j\},\, j\in U(I)} \mu\{x: x_j=a_j \quad \text{for all} \quad j\in U(I)\}\prod_{i\in I}\varphi^{b_i}(a_{i+v_1},\dots,a_{i+v_r}),\end{aligned}\right. \tag{1}$$

where $x = (x_i)$, $x \in X_m$ and $U(I) = \bigcup_{i\in I} U(i)$.

We refer to operators P_φ of the type described above as operators of type (1). A measure μ is said to be invariant for a given operator P_φ if $\mu = \mu P_\varphi$. An operator P_φ is called ergodic if it has only one, up to a multiplier, invariant measure (there is always at least one); otherwise it is called nonergodic.

The fundamental result of the article is proof of the nonergodicity of a certain class of operators P_φ of the given type and the construction for any positive integer n of operators P_φ having at least n distinct

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invariant measures $\mu_1, \ldots, \mu_n$. Here n can be any positive integer compatible with dimensions d = 2 or more, but the number m + 1 of automaton states must be made greater than or equal to n.

Nonergodic operators of the type described above or analogous thereto have been constructed in several papers [1-6]. In [1-5], however, the authors rely heavily on the fact that certain values of the functions φ^b are equal to zero, and nonergodicity is associated with the existence of "degenerate" invariant measures equal to zero on certain cylindrical sets. Dobrushin [6] has demonstrated a technique for the transformation of Ising models into Markov chains of a form analogous to that considered here, but only for continuous time t. A phase transition in the Ising models guarantees nonergodicity of the chains in this situation.

In our work the functions $\varphi^b(a_1, \ldots, a_r)$ are positive for all $a_1, \ldots, a_r$, b, so that the values of the invariant measures on all cylindrical sets are also positive. Moreover, the conditions imposed on $\varphi^b(a_1, \ldots, a_r)$ for the proof of the fundamental theorem constitute inequalities and are therefore satisfied by an entire domain in the space of sets of values of $\varphi^b(a_1, \ldots, a_r)$.

The operators P_φ constructed below depend on a parameter ε, $0 \le \varepsilon \le 1$, and their nonergodicity is proved only for sufficiently small values of ε. For $\varepsilon = 0$ the operator P_φ becomes deterministic, i.e., the functions φ^b assume only the values 0 or 1. For $\varepsilon = 0$ the invariant measures $\mu_1, \ldots, \mu_n$ go over to measure $\delta_1, \ldots, \delta_n$ concentrated in states of the type "all x_i are equal to k."

Every operator P_φ discussed below is the superposition of an operator P_f and an operator S_α (P_f is applied to the measure first, and then S_α). Here P_f is a deterministic operator, i.e., P_f maps measure δ_a into $\delta_{a'}$, where the state $a' = (a_i')$ is the following function of the state $a = (a_i)$, $i \in Z^d$.

$$a_i' = f(a_{i+v_1}, \ldots, a_{i+v_r}). \tag{2}$$

We refer to operators P_f of the type just described as operators of type (2). Here f is a deterministic function of r arguments $f: M_m^r \to M_m$, and is the same for all $i \in Z^d$.

The operators S_α is described by a stochastic matrix of order $m + 1 : \alpha = \{\alpha_{k,l}\}$, $1 \le k, l \le m + 1$. Broadly, its action may be summed up as taking every automaton found in a state k ($1 \le k \le m + 1$) into state l with probability $\alpha_{k,l}$ independently of all other automata. In precise terms, the operator S_α maps every measure δ_a into a measure in which all x_i are independent and $x_i = b$ with probability $\alpha_{a_i,b}$.

