I.5
Markov invariant measures and Gibbs fields

These last chapters are about some independent homogeneous operators on $X = \{1,2,\ldots,n\}^V$ having an invariant measure which can be expressed explicity in one or another sense.

Chapter 16 describes the set $\mathscr{P}_M$ of all operators on the graph Γ_1 having a Markov invariant measure. It proves in the case $n > 2$ that the largest dimension component of $\mathscr{P}_M$ consists of operators having a Bernoulli invariant measure (as in Example 1.3).

Chapter 17 proves that on any homogeneous graph $\Gamma(\mathbb{Z}^d, \mathscr{U})$ there is a family of operators having a Bernoulli invariant measure and that operators of this family are ergodic.

Chapter 18 discusses the notion of a Gibbs measure which is a generalisation of a Markov measure to any graph. It presents some operators which have Gibbs invariant measures. In particular, a non-ergodic non-degenerate operator on $V = \mathbb{Z}^2$ is built in Chapter 18 (which has been mentioned as Example 1.4b). This operator is reversible in the following sense: its evolution measure is such that the back transition from $t + 1$ to t can be treated as given by the same (modulo translation) independent operator. A necessary and sufficient condition is given for an operator to be reversible and some results about ergodicity of reversible operators are given too.

Chapter 19 is about evolution measures as random fields. These are always Gibbs fields. Some new kind of non-ergodicity is possible for them. Markov systems with refusals which generalise these fields are treated too.

Chapter 16
Operators on the graph Γ_1

Remember that $\Gamma_1(V, \mathcal{U})$ has $V = \mathbb{Z}$, $U(h) = \{h-1, h\}$. Here

$$X_0 = \{1,2,\ldots,n\}, X = X_0^V.$$

A homogeneous independent operator is defined by n^3 transitional probabilities

$$\theta_{pq}^s = \tilde{\mu}(x_h^{t+1} = s \mid x_{h-1}^t = p, x_h^t = q),\ p,\ q,\ s = 1,\ 2,\ \ldots,\ n.$$

which are subject to conditions

$$\sum_s \theta_{pq}^s = 1,\ \theta_{pq}^s \geqslant 0.$$

Throughout this chapter all the operators are non-degenerate, that is, all $\theta_{pq}^s > 0$. Let $\mathcal{P}$ stand for the set of non-degenerate operators of this sort which may be seen as the inner points of the $(n^3 - n^2)$-dimensional polyhedron given by the inequalities $\theta_{pq}^s > 0$. The aim of this chapter is to find out which operators of $\mathcal{P}$ have invariant homogeneous Markov (and Bernoulli among them) measures. Let us have a stochastic matrix $B = (\beta_{pq})$,

$$p, q = 1,\ 2,\ \ldots,\ n,\ \sum_q \beta_{pq} = 1,\ \beta_{pq} \geqslant 0$$

which has a row-eigenvector

$$b = (\beta_p),\ \sum_p \beta_p = 1,\ \beta_p \geqslant 0,\ bB = b.$$

This matrix defines a homogeneous Markov measure μ_B on X as follows:

$$\mu_B(x_h = s_0, x_{h+1} = s_1, \ldots, x_{h+m} = s_m) = \beta_{s_0}\beta_{s_0 s_1} \ldots \beta_{s_{m-1} s_m}. \qquad (16.1)$$

As we restricted ourselves to non-degenerate operators only, we need only non-degenerate Markov measures which have all $\beta_{pq} > 0$. This measure μ_B is Bernoulli if and only if B has range 1, or if $\beta_{pq} \equiv \beta_q$.

Using the notations introduced, we can write down the conditions for a Markov measure μ_B to be invariant of an operator $P \in \mathscr{P}$

$$\Sigma\beta_{p_0}\beta_{p_0p_1} \dots \beta_{p_{m-1}p_m}\theta^{s_1}_{p_0p_1} \dots \theta^{s_m}_{p_{m-1}p_m} = \beta_{s_1}\beta_{s_1s_2} \dots \beta_{s_{m-1}s_m} \quad (16.2)$$

where the summation is over all $p_0, p_1, \dots, p_m \in X_0$ and for all $m \in \mathbb{Z}_+$, $s_1, \dots, s_m \in X_0$. We denote

$$\alpha^s_{pq} = \beta_{pq}\theta^s_{pq}$$

and consider the n square matrices

$$A_s = (\alpha^s_{pq}),\ s \in X_0.$$

Now (16.2) may be rewritten as

$$bA_{s_1}A_{s_2} \dots A_{s_m}e = \beta_{s_1}\beta_{s_1s_2} \dots \beta_{s_{m-1}s_m} \quad (16.3)$$

where e is the column vector, all components of which equal 1. We shall use this notation throughout this chapter.

Proposition 16.1 Any Markov measure is invariant for some operator.

Proof. Take any $\tilde{\mu}_B$. First we present B as a product of two commuting stochastic matrices $\Lambda = (\lambda_{pq})$ and $K = (\varkappa_{pq})$. This can be done in many ways, say $\Lambda = B$, $K = E$. But we want all $\lambda_{pq} > 0$, $\varkappa_{pq} > 0$. For this we can take, for example,

$$\Lambda = (1 + \varepsilon)B(E + \varepsilon B)^{-1},\ K = (1 + \varepsilon)^{-1}(E + \varepsilon B),$$

where $\varepsilon > 0$ is small enough. Now Proposition 16.1 just follows from

Lemma 16.2 Let

$$B = K\Lambda = \Lambda K$$

where K and Λ are stochastic $n \times n$ matrices. Then μ_B is invariant for the operator having

$$\theta^s_{pq} = \varkappa_{ps}\lambda_{sq}/\beta_{pq} \quad (16.4)$$

as transitional probabilities.

Proof of Lemma 16.2 consists of simply checking the (16.2) equalities. Before doing it we note that

$$bKB = bBK = bK,\ be = bKe = 1.$$

So bK is a normed eigenvector of B with eigenvalue 1. But B has only one such vector (because $\beta_{pq} > 0$), which is b; so

$$b = bK.$$

Now we are ready to check (16.2). First we do it for $m = 1$:

$$\sum_{p_0, p_1} \beta_{p_0}\beta_{p_0p_1}\theta^s_{p_0p_1} = \sum_{p_0, p_1} \beta_{p_0}\varkappa_{p_0p_0s}\lambda_{sp_1} = \beta_s \sum_{p_1} \lambda_{sp_1} = \beta_s$$

where we used $bK = b$ and $\Lambda e = e$. Now for $m = 2$:

$$\sum_{p_0, p_1, p_2} \beta_{p_0}\beta_{p_0p_1}\beta_{p_1p_2}\theta^{s_1}_{p_0p_1}\theta^{s_2}_{p_1p_2} = \sum_{p_0, p_1, p_2} \beta_{p_0}\varkappa_{p_0s_1}\lambda_{s_1p_1}\varkappa_{p_1s_2}\lambda_{s_2p_2} =$$

$$= \Big(\sum_{p_0} \beta_{p_0}\varkappa_{p_0s_1}\Big)\Big(\sum_{p_0} \lambda_{s_1p_1}\varkappa_{p_1s_2}\Big)\Big(\sum_{p_2} \lambda_{s_2p_2}\Big) = \beta_{s_1}\beta_{s_1s_2}$$

where we used $bK = b$, $\Lambda K = B$, and $\Lambda e = e$. For $m \geqslant 2$ one can do the same. This proof of Lemma 16.2 is ingenuous rather than ingenious. But in Chapter 18 we shall prove Proposition 18.6, of which Lemma 16.2 is just a corollary. (Example 18.8 is relevant too.)

To formulate the main theorem of this chapter we need the following notation: $\mathscr{P}_B$ is the set of those operators of $\mathscr{P}$ for which the measure μ_B is invariant (Proposition 16.1 means that all $\mathscr{P}_B$ are non-empty); r_B stands for the defect of B:

$$r_B = n - \text{rang } B,$$
$$R_0 = \text{Ker } B = \{a \in \mathbb{R}^n \colon aB = 0\},\ R' = \mathbb{R}^n/R_0,$$

B' is the transformation of R' induced by B, and finally ℓ_B is the dimension of the space of those linear transformations of R' which commute with B'.

Theorem 16.3 (*I. I. Piatetski-Shapiro*)

(a) For μ_B to be invariant under P it is sufficient that (16.2) hold for $m \leqslant n + 1$, $s_1, \ldots, s_m \in X_0$.

(b) $\mathscr{P}_B$ is the intersection of $\mathscr{P}$ with an algebraic manifold of dimension $n(n-1)r_B + \ell_B - 1$.

(c) $\mathscr{P}_M = \bigcup_B \mathscr{P}_B$ is the intersection of $\mathscr{P}$ with an algebraic manifold of dimension $(n-1)(n^2 - n + 1)$.

Let us sketch the proof. We shall use (16.3) instead of (16.2). Let $\mathscr{A}$ stand for the algebra generated by the matrices

$$E, A_1, \ldots, A_n.$$

L stands for the set of bA where $A \in \mathscr{A}$ and L_0 stands for the set of those $a \in L$ for which

$$\forall A \in \mathscr{A}: aAe = 0$$

and

$$a_s = \beta_s^{-1} bA_s,\ s \in X_0.$$

The central part in proving our theorem is played by

Lemma 16.4 $\mu_B P = \mu_B$ if and only if the two following conditions hold:

(1) all the scalar products

$$a_s e = 1,\ s \in X_0;$$

(2) $a_{s_1} A_{s_2} - \beta_{s_1 s_2} a_{s_2} \in L_0$ for all $s_1, s_2 \in X_0$.

Proof of the lemma. First we suppose (16.3) and prove (1) and (2). In fact, (1) is just (16.3) for $m = 1$. To prove (2) we denote

$$c_{s_1 s_2} = a_{s_1} A_{s_2} - \beta_{s_1 s_2} a_{s_2} \in \mathbb{R}^n.$$

Since

$$c_{s_1 s_2} A_{s_3} \ldots A_{s_m} e = a_{s_1} A_{s_2} \ldots A_{s_m} e - \beta_{s_1 s_2} a_{s_2} A_{s_3} \ldots A_{s_m} e \quad (16.5)$$

the assumed (16.3) implies

$$c_{s_1 s_2} A_{s_3} \ldots A_{s_m} e = 0$$

whence (2) follows.

The inverse way from (1) and (2) to (16.3) uses (16.5) and induction in m.