It is easily verified that the product $P_\varphi = P_f S_\alpha$ is an operator of type (1), where

$$\varphi^b(a_1, \ldots, a_r) = \alpha_{f(a_1, \ldots, a_r), b}.$$

We denote by $\beta(\varepsilon)$ the matrix of order m + 1 formed by elements of the type

$$\beta_{k,l} = \begin{cases} \varepsilon/m & \text{if } k \ne l, \\ 1-\varepsilon & \text{if } k = l, \end{cases}$$

where $0 \le k, l \le m$, $0 \le \varepsilon \le 1$. As $\varepsilon \to 0$ the matrix $\beta(\varepsilon) \to E$ and the operator $S_{\beta(\varepsilon)}$ tends to the identity operator. Following is our fundamental result.

THEOREM. For any positive integer n it is possible to construct an operator P_f such that for sufficiently small (but positive) values of ε the operator $P_\varphi = P_f S_{\beta(\varepsilon)}$ has at least n distinct invariant measures.

The fundamental lemma proved in §2 below is central to the proof of the theorem. In §3 we construct the operator P_f and complete the proof of the theorem. In all the proofs the matrix $\beta(\varepsilon)$ can be replaced by any matrix α in which $\alpha_{k,l} \le \varepsilon/m$, as long as $k \ne l$.

What we actually do in §3 is to construct an operator P_f having the property designated by the formula

$$\lim_{\varepsilon \to 0} \max_{0 \le k \le m} \sup_{t=0,1,2,\ldots} \delta_k (P_f S_{\beta(\varepsilon)})^t (x : x_i \ne k) = 0. \tag{3}$$

Relation (3) implies that for sufficiently small values of ε the measures δ_k are "stable" in the sense that they change very little after any number of applications of the operator $P_\varphi = P_f S_{\beta(\varepsilon)}$. Hence, applying the fixed-point theorem, we readily infer the existence of invariant measures $\mu_0, \ldots, \mu_m$ that tend w ly to $\delta_0, \ldots, \delta_m$ as $\varepsilon \to 0$ and are therefore distinct for sufficiently small $\varepsilon > 0$.

§2. The Fundamental Lemma

We limit the discussion of this section to the case m = 1, i.e., to automata having two states: 0 and 1. Let the operator $S_{\beta^0(\varepsilon)}$ be given by the following matrix $\beta^0(\varepsilon)$ of order two:

$$\beta^0_{0,0}=1-\varepsilon, \quad \beta^0_{0,1}=\varepsilon, \quad \beta^0_{1,0}=0, \quad \beta^0_{1,1}=1.$$

Broadly, the action of $S_{\beta^0(\varepsilon)}$ may be summed up as leaving all automata found in the state 1 in that state and taking all automata found in the state 0, each independently of the other, into the state 1 with probabilities equal to ε.

LEMMA. Let an operator P_f of type (2) be specified for the case m = 1 by the set of vectors $U(0) = \{V_1, \ldots, V_r\}$ and the function $f(\sigma_1, \ldots, \sigma_r)$, which is Boolean in the given case. Let a real number c and a nondegenerate linear functional L(i) on Z^d exist such that if

$$[(x_{V_k}=0 \text{ for all } V_k \in U(0), \text{ such that } L(V_k)\leq c), \text{ or } (x_{V_k}=0 \text{ for all } V_k \in U(0), \text{ such that } L(V_k)>c)], \tag{4}$$

then $f(x_{V_1}, \ldots, x_{V_r}) = 0$. For all t = 0, 1, 2, ... and all $i \in Z^d$ there exists an upper bound of the form

$$\delta_0 P_\varphi^t\{x : x_i=1\} \leq \sigma(\varepsilon), \tag{5}$$

where the function σ(ε) is defined for $0 \leq \varepsilon < \varepsilon_0$ and $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$.

Here $P_\varphi = P_f S_{\beta^0(\varepsilon)}$, and δ_0 is a measure concentrated in the "all-zeros" state.

Proof of the Lemma. Together with the function f we consider a function f' such that to have $f'(x_{V_1}, \ldots, x_{V_r}) = 0$ condition (4) is not only sufficient, but necessary as well. Clearly, $f' \geq f$ (for any set of values of the arguments). Consequently, $P_{f'} > P_f$ in the sense of Mityushin [4], and it suffices to prove inequality (3) for $P_{\varphi'} = P_f S_{\beta^0(\varepsilon)}$. Clearly, $f'(x_{V_1}, \ldots, x_{V_r})$ can be represented in the form

$$\begin{cases} f'(x_{V_1}, \ldots, x_{V_r}) = \left(\bigvee_{k:L(V_k)\leq c} x_{V_k}\right)\left(\bigvee_{l:L(V_l)>c} x_{V_l}\right) \\ = \bigvee_{k,l:L(V_k)\leq c,\, L(V_l)>c} x_{V_k} x_{V_l}. \end{cases} \tag{6}$$

We denote

$$c' = \min_{l:L(V_l)>c} L(V_l) > c.$$

We rewrite (6), using the c' just defined and enumerating all pair products in (6) in sequence by numbers from 1 to q, q being equal to the product of the number of values of k, where $L(V_k) \leq c$, by the number of values of l, where $L(V_l) > c$:

$$f'(x_{V_1}, \ldots, x_{V_r}) = \bigvee_{k=1}^{q} x_{V'_k} x_{V''_k}, \tag{7}$$

$L(V'_k) \leq c$, $L(V''_k \geq c')$ for all k, $1 \leq k \leq q$. Obviously, the list of vectors $V'_1, \ldots, V'_q$ consists of the vectors V_k for which $L(V_k) \leq c$ with repetitions, and the list of vectors $V''_1, \ldots, V''_q$ consists of the vectors V_l for which $L(V_l) \geq c'$ with repetitions.

We represent the operator $S_{\beta^0(\varepsilon)}$ as follows. We introduce stochastic variables γ_i, which are independent of one another and are equal to 0 with probability 1−ε and to 1 with probability ε. Then the measure $\delta_a S_{\beta^0(\varepsilon)}$ is induced by the distribution of γ_i under the map

$$x_i = a_i \vee \gamma_i, \quad i \in Z^d.$$

At the initial time t = −T (T being a positive integer) let the measure $\delta_{x^{-T}}$ be given. Let the operator $P_{\varphi'}$ then be applied to it repeatedly. We denote by γ_i^t the variables γ_i participating in an application of $P_{\varphi'}$ that maps the measure at time t−1 into the measure at time t. Of course, all the γ_i^t are mutually independent. Clearly, the state x_0^0 of the automaton x_0 at time t = 0 is a deterministic function of a finite number of γ_i^t and x_i^{-T}. We denote that function by F_T:

$$x_0^0 = F_T(\{\gamma_i^t, |i|\leq -Rt, -T<t\leq 0\}, \{x_i^{-T}, |i|\leq RT\}),$$

where |i| is the sum of the moduli of the components $i \in Z^d$, $R = \max_{1\leq k\leq r} |V_k|$. At T = 0 the function F_T goes over to F_0, which is equal to

$$x_0^0 = F_0(x_0^0) = x_0^0.$$

For $T \geq 1$ the expression for F_T is obtained by substitution of the expressions

$$\gamma_i^{-T+1} \vee f'(x_{i+v_1}^{-T}, \ldots, x_{i+v_r}^{-T})$$

for all x_i^{-T+1} contained in the expression for F_{T-1}. It is obvious that for all T the function F_T is monotone Boolean. It is therefore representable (in several ways) as the disjunction of conjunction-terms of its arguments. We define one such representation

$$F_T = \bigvee_{G \in H_T} \bigwedge_{(i,t) \in G} (\gamma_i^t, \text{where } t > -T, \text{ or } x_i^{-T}) \tag{8}$$

inductively on T. Here G is the set of pairs (i, t) enumerating those γ_i^t or x_i^{-T} occurring in the given term, and H_T is the set of sets G corresponding to all terms of the disjunction.