The proof of the lemma also shows that (16.3) being fulfilled for all $m \leqslant M$ is equivalent to

$$\forall s_1, \ldots, s_m \in X_0,\ m \leqslant M: c_{s_1 s_2} A_{s_3} \ldots A_{s_m} e = 0.$$

Now let us prove the assertion (a) of Theorem 16.3. It is sufficient to prove the following: if

$$cA_{s_1} \ldots A_{s_m} e = 0 \quad \text{for all} \quad m \leqslant n - 1$$

then

$$cAe = 0 \quad \text{for all} \quad A \in \mathscr{A}.$$

To do it, we form the sequence of subspaces

$$L_1 \subset L_2 \subset L_3 \subset \ldots$$

by the rule:

$$L_1 = \{\lambda c, \lambda \in \mathbb{R}\},$$

L_{m+1} is the linear hull of L_m and all $L_m A_s$, $s \in X_0$. Since dimensions of all L_m do not exceed n, they become one and the same at some number which does not exceed n. Hence the union of all L_m is generated by vectors

$$cA_{s_1} \dots A_{s_m}, \ m \leqslant n - 1.$$

Thus,

$$cA_{s_1} \dots A_{s_m} e = 0 \quad \text{for all} \quad m \leqslant n - 1$$

implies

$$(a,e) = 0 \quad \text{for all} \quad a \in \bigcup_m L_m$$

which proves assertion (a) of the theorem.

At the same time we have proved that all $\mathscr{P}_B$ and their union $\mathscr{P}_M$ are algebraic manifolds defined by $(n + 1)$ equalities (16.2) with $m \in \{1,\dots,n+1\}$ (we mean intersection of these manifolds with the polyhedron $\mathscr{P}$), where parameters β_{pq} are fixed for $\mathscr{P}_B$ and arbitrary for $\mathscr{P}_M$.

The most cumbersome and technically complicated part of the proof remains: calculation of dimensions. Note that $\mathscr{P}_B$ is reducible in the general case and its structure depends on the Jordan form of the B matrix. Let us first examine in detail the two extremal cases in which the range of B equals n or 1.

Proposition 16.5 Let rang $B = n$. Then all the operators P having μ_B as invariant are those given by (16.4) where K and Λ are any stochastic matrices such that

$$K\Lambda = \Lambda K = B.$$

Proof. First let us prove that a_s $(s=1,2,\dots,n)$ of Lemma 16.4 are linearly independent and that $L_0 = 0$. This follows from linear independence of a_s modulo L_0. Suppose the contrary. Then there are γ_p some of which differ from 0, for which

$$a_0 = \sum_p \gamma_p a_p \in L_0.$$

For every $s \in X_0$

$$0 = a_0 A_s e = \sum_p \gamma_p a_p A_s e = \sum_p \gamma_p \beta_{ps}.$$

Since rang $B = N$, this implies $\gamma_p \equiv 0$. This contradiction proves that $L_0 = 0$ and a_s are linearly independent.

Now let e_s stand for the row-vector which has 1 at the sth place and zeros at all the other places. Let Λ stand for such a matrix that

$$e_s\Lambda = a_s \quad \text{for all} \quad s \in X_0.$$

Of course, $(a_s,e) = 1$. which implies $c\Lambda e = (c,e)$ for all $e \in \mathbb{R}^n$.

The proved $L_0 = 0$ implies $\alpha_p A_s = \beta_{ps} a_s$. This means that every A_s projects $\mathbb{R}^n$ to the one-dimensional linear hull of a_s. Hence there are such $\varkappa_{pq}$ that

$$\alpha^s_{pq} = \varkappa_{ps}\lambda_{sq}.$$

Finally

$$a_p B = a_p \sum_s A_s = \sum_s \beta_{ps} a_s$$

which implies $e_p\Lambda B = e_p B\Lambda$ and $\Lambda B = B\Lambda$. This and non-degeneracy of B leads to

$$B = K\Lambda = \Lambda K$$

and

$$\theta^s_{pq}\beta_{pq} = \alpha^s_{pq} = \varkappa_{ps}\lambda_{sq}$$

which are the (16.4) we had to prove.

The inverse assertion has been proved with Lemma 16.2.

Proposition 16.6 Let rang $B = 1$, μ_B be invariant for P, and dim L equal 1 or n. Then b is:

either a common eigenvector of all the n matrices A'_q whose elements are $\alpha'_{ps} = \theta^s_{pq}$,
or a common eigenvector of all the n matrices A''_p whose elements are $\alpha''_{qs} = \theta^s_{pq}$,

where θ^s_{pq} are transitional probabilities of P.

Proof. First case: dim $L = 1$. Lemma 16.4 implies

$$bA_s = \beta_s b$$

that is

$$\sum_p \beta_p\beta_q\theta^s_{pq} = \beta_s\beta_q.$$

Reduction at β_q yields

$$\sum_p \beta_p\theta^s_{pq} = \beta_s.$$

This just means that all the linear transformations A'_q have the common eigenvector b.

Second case: dim $L = n$. Lemma 16.4 implies that L_0 and a_s generate all the L. Hence

$$aA^s e = \beta_s(a,e)$$

for all $a \in \mathbb{R}^n$, $s \in X_0$. This implies

$$\sum_q \beta_q \theta^s_{pq} = \beta_s$$

for $a = e_q$. This just means that all the linear transformations A''_p have the common eigenvector b.

The assumptions of Proposition 16.6 allow us to prove some more: P is ergodic and has a Bernoulli invariant measure. A more general assertion will be proved in Chapter 17.

In the case $n = 2$ the classes of operators built in Propositions 16.5 and 16.6 exhaust $\mathscr{P}_M$ since rang B equals only 2 or 1. The corresponding equalities have been found in [2]. These are:

If $\theta_{01}\theta_{10}(1 - \theta_{00} - \theta_{11}) = \theta_{00}\theta_{11}(1 - \theta_{01} - \theta_{10})$ the operator has a Markov invariant measure.

If $\theta_{10}(1 - \theta_{01}) = \theta_{11}(1 - \theta_{00})$ or $(1 - \theta_{10})\theta_{01} = \theta_{11}(1 - \theta_{00})$ the operator has a Bernoulli invariant measure.

The points

$$\theta_{00} = \theta_{11} = \varepsilon,\ \theta_{01} = \theta_{10} = 1 - \varepsilon$$

of our Example 1.3 belong to the intersection of all three manifolds.

Let us sketch a way to find the dimension of $\mathscr{P}_B$, that is, to prove the theorem's assertion (b). Remember L_0 of Lemma 16.4 and introduce $\overline{L}_0$ which is the set of such vectors

$$\sum_p \lambda_p a_p \quad \text{for which} \quad \forall q \in X_0\colon \sum_p \lambda_p \beta_{pq} = 0.$$

Of course

$$L_0 \subset \overline{L}_0 \subset R_0.$$

So operators induced by B on $L' = L/\overline{L}_0$ and on $R' = \mathbb{R}^n/R_0$ act in the isomorphic way. If the B-invariant $\overline{L}_0$ and L are fixed, the set of A_s (needed) is defined by the three following conditions:

(1) $\sum_s A_s = B$,

(2) L and L_0 are A_s-invariant for all s,

(3) there is such an invertible linear operator Λ': $R' \to L'$ which commutes with the action on R' and L' of the functionals and operators induced by the functional $a \to (a,e)$ (the sum of coordinates) and the operator $a \to aB$ on $\mathbb{R}^n$.

Then the equalities

$$a'_p A'_p = \beta_{pq} a'_q$$

hold for all p, q, where the matrix A'_p is induced by A_p on L' and $e'_p \in R'$ is the image of the row e_p, and $a'_p = e'_p\Lambda'$. Using this construction of the set of A_s, one can check that the maximal dimension among components of $\mathscr{P}_B$ equals $n(n-1)r_B + \ell_B - 1$ and is reached if $\overline{L}_0 = \mathbb{R}$, or in the case Ker B^2 = Ker $B = R_0$ if $\overline{L}_0 = 0$.

Now we shall count the dimension of $\mathscr{P}_M$, that is, prove assertion (c) of the theorem. First we count the dimension of those components of $\mathscr{P}_M$ considered in Proposition 16.6. The vector which defines a Bernoulli measure has $(n-1)$ parameters, and there are n matrices preserving this vector, each having $(n-1)^2$ parameters. The total dimension of every component in question equals

$$(n-1) + n(n-1)^2 = (n-1)(n^2-n+1). \tag{16.6}$$

This counting is correct because Theorem 17.1 proves that all operators treated in Proposition 16.6 are ergodic, whence none of them can have two different invariant measures.

Now about the other components of $\mathscr{P}_M$. All their dimensions are no more (with $n > 2$, actually less) than (16.6). In fact, remember assertion (b) and note the following. The set of $n \times n$ stochastic matrices of range $n-r$ has $n(n-1)-r^2$ parameters, and ℓ_B of these matrices equals $n-r$ in the general case. So

$$\dim \bigcup_{\text{rang } B=n-2} \mathscr{P}_B \leqslant n(n-1) - r^2 + n(n-1)r + (n-r) - 1 =$$
$$= (n-1)(n^2-n+1) - (n-r-1)(n^2-2n-r) \tag{16.7}$$

(because the set of those B which have $\ell_B > n - r_B$ has still lower dimension). You can see that (16.7) is maximal with $r = n-1$. (Note that the dimension of the set of matrices of range $n-r$ is maximal in the other extreme case $r = 1$.)

Note 16.7 This chapter has been about operators on just one graph Γ_1, the simplest one. In some sense the case of homogeneous operators on $V = \mathbb{Z}$ can be reduced to this one. Having any

$$U(0) = \{-R,\ldots,R\} \tag{16.8}$$

we can separate $\mathbb{Z}$ into segments of length $2R + 1$ and claim the states of all the segments to be states of a super-automaton. The resulting system of super-automata is an operator on Γ_1. However, the resulting operators are not general but special. So the general description of operators having Markov invariant measures, as was given by Theorem 16.3, is not suitable in this case.

Moreover, the notion of Markov measure is still more insufficient in the case (16.8) than it was with Γ_1. The notion of a k-Markov measure seems more relevant. A measure μ on $X_0^{\mathbb{Z}}$ is termed k-Markov if $x_{h+1}, \ldots, x_{h+k}$ having been fixed, the x_i, $i \leqslant h$ become independent from x_j, $j \geqslant h + k + 1$. The described reduction of the case (16.8) to Γ_1 reduces k-Markov measures to Markov ones, but the resulting Markov measures are special, and this method did not help us to describe all operators on $V = \mathbb{Z}$ having k-Markov invariant measures, as we wanted.