Let us suppose that an expression of the form (8) has already been obtained for F_{T-1}. To obtain an expression of the form (8) for F_T we need to replace every x_i^{-T+1} in the expression (8) for F_{T-1} by

$$\gamma_i^{-T+1} \vee \bigvee_{k=1}^{q} x_{i+V'_k}^{-T} x_{i+V''_k}^{-T}$$

and develop the parentheses. This defines an expression of the form (8) inductively (we do not know whether it is a disjunctive normal form).

<u>Definition.</u> A cycle Q is a combination formed by a finite sequence of pairs of the form $(i, t) \in Z^{d+1}$ (some of the pairs can be identical) and a set G_Q containing certain pairs entering into that sequence:

$$Q = (\{(i_k, t_k), 0 \leq k \leq S_Q - 1\}, G_Q), \tag{9}$$

where the following conditions are satisfied:

a) $(i_0, t_0) = (0, 0)$;

b) all pairs in G_Q enter into the sequence in (9).

We define

$$\Delta_k = \begin{cases} (i_{k+1}, t_{k+1}) - (i_k, t_k), & \text{if } 0 \leq k \leq S_Q - 2, \\ (i_0, t_0) - (i_{S_Q-1}, t_{S_Q-1}), & \text{if } k = S_Q - 1; \end{cases}$$

c) for any k, $0 \leq k \leq S_Q - 1$ the difference vector Δ_k has one of the alternative forms $(V'_l, -1)$, $(V''_l - V'_l, 0)$, or $(-V''_l, 1)$, where $1 \leq l \leq q$;

d) if Δ_k, $0 \leq k \leq S_Q - 1$ has the form $(V''_l - V'_l, 0)$, then the pair (i_k, t_k) enters into the set G_Q.

<u>Remark.</u> Suppose that the sequence involved in a cycle of the form (9) satisfies conditions a) and c).

We estimate the lower bound of the number of indices $0 \leq k \leq S_Q - 1$ for which Δ_k has the form $(V''_l - V'_l, 0)$. We introduce the following linear functional on Z^{d+1}: $L'(i, t) = L(i) + (1/2)(c' + c)t$.

It is readily shown by calculations that $L'(\Delta_k) \leq (1/2)(c - c')$ if Δ_k has the form $(V'_l, -1)$ or $(-V''_l, 1)$. We define

$$D = \max_{1 \leq l \leq q} L'(V''_l - V'_l, 0).$$

Clearly,

$$\sum_{k=0}^{S_Q-1} \Delta_k = 0,$$

whereupon

$$\sum_{k=0}^{S_Q-1} L'(\Delta_k) = 0.$$

We separate the latter sum into two parts. We assign to the first sum those terms for which Δ_k has the form $(V''_l - V'_l, 0)$, $1 \leq l \leq q$, and to the second sum all remaining terms. Suppose that the first sum comprises h terms. Then it is not greater than Dh. The second sum is not greater than $(1/2)(c - c')(S_Q - h)$. Therefore, $0 \leq Dh + (1/2)(c - c')(S_Q - h)$, whence

$$h \geq \frac{c'-c}{2D+c'-c} S_Q. \tag{10}$$

We now prove the following statement by induction on T. For every $G \in H_T$, where H_T is defined insofar as the form (8) is defined, there is a cycle Q such that $G = G_Q$.

For T = 0 the set H_T comprises only one set G, which consists of the single pair (0, 0). For this G we choose $S_Q = 3$ and the sequence of pairs (0, 0), $(V_1''-V_1', 0)$, $(-V_1', 1)$. (Instead of V_1', V_1'' we could have chosen V_l', V_l'' for any l, $1 \le l \le q$.) Next we go from T−1 to T. We have described an algorithm for the derivation of the form (8) for F_T from the form (8) for F_{T-1}. Every term of the form for F_T is obtained from some term of the form for F_{T-1} in such a way that every x_i^{-T+1} is replaced either by γ_i^{-T+1} or by some product $x_{i+V_l'}^{-T} x_{i+V_l''}^{-T}$, where $1 \le l \le q$. By the induction hypothesis the leading term in the form (8) for F_{T-1} has associated with it a cycle Q of the form (9). We transform it into the cycle Q' corresponding to a new term entering into the form (8) for F_T.

We call an index k, $0 \le k \le S_Q-1$ singular (for a given leading term and given new term derived therefrom) if the term (i_k, t_k) of the sequence in Q meets the following conditions: $t_k = -T+1$; the factor $x_{i_k}^{t_k}$ is included in the leading term; in transition to the new term the indicated factor is replaced by a product of the form

$$x_{i_k+V_l'}^{-T} x_{i_k+V_l''}^{-T}, \text{ where } 1 \le l \le q. \tag{11}$$

The sequence involved in Q' differs from the sequence (i_k, t_k), $0 \le k \le S_Q-1$ to the extent that after every term (i_k, t_k) thereof with a singular index k three new terms are inserted into the sequence in the following order:

$$(i_k+V_l', -T),\ (i_k+V_l'', -T),\ (i_{k+1}+V_l'', -T), \tag{12}$$

where the index l coincides with the index l in (11). The resulting sequence lengthened by three times the number of singular indices is reenumerated by integers beginning with zero.

The set $G_{Q'}$ is defined as the set of pairs (i, t) enumerating all factors of the new term. It is easily proved that the Q' so constructed satisfies conditions a) through d).

We now deduce the bound (5), which is the substance of the lemma. We must find an upper bound for the probability that $x_0^0 = 1$ when the "all zeros" measure δ_0 is specified at time −T. (Obviously, for any $i \in Z^d$ the probability that $x_i^0 = 1$ is the same as for i = 0.) In other words, we must find an upper bound for the probability that $F_T(\{\gamma_i^t, |i| \le -Rt, -T < t \le 0\}, \{x_i^{-T}, |i| \le RT\}) = 1$, when all x_i^{-T} are identically zero, while all γ_i^t are independent and equal to zero with probability $1-\theta$ and to unity with probability θ. We denote by $F_T^0(\{\gamma_i^t, |i| \le -Rt, -T < t \le 0\})$ the function obtained from F_T by the substitution of zeros for all x_i^{-T}. Clearly,

$$F_T^0 = \bigvee_{G \in H_T^0} \bigwedge_{(i,t) \in G} \gamma_i^t, \tag{13}$$

where the set H_T^0 is obtained from H_T in (8) by the deletion of all sets included therein containing at least one pair of the form (i, −T).

We define a complex K as a nonvacuóus finite set (N-tuple) of cycles

$$K = (Q_1, \ldots, Q_N). \tag{14}$$

We say that a complex K generates a Boolean function F of a finite number of variables γ_i^t if

$$F = \bigvee_{l=1}^{N} \bigwedge_{(i,t) \in G_{Q_l}} \gamma_i^t. \tag{15}$$

Here G_{Q_l} are the sets entering into the cycles Q_l forming K. We have proved above that the function F_T^0 is representable as generated by a certain complex K of the form (14), where $Q_1, \ldots, Q_N$ are the cycles corresponding to all sets G entering into H_T^0 in (13). (Since T is now fixed, we do not attach an index to it.)

We transform the complex K into another complex $K' = (Q_1', \ldots, Q_{N'}')$ in such a way as to meet conditions a) through d) below. We denote by F_T' the function generated by K'.

a) $F_T \le F'_T$ (on all sets of values of γ_i^t);

b) for each cycle Q'_l, $1 \le l \le N'$, all terms of its member sequence are distinct;

c) all sets $G_{Q'_l}$ entering into the cycles Q'_l, $1 \le l \le N'$ are not subsets, any one of any other. (In particular, therefore, all $G_{Q'_l}$ are distinct, so certainly all Q'_l are distinct);

d) the sequence entering into each cycle Q'_l, $1 \le l \le N'$ contains at least three terms.