The class of asynchronous (continuous-time) interacting Markov processes having an invariant Markov measure, is larger than that of synchronous ones. More is known about the description of such systems; in particular, it has been proved that a homogeneous process in $\{0,1\}^{\mathbb{Z}}$ with the transition probabilities (flip-rates)

$$\lambda(y_i | x_{i+R}),\ R = [-r,r] \subset \mathbb{Z}$$

may preserve a k-Markov measure only if $k \leqslant r$. A more exact and general result will be described in Chapter 18.

We shall finish this chapter with a negative result which claims that some operators on $X = \{0,1\}^{\mathbb{Z}}$ have 'bad' invariant measures. Before formulating it we introduce a set of 'good' measures. A measure $\mu \in \mathscr{M}(X)$ belongs to $\mathscr{W}$ if it is induced by some homogeneous Markov measure on $X_0^{\mathbb{Z}}$ with X_0 finite with some space mapping $f^{\mathbb{Z}}\colon X_0^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ where $f\colon X_0 \to \{0,1\}$.

Proposition 16.8 (published in a slightly more general form in [86]). $X = \{0;1\}^{\mathbb{Z}}$, $P \in \mathscr{P}_1(X)$, that is, P is a homogeneous independent operator on X with δ_1 as invariant measure. Then any invariant measure of P different from δ_1 does not belong to $\mathscr{W}$.

This proposition can be applied to operators of Example 1.2, Theorem 8.4, Theorem 10.1 and Proposition 11.2, in particular.

Historically, the first proof of non-ergodicity of our Example 1.2 operator P_ε with small ε was based not on percolation, but on quite another idea: on constructing a non-trivial convex invariant set of measures [71]. This proof was more cumbersome than the percolation one, but it is interesting in its own way, since it provides some information about the non-trivial invariant measure of P_ε. In fact, Proposition 16.8 follows the idea of this proof.

Chapter 17
Bernoulli invariant measures

This chapter is about homogeneous operators on

$$X = \{1,\ldots,n\}^V \quad \text{where} \quad V = \mathbb{Z}^d.$$

Here we generalise Proposition 16.6 and find sufficient conditions for both ergodicity and having a Bernoulli invariant measure.

Let us have an operator P. Take two finite subsets $I, K \subset \mathbb{Z}^d$ and fix the state beyond one of them: $z \in X_{\mathbb{Z}^d \setminus I}$. The operator P induces some operator $P^{(z)}_{I,K}: X_I \to X_K$ defined as follows:

$$\mu P^{(z)}_{I,K} = (\delta_z \times \mu)P|_K$$

where $\mu \in \mathcal{M}(X_I)$ and $|_K$ means projection to X_K. In fact, the result does not depend on all components of z, but only those in $U(K)\setminus I$.

Theorem 17.1 (A. V. Vasilyev, L. G. Mityushin and I. I. Piatetski-Shapiro). We have a homogeneous operator P on $X = X_0^{\mathbb{Z}^d}$ with a translation v of $\mathbb{Z}^d$ and a Bernoulli measure $\mu = \Pi\mu_h$ on $X = \prod_{h \in \mathbb{Z}^d} X_h$.

Assume the following:

(a) For any finite $K \subset \mathbb{Z}^d$ there is $h \in K$ such that $v(h)$ does not belong to $U(K\setminus\{h\})$; (17.1a)

(b) For any $h \in \mathbb{Z}^d$ and any $z \in X_{U(h)\setminus\{v(h)\}}$,

$$\mu_{v(h)} P^{(z)}_{v(h),h} = \mu_h. \tag{17.1b}$$

Then μ is invariant under P. If P is non-degenerate, it is ergodic.

Note. The set of operators in question for this theorem is non-trivial only if $v(0) \in U(0)$. This is possible even in the simplest case of Γ_1, where we can put $\nu(h) \equiv h$ or $\nu(h) \equiv h - 1$; that gives us just those sets of operators referred to in Proposition 16.6.

Proof. The simpler part is to prove invariance of μ. We take any $K \subset \mathbb{Z}^d$, $I = \nu(K)$, $z \in X_{\mathbb{Z}^d \setminus I}$ and prove

$$\mu_I P_{I,K}^{(z)} = \mu_K \tag{17.2}$$

where μ_I and μ_K are projections of μ to I and K. We prove it by induction over the number $|K|$ of points in K. In the starting case $|K| = 1$ (17.2) coincides with (17.1b). Suppose we have proved (17.2) for all $|K| < m$. Take some K having $|K| = m$. We have $h \in K$ which fulfils (17.1a). Denote $j = \nu(h)$, $I' = I \setminus \{j\}$, $K' = K \setminus \{h\}$. We shall write states of X in the form (x,y,z) where $x \in X_{I'}$, $y \in X_j$, $z \in X_{\mathbb{Z}^d \setminus I}$. From the induction supposition,

$$\mu_{I'} P_{I',K'}^{(y,z)} = \mu_{K'}.$$

Since P is independent, and j does not belong to $U(K')$, we have

$$\begin{aligned}\delta_{(x,y)} P_{I,K}^{(z)} &= \delta_{(x,y,z)} P|_K = \\ &= \delta_{(x,y,z)} P|_{K'} \times \delta_{(x,y,z)} P|_h = \delta_x P_{I',K'}^{(z)} \times \delta_{(x,y)} P_{I,h}^{(z)}.\end{aligned}$$

Multiplying the left-hand and right-hand sides by $\mu_j(y)$, then summing over $y \in X_j$ and using (17.1a), (17.1b) leads us to the following:

$$(\delta_x \times \mu_j) P_{I,K}^{(z)} = \delta_x P_{I',K'}^{(z)} \times (\delta_x \times \mu_j) P_{I,h}^{(z)} = \delta_x P_{I',K'}^{(z)} \times \mu_h.$$

Multiplying the left-hand and right-hand sides by $\mu_{I'}(x)$, then summing over $x \in X_{I'}$ and using the induction supposition leads to (17.2).

Thus, (17.2) is proved. Hence $(\nu \times \mu_I) P|_K = \mu_K$ for any $\nu \in X_{\mathbb{Z}^d \setminus I}$ and $\mu P|_K = \mu_K$ for any K, which means $\mu P = \mu$.

Invariance of μ is proved. This part of the proof works as well for a much more general case: any graph $\Gamma(V, \mathscr{U})$, any mapping ν: $V \to V$, and any non-homogeneous (but still independent) operator P on $X = \prod_{h \in V} X_h$.

But ergodicity of P is more difficult to prove. Take first, the simplest case of Γ_1: $\mathbb{Z}^d = \mathbb{Z}$, $U(h) = \{h-1, h\}$, and $\nu(h) \equiv h$. The following proposition is crucial in this case:

Proposition 17.2 Let the assumption of Theorem 17.1 hold for a non-degenerate operator P on Γ_1 with $\nu(h) \equiv h$, $K = \{1, \ldots, m\}$, $x \in X$ and μ is invariant of P. Then $(xP^t - \mu)|_K$ tend to 0 with $t \to \infty$ uniformly in $x \in X$.

Proof. Let x^t and z^t stand for the states of set K and point 0 at time t. Of course,

$$\tilde{\mu}(x^T) = \sum_{z^{[0,T-1]}} \tilde{\mu}(z^{[0,T-1]}) \cdot \tilde{\mu}(x^T | z^{[0,T-1]}) \tag{17.3}$$

where the sum is taken over all $z^{[0,T-1]} = (z^0, \ldots, z^{T-1})$, $z^0 = x_0^0$. The following equality can be proved too:

$$\tilde{\mu}(x^T | z^{[0,T-1]}) = x^0 P_{K,K}^{(z^0)} \ldots P_{K,K}^{(z^{T-1})}(x^T). \tag{17.4}$$

This can be reworded as follows: If we modify the process by 'freezing' the states $z^0, \ldots, z^{T-1}$ and preserving the transition probability in K, all the parameters of $\tilde{\mu}$ remain as before. This is not trivial, and becomes wrong with another U, say $U(h) = \{h-1, h, h+1\}$. To prove (17.4) we have no better way than to express the relevant values in transitional probabilities.

We assumed P to be non-degenerate, whence $\overline{P}^{(z)}$ contractive with some $\varkappa < 1$:

$$\|\mu\overline{P}^{(z)} - \nu\overline{P}^{(z)}\| \leqslant \varkappa\|\mu - \nu\|,$$

whence

$$\|\overline{\nu}_x^T - \overline{\mu}\| \leqslant \varkappa^T$$

and tends to 0. The simplest case is over.

The more general case, with $V = \mathbb{Z}^d$, $v(h) \equiv h$ and all points of $U(0)$ being non-negative in some coordinate system, can be proved in the similar way.

To reduce the general case to that mentioned just now, we need the following lemma.

Lemma 17.3 Let us have a finite set $U(0) = U_0 \in \mathbb{Z}^d$ and a shift $v(h) = v + h$ with $v \in U_0$ for which (17.1a) holds. Then v is a boundary point of the convex hull of U_0.

Proof. Assume the contrary: v is inside the convex hull of U_0. Then there are such points $u_1, \ldots, u_R \in U_0$ and such naturals $M_1, \ldots, M_R$ that

$$M_1(u_1 - v) + \ldots + M_R(u_R - v) = 0.$$

We are going to find K for which (17.1a) does not hold. We take K as sums

$$m_1(u_1 - v) + \ldots + m_R(u_R - v) \tag{17.5}$$

where $m_1, \ldots, m_R$ are integers in the ranges

$$0 \leqslant m_r \leqslant M_r.$$

In fact, any $h \in K$ is presentable as (17.5) with some $m_r > 0$ (for $h \neq 0$ it is obvious; for $h = 0$ we can take all $m_r = M_r$). Hence any $h \in K$ is presentable as

$$h = u_r - v + i, \ i \in K.$$

This contradicts (17.1a). The lemma is proved.

Now reduce the general case to this one. Let us have P. Find some reversible affine transformation $f\colon \mathbb{Z}^d \to \mathbb{Z}^d$ to make all coordinates of all the points in $f(U(0))$ non-negative and $f(v) = 0$ and denote $P' = f^{-1}(P)$. Then P' is subject to the already examined case and ergodic. So is P.