We construct in succession complexes $K^0, K^1, K^2, \ldots$, where $K^0 = K$, until we obtain a complex K^n that can be adopted as K'. The complexes K^n are formed inductively. Suppose that the following complex has already been constructed:

$$K^n = (Q_1^n, \ldots, Q_{N^n}^n). \qquad (16)$$

If K^n satisfies conditions b) and c), we can adopt it as K'. Incidentally, conditions a) and d) are satisfied by all K^n, $n = 0, 1, 2, \ldots$.

Let K^n not satisfy condition b). Then in one cycle Q_l^n of the form $Q_l^n = (\{i_k, t_k), 0 \le k \le S_{Q_l}^n - 1\}, G_{Q_l}^n)$ two pairs entering into the member sequence are identical: $(i_{k_1}, t_{k_2}) = (i_{k_2}, t_{k_2})$, where $k_1 < k_2$.

In this case we delete from the sequence in Q_l^n all pairs (i_k, t_k), $k_1 < k \le k_2$. We enumerate the remaining pairs in succession by integers beginning with zero. We also eliminate from the set $G_{Q_l}^n$ those elements (i, t) that are now included in the sequence. Clearly, we obtain as a result another cycle, which we call $(Q_l^n)'$. Replacing the cycle Q_l^n in K^n by $(Q_l^n)'$, we obtain a new complex K^{n+1}.

Let K^n not satisfy condition c). Then for a certain pair of cycles $Q_{l_1}^n$, $Q_{l_2}^n$ entering into K^n the set $G_{Q_{l_1}^n}$ is a subset (possibly not a proper subset) of the set $G_{Q_{l_1}^k}$. In this case the list of cycles defining the complex K^{n+1} is obtained from the list (16) by deletion of the cycle $Q_{l_2}^n$.

It is easily verified that after a finite number of such reconstructions of the complex K a complex satisfying conditions a) through d) is obtained, which is then taken as K'. From condition a) and the definition of F'_T we have

$$\left.\begin{aligned} &\Pr(F_T^0=1) \leqslant \Pr(F_T'=1) \\ &= \Pr\Big(\bigvee_{1 \leqslant l \leqslant N'} \bigwedge_{(i,t) \in G_{Q'_l}} \gamma_i^t = 1\Big) \\ &\leqslant \sum_{1 \leqslant l \leqslant N'} \Pr\Big(\bigwedge_{(i,t) \in G_{Q'_l}} \gamma_i^t = 1\Big) \\ &= \sum_{1 \leqslant l \leqslant N'} \varepsilon^{|G_{Q'_l}|}, \end{aligned}\right\} \qquad (17)$$

where $|G|$ is the number of elements in G.

From condition b) for the complex K' and condition d) in the definition of a cycle the number $|G_{Q'_{l_1}}|$ of elements in $G_{Q'_{l_1}}$ is not less than the number of difference vectors Δ_k, $0 \le k \le S_{Q'_l} - 1$ in the Q'_l sequence of the form $(V''_j - V'_j, 0)$, where $1 \le j \le q$. We therefore obtain from relation (10)

$$|G_{Q'_l}| \geqslant \frac{c'-c}{2D+c'-c} S_{Q'_l}. \qquad (18)$$

We separate the last sum in (17) into a sum of sums. We assign to the j-th sum those terms for which the number $S_{Q'_l}$ of the elements in the Q'_l sequence is equal to j. By condition d) for the complex K' only the sets of terms of sums for $j \ge 3$ are nonvacuous. Consequently, taking (18) into account, we obtain

$$\left.\sum_{1 \leqslant l \leqslant N'} \varepsilon^{|G_{Q'_l}|} = \sum_{j=3}^{\infty} \sum_{\substack{1 \leqslant l \leqslant N' \\ S_{Q'_l}=j}} \varepsilon^{|G_{Q'_l}|} \leqslant \sum_{j=3}^{\infty} \sum_{\substack{1 \leqslant l \leqslant N' \\ S_{Q'_l}=j}} \Big(\varepsilon^{\frac{c'-c}{2D+c'-c}}\Big)^j. \right\} \qquad (19)$$