Some generalisations of Theorem 17.1 are possible. Even if the mapping $v\colon \mathbb{Z}^d \to \mathbb{Z}^d$ is not a shift and μ is non-homogeneous, the homogeneity of P provides their ergodicity (whence their invariant measures have to be homogeneous).

We attempted to prove ergodicity in the case of general graphs, but met difficulties with proving (17.4). Even in the case of $v(h) \equiv h$ we need some more suppositions. Let us term a set $K \subset V$ 'passable' if for any sequence $h_1, \ldots, h_m \in V$ where all $h_{\ell+1} \in U(h_\ell)$, the conditions $h_1 \in K$ and $h_m \in K$ imply that all $h_\ell \in K$. For all passable sets, (17.4) can be proved. So we can prove ergodicity for such V where any finite K is contained in some passable finite set. But it is easy to present a graph $\Gamma = (V, \mathcal{U})$ for which (17.1a) holds with $v(h) \equiv h$, but passable sets cannot be arbitrarily large. In fact we take $V = \mathbb{Z}$, and

$$U(h) = \begin{cases} \{h-2, h-1, h\} & \text{for} \quad h \text{ even,} \\ \{h, h+2\} & \text{for} \quad h \text{ odd.} \end{cases}$$

Then any finite set which includes an even point and an odd point is not passable.

Unfortunately, Theorem 17.1 does not yield interesting examples of non-ergodic non-degenerate operators, because any two pairs (v,μ), (v',μ') satisfying all the conditions of the theorem with some non-degenerate P, have $\mu = \mu'$.

The main results of this section belong to A. V. Vasilyev [97] (the criterion of existence of a Bernoulli invariant measure) and to L. G. Mityushin and I. I. Piatetski-Shapiro (proof of ergodicity).

The following theorem gives a necessary and sufficient condition for a continuous-time homogeneous interaction Markov process to have a Bernoulli invariant measure.

We restrict ourselves to the case when $X_0 = \{-1,1\}$. Our process on $\{-1,1\}^{\mathbb{Z}^d}$ is defined by its transitional rates (flip-rates) $\lambda_h(x_h | x_{h+K})$. A point $h \in \mathbb{Z}^d$ changes its state from x_h to the other state $-x_h$ with probability

$$\Delta t \cdot \lambda_h(x_h | x_{h+K}) + o(\Delta t)$$

during Δt. Homogeneity means that λ_h are the same for all $h \in \mathbb{Z}^d$ and we may write $\lambda(x_0 | x_K)$. This value always has a unique expansion in the form of a polynomial:

$$\lambda(x_0|x_K) = \Sigma a_\xi \Pi x_i + \Sigma b_\xi x_0 \cdot \Pi x_i.$$

Both sums are taken over all $\xi \subseteq K$; both products are taken over all $i \in \xi$. Empty ξ are also include, the corresponding products being equal to 1. We assume $0 \notin K$ and denote $\bar{K} = K \cup \{0\}$. Let us separate the set of subsets of $\bar{K}$ into equivalence classes, two subsets of $\bar{K}$ being equivalent if one is a translate of the other. In every class we choose one representative, which contains the point 0, if one exists. If a class fails to have an element containing 0, it has no representative according to our rule. All the representatives chosen form a new set Δ^0. Let us have a Bernoulli measure μ, and assume

$$\mu(x_h = 1) = p > 0, \mu(x_h = -1) = q > 0.$$

Of course, $p + q = 1$. Denote

$$C_{\xi \cup 0} = (q - p)a_\xi - b_\xi,$$
$$C_\xi = (q - p)C_{\xi \cup 0}$$

for any $\xi \subseteq K$.

Theorem 17.4 [77]. The Bernoulli measure μ is invariant over the Markov process in question if and only if $\Sigma C_{\zeta + i} = 0$ for all $\zeta \in \Delta^0$. The sum is taken over all $i \in \mathbb{Z}^d$ such that $\zeta + i \subseteq \bar{K}$. All these equations are linearly independent.

Chapter 18

Gibbs measures and reversible Markov chains

In this chapter V is finite or countable, X_h may differ from each other (but each must remain finite), and $X = \prod_{h \in V} X_h$. This chapter is about operators on X having Gibbs invariant measures. We start with introducing the necessary definitions which are much the same as in [1,9,10,15,40,65,81]. We denote by $\mathscr{H}$ some family of finite subsets of V. For any $K \subseteq V$, $k \in V$

$$\mathscr{H}(K) = \{H \in \mathscr{H}: K \cap H \neq \varnothing\},$$
$$\mathscr{H}(k) = \{H \in \mathscr{H}: k \in H\}.$$

$\mathscr{H}(k)$ is assumed to be finite for any point $k \in V$, whence the union $U(k)$ of all $\mathscr{H}(k)$ is finite too; we term it the 'neighbourhood' of the point k. The neighbourhood $H \in U(K)$ of any $K \subset V$ is the union of all $H \in \mathscr{H}(K)$; it is finite for any finite K.

Definition 18.1 If for any $H \in \mathscr{H}$ we have a positive function α_H: $X_H \to \mathbb{R}_+$, we term these a potential. A measure $\mu \in \mathscr{M}(X)$ is termed Gibbs with $\{\alpha_H, H \in \mathscr{H}\}$ potential if for any $x \in X$ and $K, I \subseteq V$ with $U(K) \subseteq I$ there is such a function ψ: $X_{I \setminus K} \to \mathbb{R}$ that

$$\mu(x_I) = \psi(x_{I \setminus K}) \cdot \prod_{H \in \mathscr{H}(K)} \alpha_H(x_H). \tag{18.1a}$$

In other words, if $x_{V \setminus K} = y_{V \setminus K}$ then

$$\frac{\mu(x_I)}{\mu(y_I)} = \prod_{H \in \mathscr{H}(K)} \frac{\alpha_H(x_H)}{\alpha_H(y_H)}. \tag{18.1b}$$

Our potential is multiplicative; logarithms of α_H form the corresponding additive potential, which is more usual in physics; the difference is not important. But the finiteness of every $H \in \mathscr{H}$ is essential. Infinite-range potentials are interesting (even necessary if one wants to fit the real physical world) but they are beyond the scope of this survey. The case when some values of $\mu(.)$ or $\alpha(.)$ may equal zero is interesting but is avoided in Chapters 18–20.

Another equivalent form of (18.1a) and (18.1b) is in terms of conditional probabilities

$$\mu(x_K | x_{I \setminus K}) = \varphi(x_{I \setminus K}) \prod_{H \in \mathscr{H}(K)} \alpha_H(x_H) \tag{18.1c}$$

where $\varphi(.)$ can be expressed as a 'statistical sum'

$$\frac{1}{\varphi(x_{I \setminus K})} = \sum_{x_K \in X_K} \left(\prod_{H \in \mathscr{H}(K)} \alpha_H(x_H) \right). \tag{18.2}$$

To check (18.1a) or (18.1b) or (18.1c) (which are equivalent) one has to do it for those K that consist of one point only. It is most obvious for (18.1b). In fact, we take any $x, y \in X$ with $x_{V \setminus K} = y_{V \setminus K}$ and build a sequence

$$x^{(0)} = x, x^{(1)}, \ldots, x^{(m)} = y$$

in which adjacent members $x^{(\ell)}$ and $x^{(\ell+1)}$ differ in just one point. Multiplying the (18.1b) for these one-point cases gives (18.1b) for x and y.

If V is finite, the formulae (18.1c) and (18.2) for $K = V$ define the unique Gibbs measure.

If V is infinite, there are only explicit expressions in potential of conditional probabilities $\mu(x_K | x_{I \setminus K})$ but not of $\mu(x_K)$, and indeed for a given potential there may be more than one measure which is essential in the phase transition theory.

At least one Gibbs measure exists for any potential. Take such a sequence $K_m \subset V$, $m = 1, 2, \ldots$ that $K_m \subset K_{m+1}$ and $\cup K_m = V$. For any K_m fix any state $x^{(m)}$ beyond it and define

$$\mu^{(m)}(x_{K_m} | x^{(m)})$$

by the formulae (18.1c) and (18.2). Thus for any m we have a measure on the whole X which is $\mu^{(m)}$ in the K_m and concentrated in $x^{(m)}$ beyond K_m. Since $\mathscr{M}(X)$ is a compact, the sequence $\mu^{(m)}$ contains a convergent subsequence and its limit is a Gibbs measure with the same potential. (It has been shown [11] that the boundary points of the convex set of Gibbs measures corresponding to the given potential may be constructed in this way.)

Of course, one and the same measure can be Gibbs with various potentials and even various $\mathscr{H}$ families. This moves us to the following.

Definitions 18.2 Two potentials α_H, $H \in \mathscr{H}$ and $\alpha'_{H'}$, $H' \in \mathscr{H}$ are termed equivalent if for all $x \in X$, $h \in V$ the ratio

$$\prod_{H \in \mathscr{H}(h)} \alpha_H(x_H) \Big/ \prod_{H' \in \mathscr{H}'(h)} \alpha'_{H'}(x_{H'}) \tag{18.3a}$$

does not depend on x_h.

This condition has another form: for all $K \subset V$ and $x, y \in X$ with $x_{V\setminus K} = y_{V\setminus K}$

$$\prod_{H \in \mathscr{H}(h)} \frac{\alpha_H(x_H)}{\alpha_H(y_H)} = \prod_{H' \in \mathscr{H}'(h)} \frac{\alpha'_{H'}(x_{H'})}{\alpha'_{H'}(x_{H'})}. \tag{18.3b}$$

Of course, equivalent potentials have one and the same set of corresponding Gibbs measures.

The notion of a Gibbs measure is proved to be equivalent to the notion of a $\mathscr{U}$-Markov field, where $\mathscr{U} = \{U(h), h \in V\}$ is a family of finite sets $U(h) \subset V$, $h \in U(h)$. A measure μ on X^V is termed a $\mathscr{U}$-Markov field if for any finite $K \subset U(K) \subset L \subset V$, $\mu(x_K | x_{L\setminus K}) = \mu(x_K | x_{U(K)\setminus K})$; the last condition is equivalent (if $\mu(x_K)$ is positive for all x_K) to the following: μ is an $\mathscr{H}$-Gibbs measure, where $\mathscr{H}$ is the set of cliques of the graph $(V, \mathscr{U})$, that is, $H \in \mathscr{H}$ iff $H \subset U(h)$ for all $h \in H$.

To illustrate this definition take the set of homogeneous Markov measures on $X = \{1, \ldots, n\}^{\mathbb{Z}}$. One can check that any Markov measure μ_B defined by matrix B (see (16.1)) is Gibbs measure with

$$\mathscr{H} = \{\{h-1, h\}, h \in \mathbb{Z}\}$$

and

$$\alpha_{\{h-1,h\}}(p,q) = \beta_{pq}. \tag{18.4}$$

The following proposition shows that there are no other Gibbs measures with this V and $\mathscr{H}$ and a translation-invariant potential.

Proposition 18.3 An $n \times n$ positive matrix $B = (\beta_{pq})$ is given. Then any Gibbs measure on $\{1, \ldots, n\}^{\mathbb{Z}}$ with the potential defined by (18.4) is a homogeneous Markov one.

Proof. First let us substitute the potential (18.4) by its equivalent having a stochastic matrix B'. Due to (18.3a), this amounts to finding a diagonal matrix Δ and a number $\lambda > 0$ to make

$$B = \lambda \Delta B' \Delta^{-1}$$

with B' stochastic. We take the number λ equal to the greatest eigenvalue of B and the diagonal elements δ_{ss} of Δ equal to the components of the corresponding eigenvector-column (which means that $B\Delta = \lambda\Delta$) and put

$$B' = \lambda^{-1}\Delta^{-1}B\Delta.$$

So let us assume that the initial B was stochastic. Let us prove that the equalities (16.1) hold for μ. First let us prove that

$$\mu(x_0 = s) = \beta_s$$

where $b = (\beta_1,\ldots,\beta_n)$ is the eigenvector-line of the matrix B, that is, $bB = b$. For that let $\beta_{pq}^{(m)}$ stand for the elements of B^m and note that every $\beta_{pq}^{(m)}$ tends to β_q with $m \to \infty$. According to (18.1c)

$$\mu(x_0{=}s\,|\,x_{-m}{=}p,x_m{=}q) =$$

$$= \frac{\sum\limits_{(s_i),i\neq 0} \beta_{ps_{-m+1}}\beta_{s_{-m+1}s_{-m+2}}\ldots\beta_{s_{-1}s}\beta_{ss_1}\ldots\beta_{s_{m-2}s_{m-1}}\beta_{s_{m-1}q}}{\sum\limits_{(s_i)}\beta_{ps_{-m+1}}\beta_{s_{-m+1}s_{-m+2}}\ldots\beta_{s_{m-2}s_{m-1}}\beta_{s_{m-1}q}} =$$

$$= \frac{\beta_{ps}^{(m)}\beta_{sq}^{(m)}}{\beta_{pq}^{(2m)}}.$$

With m large enough the values of

$$\mu(x_0{=}s\,|\,x_{-m}{=}p,x_m{=}q)$$

approximate β_s, whence

$$\mu(x_0 = s) = \sum_{p,q}\mu(x_0{=}s\,|\,x_{-m}{=}p,x_m{=}q)\mu(x_{-m}{=}p,x_m{=}q)$$

must equal β_s.

In the same way (16.1) can be proved for any m.

Proposition 18.3 is now proved.

In the analogous way one can prove that any Gibbs measure on $X = X_0^{\mathbb{Z}}$ with a translation-invariant finite potential $\alpha_{(i,i+1,\ldots,i+k)}$ is a homogeneous k-Markov measure. Hence different measures cannot conform to one potential, which corresponds to the physical conviction that phase transitions are impossible in one-dimensional systems.

With all $d \geqslant 2$ the phase transitions are possible. The mathematical counterpart of this physical fact is the non-uniqueness of measure for a given potential, which takes place with all $d \geqslant 2$.

The notion of Gibbs measure has developed from investigation of equilibrium systems like the classical Ising model, where time is absent from the model. Our survey's theme is evolution systems where time is essential. But introducing the evolution spaces and measures makes time just another coordinate in our systems. This makes it possible to interpret our evolution measures as Gibbs measures:

Note 18.4 P is an independent operator on a graph $\Gamma(V, \mathscr{U})$, having transitional probabilities $\theta_h(y|z)$ where $y \in X_h$, $z \in X_{U(h)}$. Let $\tilde{\mu}$ be the evolution space $X^{\mathbb{Z}} = \{(x^t), t \in \mathbb{Z}\}$, its projection on any X^t being a certain invariant measure μ. Then $\tilde{\mu}$ is a Gibbs measure with $\mathscr{H}$ consisting of the sets

$$\tilde{U}(h,t+1) = \{(h,t+1);(k,t),k \in U(h)\},\ h \in V,\ t \in \mathbb{Z} \qquad (18.5)$$

and potential of the form

$$\alpha_{\tilde{U}(h,t+1)}(x_h^{t+1}, x_{U(h)}^t) = \theta_h(x_h^{t+1}|x_{U(h)}^t). \qquad (18.6)$$

Thus the problem of finding all the invariant measures of a given P is a special case of the problem of describing all the Gibbs measures on $X^{\mathbb{Z}}$.

The main use of Gibbs measures in our work is to define such a measure $\tilde{\mu}$ on $X^{\{t,t+1\}}$, the projections of which to the layers X^t and X^{t+1} are one and the same invariant measure μ. We shall use this construction in some examples below. To do it in a more general way, we need the following.

Definition 18.5 We shall say that a bipartite graph $\hat{\Gamma} = \Gamma(\hat{V}, \mathscr{H})$ is given if its set of vertices $\hat{V}$ consists of two disjoint sets V and V' and a set of pairs $\mathscr{H} = \{(j,k)\}$ is given where $j \in V$, $k \in V'$ (see Fig. 18.1).

Denote for any $j \in V$, $k \in V'$:

$$U(j) = \{k\colon (j,k) \in \mathscr{H}\},\ U(k) = \{j\colon (j,k) \in \mathscr{H}\}$$

and assume that all $U(.)$ are finite. Denote also

$$\bigcup_{k \in S} U(k) = U(S).$$

Proposition 18.6 We have a bipartite graph $\Gamma(V \cup V', \mathscr{H})$ and two spaces

$$X = \prod_{j \in V} X_j,\ X' = \prod_{k \in V'} X'_k.$$

We have a Gibbs measure $\tilde{\mu}$ on their product $\hat{X} = X \times X'$ with some potential $\{\alpha_{jk}, (j,k) \in \mathscr{H}\}$.

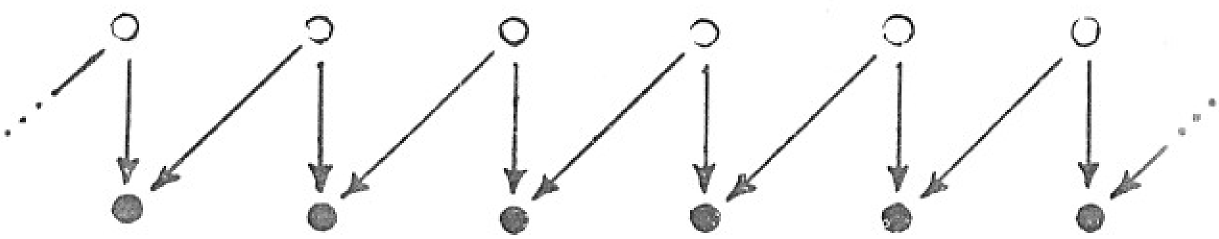

Fig. 18.1 Bipartite graph $\hat{\Gamma}_1$ corresponding to the simplest graph Γ_1 (Fig. 2.1).

Then the projection of $\tilde{\mu}$ to X is a Gibbs measure with the potential defined on the $U(k)$ sets:

$$\sigma_{U(k)}(x_{U(k)}) = \sum_{y_k \in X'_k} \Big(\prod_{j \in U(k)} \alpha_{jk}(x_j, y_k) \Big). \tag{18.7}$$

For any $I \subset V$, $k \in V'$ where $U(k) \subset I$ the conditional probabilities of the initial measure $\tilde{\mu}$ can be expressed as

$$\hat{\mu}(y_k | x_I) = \prod_{k \in K} \frac{\prod_{j \in U(k)} \alpha_{jk}(x_j, y_k)}{\sigma_{U(k)}(x_{U(k)})}. \tag{18.8}$$

Proof. Evidently

$$\mu(x_I) = \hat{\mu}(x_I) = \sum_{y_k \in X'_k} \hat{\mu}(x_I, y_k).$$

From the definition of a Gibbs measure (18.1a)

$$\hat{\mu}(x_I, y_k) = \psi(x_{J \setminus I}) \prod_{\substack{k \in S \\ j \in U(S)}} \alpha_{jk}(x_j, y_k).$$

For any $S \subset V'$ it is easy to check that

$$\sum_{y_S \in X'_S} \Big(\prod_{j \in U(k)} \alpha_{jk}(x_j, y_k) \Big) = \prod_{k \in S} \Big(\sum_{y_k \in X'_k} \Big(\prod_{j \in U(k)} \alpha_{jk}(x_j, y_k) \Big) \Big), \tag{18.9}$$

whence

$$\mu(x_J) = \psi'(x_{J \setminus I}) \prod_{k \in U(I)} \sigma_{U(k)}(x_{U(k)})$$

where $\sigma_{U(k)}$ are defined by (18.7). The last two equalities and (18.1c) prove (18.8).

Note that some Gibbs measures are not invariant under independent operators. It is proved in [7] that there is no preserving operator for any homogeneous measure on $X = \{0,1\}^{\mathbb{Z}^2}$ with the potential

$$\alpha_{jk} = \alpha_{jk}(x_j, y_k) = \alpha(x_j, y_k),\ |j - k| = 1,$$

provided this is not a Bernoulli measure.

Example 18.7 This is a non-ergodic non-degenerate operator on $\{-1,1\}^{\mathbb{Z}^2}$ which is very like Example 1.4b.

First assume that elements of $\mathscr{H}$ are pairs of adjacent points

$$\mathscr{H} = \{(j,k)\colon j,\ k \in \mathbb{Z}^2,\ |j - k| = 1\}.$$

The potential is

$$\alpha_{jk}(x_j, y_k) = \begin{cases} e^{\beta} & \text{if} \quad x_j = y_k, \\ e^{-\beta} & \text{if} \quad x_j \neq y_k. \end{cases} \tag{18.10}$$

In fact this is the classical Ising model. It is known that β being large enough, this model has two different Gibbs measures $\hat{\mu}_+$ and $\hat{\mu}_-$ where

$$\hat{\mu}_-(x_h = 1) < \tfrac{1}{2} < \hat{\mu}_+(x_h = 1).$$

More exactly, according to the Onsager–Young formulae

$$\hat{\mu}_+(x_h = -1) = \hat{\mu}_-(x_h = +1) = (1 + \varepsilon)^{1/4}(1 - \varepsilon)^{-1/2}(1 - 6\varepsilon + \varepsilon^2)^{1/8}$$

with $\varepsilon = e^{-4\beta} < 3 - 2\sqrt{2}$. Both measures are translation invariant and are limits of measures on finite pieces of the space with opposite boundary conditions: 'all minus ones' for $\hat{\mu}_-$, 'all ones' for $\hat{\mu}_+$.