From condition c) for the complex K' we know that all cycles in K' are distinct. Invoking conditions a), b), and c) in the definition of a cycle, we readily verify that there are at most $(6q)^j$ distinct cycles whose member sequences contain j terms. Therefore, the last expression in (19) is not greater than

$$\sum_{j=3}^{\infty}(6q)^j\left(\varepsilon^{\frac{\varepsilon'-\varepsilon}{2D+\varepsilon'-\varepsilon}}\right)^j$$

$$=\left(6q\varepsilon^{\frac{\varepsilon'-\varepsilon}{2D+\varepsilon'-\varepsilon}}\right)^3\Big/\left(1-6q\varepsilon^{\frac{\varepsilon'-\varepsilon}{2D+\varepsilon'-\varepsilon}}\right)=\sigma(\varepsilon). \tag{20}$$

This equation serves as a definition of $\sigma(\varepsilon)$. As $\varepsilon \to 0$ it tends to zero, whereupon we arrive at the conclusion of the lemma.

§3. Proof of the Fundamental Theorem

Let a positive integer n be given. It is required to construct an operator $P_\varphi = P_f S_{\beta(\varepsilon)}$ having at least n distinct invariant measures. We set $m + 1 = n$. We determine f. To do so we choose m + 1 linear functionals L_k, $1 \le k \le m + 1$, on Z^d. All that is required is that the functionals L_k be nondegenerate and pairwise nonproportional; otherwise they can be arbitrary. We define a finite set $U(0) = \{V_1, \ldots, V_r\}$ so that it has a nonvacuous intersection with each of the four quarter lattices into which Z^d is divided by each pair of hyperplanes $L_k = 0$, $L_l = 0$, $1 \le k \le l \le m + 1$. It is sufficient, for example, to draw through the point 0 in Z^d a two-dimensional rational plane on which each of the functionals L_k is not constant and to form the set U(0) of 2(m + 1) integer-valued points belonging to that plane, one per node, at which that plane intersects the hyperplanes $L_k = 0$, $1 \le k \le m + 1$. Then the following definition of $f(x_{V_1}, \ldots, x_{V_r})$ is noncontradictory:

$$\left.\begin{array}{l} f(x_{V_1},\ldots,x_{V_r})=k, \text{ if} \\ x_{V_l}=k \ \text{ for all } \ V_l\in U(0), \text{ such that } \ L_k(V_l)\leqslant 0 \text{ or} \\ x_{V_l}=k \ \text{ for all } \ V_l\in U(0), \text{ such that } \ L_k(V_l)>0. \end{array}\right\} \tag{21}$$

For the case in which condition (21) is not satisfied for any k, the definition of the function is augmented in any way necessary. We are required to prove that condition (3) is satisfied. We show that the following specific condition holds, which clearly guarantees (3):

$$\delta_k P_\varphi^t(x : x_i \neq k) \leqslant \sigma(\varepsilon), \tag{22}$$

where the function $\sigma(\varepsilon)$ is defined in (20).

We fix k from the outset, $0 \le k \le m$. We use the fundamental lemma proved in §2. For convenience we denote by Y the space X_1 and let the points of Y be denoted by $y = (y_i)$, $y_i \in \{0, 1\}$, $i \in Z^d$.