We shall use this well-known construction to build a non-degenerate operator having two different invariant measures. To this end, we present our $\mathbf{Z}^2$ as a bipartite graph like a chessboard (as in Fig. 18.2):

$$V = \{(h_1,h_2), \text{ where } h_1 + h_2 \text{ is even}\},$$
$$V' = \{(h_1,h_2), \text{ where } h_1 + h_2 \text{ is odd}\}.$$

Now to define the operator P on $X = \{-1,1\}^V$.

If x_j, $j = (j_1,j_2) \in V$ is the state of a point at time t, the same point's state at the next time $t + 1$ is y_k, where $k = (j_1+1,j_2) \in V'$. It remains to define $\tilde{\mu}$ on $X^{\{t,t+1\}}$; for that we use (18.10). From (18.8), this gives us an operator which has transitional probabilities

$$\theta_k(y_k|x_{U(k)}) = \frac{\exp[\beta y_k \xi(x_{U(k)})]}{\exp[\beta\xi(x_{U(k)})] + \exp[-\beta\xi(x_{U(k)})]} \qquad (18.11)$$

where

$$\xi(x_{U(k)}) = \sum_{j \in U(k)} x_j.$$

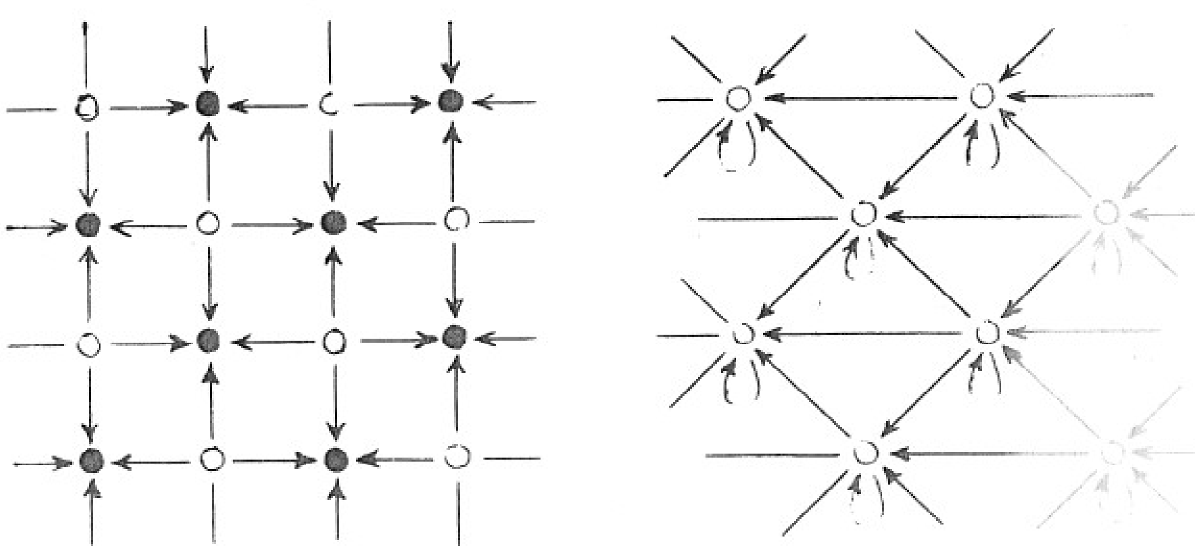

Fig. 18.2 Dependences between automata in Example 18.7.

Projections of $\hat{\mu}$ on X_V and $X_{V'}$ are translation invariant and congruent, whence each of them is an invariant measure of our operator. Since $\hat{\mu}$ was not unique (it might be $\hat{\mu}_+$ or $\hat{\mu}_-$), our operator has two invariant measures μ_+ and μ_-. The subsequent Theorem 18.13 proves that

$$\mu_- = \lim_{t\to\infty} \delta_{-1}P^t \quad \text{and} \quad \mu_+ = \lim_{t\to\infty} \delta_1 P^t.$$

From Proposition 18.6, both μ_+ and μ_- are Gibbs measures with one and the same potential

$$\sigma_{U(k)}(x_{U(k)}) = \exp[\beta\xi(x_{U(k)})] + \exp[-\beta\xi(x_{U(k)})]. \tag{18.12}$$

In the analogous way one can build a non-degenerate operator on $\{1,\ldots,n\}^{\mathbb{Z}^2}$ which has n linearly independent invariant measures which tend to $\delta_1, \ldots, \delta_n$ with $\beta \to \infty$.

Another example is also based on the construction of Proposition 18.6. It uses a Markov measure on $X_0^{\mathbb{Z}}$ which is invariant under the even translation $h \to h + 2$ to build the same class of operators having Markov invariant measures, as in Lemma 16.2 and Proposition 16.6.

Example 18.8 The bipartite graph is $\hat{V} = \mathbb{Z}$, where V consists of even numbers, V' consists of odd numbers. Take two $n \times n$ positive matrices $K = (\varkappa_{pq})$, $\Lambda = (\lambda_{pq})$ where $p, q \in X_0 = \{1,\ldots,n\}$ and $K\Lambda = \Lambda K = B$ where B is stochastic. We define $\hat{\mu}$ as the Gibbs measure with

$$\alpha_{(2m,2m+1)}(p,s) = \varkappa_{ps},$$
$$\alpha_{(2m-1,2m)}(s,q) = \lambda_{sq}.$$

Then, due to Proposition 18.6, the projections μ and μ' of $\hat{\mu}$ to X_V and $X_{V'}$ are Gibbs measures with the potential

$$\sigma_{(i,i+2)}(p,q) = \beta_{pq} = \sum_s \varkappa_{ps}\lambda_{sq}.$$

The conditional probabilities of the initial $\hat{\mu}$ are

$$\hat{\mu}(y_{\{1,3,\ldots,2m-1\}} | x_{\{0,2,\ldots,2m\}}) = \prod_{0\le i\le m-1} \theta(y_{2i+1} | x_{2i}, x_{2i+2})$$

where $\theta(s|p,q) = \theta_{pq}^s$ are defined in (16.4). Thus, the Markov measure μ_B is invariant under the operator on Γ_1 which has θ_{pq}^s transitional probabilities. As we have said in Chapter 16, this example shows that $\mathscr{P}_B$ is non-empty for any B; in the case rang $B = n$, this example exhausts $\mathscr{P}_B$.

In Examples 18.7 and 18.8 both transitions from t to $t + 1$ and from $t + 1$ to t were presentable as actions of some independent operator. The following Theorem 18.10 describes all the operators which make the backward step possible.

But first we need a definition.

Definition 18.9 A Markov chain with the initial measure μ and operator P is termed reversible if its combined measure at two adjacent times $\hat{\mu} \in \mathcal{M}(X \times X)$ is symmetric with respect to the permutation of arguments $(x,x') \to (x',x)$ in $X \times X$.

In other words the probability of being in some set A at time t and being in B at time $t + 1$ equals that of being in B at t and being in A at $t + 1$. Of course, reversibility implies invariance of μ. We also call reversible the evolution measure of reversible Markov chain.

Theorem 18.10 [49,75]. An independent operator P on $\Gamma(V,\mathcal{U})$, $X = \prod_{h \in V} X_h$ has a reversible evolution measure if and only if its parameters $\theta_h(y_h|z)$ can be expressed as

$$\theta_k(y_k|z) = \frac{\beta_k(y_k) \prod_{j \in U(k)} \alpha_{jk}(z_j,y_k)}{\sigma_{U(k)}(z)} \tag{18.13}$$

where

$$\alpha_{j,k}\colon X_j \times X_k \to \mathbb{R}_+,\ \beta_h\colon X_h \to \mathbb{R}_+,$$

$j,\ k,\ h \in V,\ j \in U(k)$

are some functions, $z \in X_{U(k)}$, $y_k \in X_k$ and

$$\sigma_{U(k)}(x_{U(k)}) = \sum_{y_k \in X_k} \beta_k(y_k) \prod_{j \in U(k)} \alpha_{jk}(x_j,y_k) \tag{18.14}$$

with the symmetry condition

$$\alpha_{jk}(x_j,y_k) = \alpha_{kj}(y_k,x_j). \tag{18.15}$$

The corresponding invariant measure (projection of evolution measure on any X^t) is a Gibbs one with the potential (18.14).

We shall not prove Theorem 18.10 here. Note only that in the assumptions of this theorem the graph of P may be treated as symmetric, that is,

$$j \in U(k) \Leftrightarrow k \in U(j).$$

Both Examples 18.7 and 18.8 are subject to the following theorem, similar to Theorem 18.10.

Theorem 18.11 [99]. We have a bipartite graph

$$\Gamma(V \cup V',\mathcal{H})$$

and spaces

$$X = \prod_{j \in V} X_j \quad \text{and} \quad X' = \prod_{k \in V'} X'_k.$$

A measure $\hat{\mu}$ on $X \times X'$ is $\mathcal{H}$-Gibbs if and only if

$$\left.\begin{aligned}\hat{\mu}(y_K|x) &= \prod_{k\in K}\hat{\mu}(y_k|x_{U(k)}),\\ \hat{\mu}(x_J|y) &= \prod_{j\in J}\hat{\mu}(x_j|y_{U(j)}),\end{aligned}\right\} \tag{18.6}$$

for any $x \in X$, $y \in X'$, $J \subset V$, $K \subset V'$.

This theorem can be interpreted as follows. Let us have such a measure $\hat{\mu}$ on $X \times X'$ that both transitions from its projection μ on X to its projection μ' on X' and back are presentable as performed by independent operators: $\mu P = \mu'$ and $\mu' P' = \mu$. Then $\hat{\mu}$ is a Gibbs measure with some potential $\{\alpha_{jk}\}$ where $j \in U(k) \subset V$, $k \in V'$.

Porposition 18.6 was about such measures. It implied that P had transitional probabilities (18.13) and μ was a Gibbs measure with the potential (18.14), but the symmetry condition (18.15) was not necessary.

Example 18.12 Let

$$V = \mathbb{Z},\ U(h) = \{h-1,h,h+1\},\ X_h = \{0,1\}.$$

According to Theorem 18.10 the family of reversible operators on this graph has three parameters, say α_1, α_2, $\beta > 0$ and

$$\theta^y_{z_1 z_2 z_3} = \frac{\beta^y \alpha_1^{y(z_1+z_3)}\alpha_2^{yz_2}}{1 + \beta\alpha_1^{z_1+z_3}\alpha_2^{z_2}}.$$

Their invariant measures are 2-Markov. In the case

$$\alpha_1 = \alpha_2 = \frac{1}{\varepsilon},\ \beta = 1,$$

this example turns into Example 1.4a.

Finally we shall formulate two theorems about ergodicity or the set of invariant measures of operators having reversible evolution measures. They are proved in [49] and have continuous-time analogues in [37,38].

Theorem 18.13 Under the assumptions of Theorem 18.10 with $X_h = \{-1,1\}$ and P monotone

$$\lim_{t\to\infty}\delta_{-1}P^t \quad \text{and} \quad \lim_{t\to\infty}\delta_1 P^t \tag{18.17}$$

are the minimal and the maximal among the Gibbs measures with potential (18.14).

Due to this theorem the equality of these limits implies the ergodicity of P.

Theorem 18.14 Under the assumptions of Theorem 18.10′ with $V = \mathbb{Z}^d$ and with homogeneity of the space, graph, potential and measure the following holds. For any homogeneous ν the sequence νP^t weakly converges to the set of homogeneous Gibbs measures with potential (18.14).

Due to this theorem, the uniqueness of Gibbs measure implies ergodicity of P in the space of homogeneous measures.

Of course, operators treated in this chapter (reversible or having a Gibbs invariant measure with a finite potential) are rare among all independent operators. But the reversible case has a special interest because it seems to allow more complete investigation (as in Examples 18.7 and 18.8) and can be used as a start for further advances, as in the continuous-time case [81,82].

As stated at the end of Chapter 16, the class of asynchronous (continuous-time) interacting Markov processes having an invariant Markov (or Gibbs) measure, is larger than that of the synchronous ones. This class was used as a tool of investigation of equilibrium states of physical models, such as the Ising system [13,74,81]. For asynchronous systems, the following algebraic theorem has been proved by Olga Stavskaya.

Theorem 18.15 (published in a more general form in [76]). Let a $\mathscr{U}$-Markov field μ on $\mathbb{Z}^d$ be invariant under a local interacting homogeneous process on $X_0^{\mathbb{Z}^d}$ with $\mathscr{U} = \{U(h), h \in \mathbb{Z}^d\}$ and transition rates $\lambda\{x_h | x_{h+K})$, $h \in \mathbb{Z}^d$ where $K = R \cap \mathbb{Z}^d$ and R is a convex set which is symmetric with the centre 0. Then μ is homogeneous too, and $U(h) = U_0 + h$, where $U_0 \subseteq K$.

Chapter 19
Markov systems with refusals

This chapter is about systems where 'refusals' are possible. To make this notion clearer, take a finite Markov chain with set of states X_0, transitional probabilities $p(x|y)$ where $x, y \in X_0$, and an initial measure μ on X_0. Let $X_0 = Y \cup W$ where $Y \cap W = \varnothing$. We interpret Y as the set of working states, W as the set of refusal states. Reading any of the states of W means that the system's behaviour terminates.

If the chain is known to have been in the working states at N first times $t = 1, \ldots, N$, the probability of having been at certain states $x^1, \ldots, x^N \in Y$ at these times is

$$P(x^t,t{=}1,\ldots,N) = \mu(x^1)p(x^2|x^1) \ldots p(x^N|x^{N-1})/Z_N \tag{19.1}$$

where Z_N is the probability of having been in Y during $[1,N]$.

Now to consider interacting Markov systems. In every point of the integer axis $\mathbb{Z}$ there is an automaton with the finite set X_0 of states. As before, $X_0 = Y \cup W$, where $Y \cap W = \varnothing$, with the same meaning. Every automaton's state x_h^{t+1} at time $t + 1$ depends on its state x_h^t and its left neighbour x_{h-1}^t at time t (which is a special case, to begin with). All the states of the automata at time t are fixed, and their states at the next time $t + 1$ are computed independently (however, this is not necessary – see [15]). Thus, we have an operator at our standard graph Γ_1 defined by transitional probabilities $\theta(x_h^{t+1}|x_{h-1}^t,x_h^t)$, which are the same for all h and t, making the system homogeneous in space and time. Now we are interested in the behaviour of a finite segment of the system with $h \in [-L+1,L-1]$ during a segment of time $t \in [-T+1,T]$. If the initial states

$$(x_h^{-T}, -L<h<L)$$

and boundary conditions

$$(x^t_{-L}, -T \leqslant t < T)$$

are given, the sequence of the segment's states

$$(x^t_h, -L<h<L),\ t = -T+1, \ldots, T$$

is a finite Markov chain. The standard approach in the theory of locally interacting systems would be to go to limit L which would turn the finite Markov chain into an operator on the infinite space $X_0^{\mathbb{Z}}$. But in the present case of possible refusals this is preceded by the assumption that during the time $t \in [-T+1,T]$ there was no refusal in our segment $[-L+1,L-1]$. After that we go to the limits $T \to \infty$, $L \to \infty$. Thus we come to the following definitions.

Definition 19.1 $P_{L,T}$ stands for the probability distribution of the states

$$(x^t_h, -L<h<L, -T<t \leqslant T)$$

with the initial conditions

$$(x_h^{-T}, -L<h<L)$$

and boundary conditions

$$(x^t_{-L}, -T \leqslant t \leqslant T),\ (x^t_L, -T \leqslant t \leqslant T)$$

given by the formula

$$P_{L,T}\{x^t_h, -L<h<L, -T<t \leqslant T\} = \prod_{t=-T}^{T-1} \prod_{h=-L+1}^{L} \theta(x_h^{t+1} | x^t_{h-1}, x^t_h)/Z_{L,t} \tag{19.2}$$

where $Z_{L,T}$ is the normalising factor.

It is essential that all $x^t_h \in Y$ in (19.2) (as in (19.1)).

Definition 19.2 The probability distribution P_L of the states

$$(x^t_h, -L<h<L, t \in \mathbb{Z})$$

with the boundary conditions

$$(x^t_{-L}, t \in \mathbb{Z}),\ (x^t_L, t \in \mathbb{Z})$$

stands for any limit point (in the sense of covergence on cylinder sets) of the distribution $P_{L,T}$ with $T \to \infty$.

Note 19.3 Except for degenerate cases when some transitional probabilities equal 0, the distribution P_L is unique because the limit of $P_{L,T}$ as $T \to \infty$ exists and does not depend on the initial state. (We shall make this clearer below.)

Definition 19.4 The limit probability distribution P of $(x_h^t, h \in \mathbb{Z}, t \in \mathbb{Z})$ is any limit point of the sequence P_L with $L \to \infty$ and any boundary conditions.

Note 19.5 The distribution $P_{L,T}$ can be written as the Gibbs distribution

$$P_{L,T}(x) = \exp\left(-\sum_{t=-T}^{T-1} \sum_{h=-L+1}^{L} U_{h,t}(x)\right)\Big/ Z_{L,T} \tag{19.3}$$

with the potential

$$U_{h,t}(x) = -\ln \theta(x_h^{t+1} | x_{h-1}^t, x_h^t), \tag{19.4}$$

where

$$x = (x_h^t \in Y, -L<h<L, -T<t\leqslant T)..$$

Hence P_L is a limit Gibbs distribution (as in [73]) in the band $-L < h < L$ with the boundary conditions

$$(x_{-L}^t, t \in \mathbb{Z}),\ (x_L^t, t \in \mathbb{Z}).$$

The band being essentially one-dimension, for any boundary conditions there is just one limit Gibbs distribution except for degenerate cases.

The distribution P as our definition 19.4 puts it, is a limit Gibbs distribution on the integer plane $\mathbb{Z}^2$ with the potential (19.3). It is non-unique in the general case, and we shall present specific automata systems which have P depending on the boundary conditions.

Example 19.6 The symmetric Stavskaya system with refusals. To the former states 0 and 1 we add the new refusal state D: $X_0 = Y \cup W$, where $Y = \{0,1\}$, $W = \{D\}$.

The transitional probabilities are given in Table 19.1, where $0 \leqslant \varepsilon$, δ,

Table 19.1

x_h^{t+1}	x_{h-1}^t	x_h^t	θ
1	1	1	$1-\varepsilon$
0	0	0	$1-\varepsilon$
0	1	1	ε
1	0	0	ε
D	1	0	$1-\delta$
D	0	1	$1-\delta$
0	1	0	$p\delta$
1	0	1	$p\delta$
1	1	0	$q\delta$
0	0	1	$q\delta$

$p, q \leqslant 1$ and $p + q = 1$. If at least one of the x^t_{h-1} or one of the x^t_h equals D, the value of θ does not matter, as we reject the entire system.

Theorem 19.7 If ε and δ are small enough in this system, the boundary conditions $x^t_{-L} = x^t_L \equiv 0$ and $x^t_{-L} = x^t_L \equiv 1$ lead to limit distributions P_0 and P_1 which are different.

Proof. Remember that the potential U equals

$$U(x^{t+1}_h, x^t_{h-1}, x^t_h) = -\ln \theta(x^{t+1}_h | x^t_{h-1}, x^t_h). \tag{19.5}$$

You can check that ε and δ being small, U has the following three properties:

(1) There are just two periodic ground (that is, having the minimal specific energy) states, namely $x^t_h \equiv 0$ and $x^t_h \equiv 1$.
(2) U is symmetric with changing all automata states from 0 to 1 and from 1 to 0.
(3) U satisfies the Peierls condition [73] with some Peierls constant, large enough.

From these three properties one can prove in the well-known way (as in [73] in particular) that P_0 and P_1 are different.

Example 19.8 The non-symmetric Stavskaya system with refusals. As in the previous example $X_0 = Y \cup W$, where $Y = \{0,1\}$, $W = \{D\}$, but the transitional probabilities are non-symmetric as in Table 19.2, where $0 \leqslant \varepsilon_0, \varepsilon_1, \delta_0, \delta_1, p_0, p_1, q_0, q_1 \leqslant 1$ and $p_0 + q_0 = p_1 + q_1 = 1$.