Along with the operator P_φ constructed above we introduce the operator $P_{\varphi_0} = P_{f_0} S_{\beta^0(\varepsilon)}$, which acts in the measure space on Y and has the form designated in the fundamental lemma. Here the Boolean function f_0 of r arguments is defined by the relation

$$\left.\begin{array}{l} f_0(y_{V_1},\ldots,y_{V_r})=0, \text{ if and only if} \\ y_{V_l}=0 \ \text{ for all } \ V_l, \\ \qquad \text{such that } L_k(V_l)\leqslant 0, \text{ where } 1\leqslant l\leqslant r, \text{ or} \\ y_{V_l}=0 \ \text{ for all } \ V_l, \\ \qquad \text{such that } L_k(V_l)>0, \text{ where } 1\leqslant l\leqslant r. \end{array}\right\} \tag{23}$$

We also define the map $H : X_m \to Y$ as $H = h^{Z^d}$, where $h : M_m \to M_1$ is as follows: $h(k) = 0$, $h(a) = 1$, if $a \neq k$. If μ is a measure on X_m, we denote by μH the measure induced on Y by μ under the map H. It now suffices to show that for any t

$$P_\varphi^t H \prec H P_{\varphi_0}^t \tag{24}$$

in the sense of Mityushin [4]; note that (22) follows from the fundamental lemma and (24).

The operator P_{φ_0} is clearly monotone (since f_0 is monotone; see [4]). We therefore prove (24) by induction on the basis of the relation

$$P_\varphi H \prec H P_{\varphi_0}. \tag{25}$$

It suffices to prove that $\delta_a P_\varphi H \prec \delta_a H P_\varphi$ for any $a \in X_m$, i.e.,

$$\delta_a P_f S_{\beta(\varepsilon)} H \prec \delta_a H P_{f_0} S_{\beta^0(\varepsilon)}. \tag{26}$$

Both measures in (26) are such that all y_i in it are independent; consequently, it is sufficient to prove the inequality only for the projections of those measures onto the i-th coordinate:

$$\delta_a P_f S_{p(\varepsilon)} H(y : y_i = 1) \leqslant \delta_a H P_{f_0} S_{p^*(\varepsilon)}(y : y_i = 1), \text{ i.e.,}$$

$$(1 - \varepsilon m^{-1}) \delta_a P_f (x : x_i \neq k) + \varepsilon \delta_a U_f (x : x_i = k) \leqslant$$

$$\leqslant \delta_a H P_{f_0} (y : y_i = 1) + \varepsilon \delta_a H P_{f_0} (y : y_i = 0). \qquad (27)$$

Clearly, each of the measures $\delta_a P_f$, $\delta_a HP_{f_0}$ is concentrated in a certain state. If condition (21) is satisfied for $a_{i+V_1}, \ldots, a_{i+V_r}$, then both sides of (27) are equal to ε. Otherwise the left-hand side is either $(1-\varepsilon m^{-1})$ or ε, and the right-hand side is equal to 1. In either case (27) is satisfied. This proves the theorem.

LITERATURE CITED

1. O. N. Stavskaya and I. I. Pyatetskii-Shapiro, "Homogeneous networks of spontaneously active elements," in: Problems of Cybernetics [in Russian], No. 20, Nauka, Moscow (1968), pp. 91-106.
2. M. A. Shnirman, "Ergodicity of a particular Markov chain," ibid., pp. 115-122.
3. A. L. Toom, "A family of homogeneous networks of formal neurons," Dokl. Akad. Nauk SSSR, 183, No. 1, 49-53 (1968).
4. L. G. Mityushin, "Nonergodicity of homogeneous threshold networks with low self-excitation," Probl. Peredachi Inform., 6, No. 3, 99-103 (1970).
5. A. L. Toom, "Nonergodicity in homogeneous random media," in: Probabilistic Research Methods [in Russian], No. 41, Izd. MGU, Moscow (1972), pp. 34-42.
6. R. L. Dobrushin, "Markov processes with many locally interacting components — the reversible case and some generalization," Probl. Peredachi Inform., 7, No. 3, 57-66 (1971).