Theorem 19.9 If δ_0, δ_1 and ε_0 are small enough in this system, there is ε_1 such that the two boundary conditions $x^t_{-L} = x^t_L \equiv 0$ and $x^t_{-L} = x^t_L \equiv 1$ lead to limit distributions P_0 and P_1 which are different.

Table 19.2

x^{t+1}_h	x^t_{h-1}	x^t_h	U
1	1	1	$1 - \varepsilon_1$
0	0	0	$1 - \varepsilon_0$
0	1	1	ε_1
1	0	0	ε_0
0	1	0	$p_0\delta_0$
1	0	1	$p_1\delta_1$
1	1	0	$q_0\delta_0$
0	0	1	$q_1\delta_1$

Suppose that δ_0, δ_1 and $\varepsilon_0 = \varepsilon_1$ are small enough. This makes U possess two properties like those in the symmetric example: U has two ground states $x_h^t \equiv 0$ and $x_h^t \equiv 1$ and satisfies the Peierls condition with some Peierls constant ρ, large enough.

Now let us fix δ_0, δ_1, ε_0, q_0 and q_1, and see how ε_1 influences U. Introduce

$$b = \ln(1 - \varepsilon_0) - \ln(1 - \varepsilon_1),$$

the parameter analogous to the external field in the Ising model. From [63] (see equation (7) there) we know that

$$b = s(F_1) - s(F_0) \tag{19.5}$$

is that singular value of b which results in two different limit distributions. But, in our case, unlike [63], the contour functionals F_1 and F_0, which can be found from some equations which are not written here but are analogous to (6) in [63], depend on b because one value of U depends on b:

$$U(0,1,1) = -\ln \varepsilon_1.$$

This dependence may not be assumed weak since the derivative is large:

$$\frac{\partial U}{\partial b} = -\frac{1 - \varepsilon_1}{\varepsilon_1} \to \infty \quad \text{with} \quad \varepsilon_1 \to 0$$

(unlike [60,62] where the field b entered the potential in the linear way).

Thus we need the following estimations which are more elaborate than in [62,63]. The functionals $\Phi(\Gamma)$ defined as in [62,63] depend on b and the derivative has the order of $1/\varepsilon_1$ for any contour Γ which contains some site of the type (0,1,1) and is bounded if Γ has no such site. Also we have the Peierls condition

$$\Phi(\Gamma) \geqslant \rho|\Gamma|$$

for any Γ and the other condition

$$\Phi(\Gamma) \geqslant \rho|\Gamma| + |\ln \varepsilon_1|$$

for Γ containing the site (0,1,1).

Introduce the space $B(\rho,\varepsilon_1)$ of pairs $F_0(\Gamma)$, $F_1(\Gamma)$ where both functionals $F_0(\Gamma)$ and $F_1(\Gamma)$ depend on b and possess the properties mentioned above. The equations (6) of [63] have the form

$$\hat{F} = \hat{\Phi} + T(\hat{F}) \tag{19.6}$$

where

$$\hat{F} = (F_0, F_1),\ \hat{\Phi} = (\Phi_0, \Phi_1),$$

and T is a nonlinear operator which can be expressed in the logarithms of contour statistical sums [62,63] and is contractive in the sense that its partial derivative $\partial T(\hat{F})/\partial F(\Gamma)$ has an estimator of the form $\exp(-F(\Gamma))$ (see [51,62,63]). If $\hat{F} \in B(\rho,\varepsilon_1)$, the value of

$$\frac{dT(\hat{F})}{db} = \sum_{\Gamma} \frac{\partial T(\hat{F})}{\partial F(\Gamma)} \cdot \frac{dF(\Gamma)}{db}$$

is bounded with a small constant (provided that ρ is large enough, as it is). Hence the solution of (19.6), which can be found by the same method of iterations as in [62,63], certainly belongs to $B(\rho,\varepsilon_1)$. This makes

$$\frac{dS(F)}{db} = \sum_{\Gamma} \frac{\partial S(F)}{\partial F(\Gamma)} \cdot \frac{dF(\Gamma)}{db} -$$

small too, whence the equation (19.5) has a (unique) solution

$$b = b(\delta_0,\delta_1,\varepsilon_0).$$

The original Stavskaya system without refusals (Example 1.2) had the transitional probabilities

$$\theta(1\,|\,x^t_{h-1},x^t_h) = \begin{cases} 1 & \text{if } x^t_{h-1} = x^t_h = 1, \\ \theta & \text{otherwise.} \end{cases}$$

Remember that it had a phase transition in θ ($\theta^* \approx 0.3$). The following is its version with refusals which has not only phase transition in θ but another phase transition in p (at $p = 0$), probability of refusal. We assume a refusal possible only if

$$x^t_{h-1} = x^t_h = 0.$$

So, $X_0 = Y \cup W$, $Y = \{0,1\}$, $W = \{D\}$ now and the new transitional probabilities are as in Table 19.3, the other being the same as without refusals. Denote

$$b = \ln(1 - p - \theta)$$

and consider the family of potentials with various b. In particular,

$$b = \ln(1 - \theta)$$

gives the original Stavskaya system and $b = 0$ gives a potential having two ground states

Table 19.3

x^{t+1}_h	x^t_{h-1}	x^t_h	$\theta(.)$
0	0	0	$1 - p - \theta$
1	0	0	θ
D	0	0	p

$$x_h^t = 0 \quad \text{and} \quad x_h^t = 1$$

which satisfy the Peierls condition with a large Peierls constant for θ small.

Theorem 19.10 For θ small enough and $p > 0$ the limit distribution of this system is unique and concentrated in the state 'all ones'.

Proof. We need definitions as in [62]. Term a triplet

$$((h,t+1),(h-1,t),(h,t))$$

of points in the evolution space *even* if its state is (0,0,0) or (1,1,1) and *odd* in the other cases. The *oddness* of a configuration $x = (x_h^t)$ is the union of its odd triplets. The *contours* are the finite connected components of the oddness. The Ising model [73] had two kinds of contours: contours with 'all zeros' boundary conditions and contours with 'all ones' boundary conditions. But the present model has only one kind of contours, namely those with boundary conditions 'all zeros', because any configuration having 'all ones' at its contour contains a place where the potential is infinite.

We are going to write down some statistical sums. As in [62,63], the 'background' configuration 'all zeros' will be assumed to have zero energy, serving the point of departure to calculate the other configurations' energies. Thus a configuration having only one contour Γ^0 (where the mark 0 signifies the boundary conditions) has the relative energy

$$H(\Gamma^0) = \Phi(\Gamma^0) + bV(\Gamma^0) \tag{19.7}$$

where $V(\Gamma^0)$ is the number of triplets having other states than (0,0,0) in this configuration, and

$$\Phi(\Gamma^0) = N_1|\ln\theta| + N_2|\ln(1-\theta)|, \tag{19.8}$$

where N_1 is the total number of triplets having states (0,1,1), (1,0,1), (0,0,1) and N_2 is the total number of triplets having states (0,1,0), (1,0,0) in this configuration.

The following estimation is evident with $\theta < \frac{1}{2}$:

$$|\ln\theta||\Gamma^0| > \Phi(\Gamma^0) > \tfrac{1}{3}|\ln\theta||\Gamma^0|, \tag{19.9}$$

where $|\Gamma^0| = N_1 + N_2$.

As in [62] a τ-functional stands for such a functional Φ that

$$\Phi(\Gamma^0) > \tau|\Gamma^0|.$$

Denote also

$$\|\Phi\| = \sup_{\Gamma} \frac{|\Phi(\Gamma)|}{|\Gamma|}$$

and write $\|\Phi\| < \infty$ to mean that $\|\Phi\|$ is bounded.

Since the contours Γ^1 are impossible in our system, any contour Γ^0 certainly is external, which means that it is filled with ones (see [62]). Hence the definition 3.5 in [62] boils down to the formula:

$$\Xi(\Gamma^0 | H) = \exp(-H(\Gamma^0)). \tag{19.10}$$

This allows us to formulate the following lemma.

Lemma 19.11 For θ small enough and any $b < \ln(1-\theta)$ there are such $a > 0$ and such a τ-functional F_0 with $\|F_0\| < \infty$ that

$$\Xi(\Gamma^0 | H) = \exp(aV(\Gamma^0))\Xi(\Gamma^0 | F_0). \tag{19.11}$$

In other words, this lemma claims that the statistical sums $\Xi(\Gamma^0 | H)$ can be described by a contour model with a parameter [60,62].

Proof. We are going to prove that the equation (19.11) has a (unique) solution. For that we substitute the expansion of the logarithm of the contour statistical sum [62]:

$$\ln \Xi(\Gamma^0 | F_0) = S(F_0)V(\Gamma^0) - F_0(\Gamma^0) + \nabla(\Gamma^0 | F_0) \tag{19.12}$$

and the definition of $\Xi(\Gamma^0 | H)$ into (19.11). This gives us:

$$-\Phi(\Gamma^0) - bV(\Gamma^0) = aV(\Gamma^0) + S(F_0)V(\Gamma^0) - F_0(\Gamma^0) + \nabla(\Gamma^0 | \mathrm{F}_0). \tag{19.13}$$

Separation of volume and boundary terms in this equation yields

$$-b = a + S(F_0), \tag{19.14}$$

$$F_0(\Gamma^0) = \Phi(\Gamma^0) + \nabla(\Gamma^0 | F_0). \tag{19.15}$$

The equation (19.15) has a (unique) solution F_0 with $\|F_0\| < \infty$ in the class of τ-functionals [62,63] and (19.14) allows us to find a. Since a is a monotone (in fact linear) function of b, it is sufficient to prove that $a \geqslant 0$ in the case of

$$b = \ln(1-\theta)$$

which is the original Stavskaya system. (In fact $a = 0$, as we shall see.)

Assume the contrary, $a < 0$, and prove the following lemma.

Lemma 19.12 If $a < 0$ and F_0 is a τ-functional with $\|F_0\| < \infty$, the expression

$$\exp(aV(\Gamma^0))\Xi(\Gamma^0 | F_0)$$

can be presented as

$$\Xi(\Gamma^0 | F_0^a)$$

where F_0^a is a τ-functional too, but $\|F_0^a\| = \infty$, and moreover F_0^a has a positive volume part, that is,

$$\lim \frac{F_0^a(\Gamma^0)}{V(\Gamma^0)} = \varphi(a) > 0 \quad \text{with} \quad \frac{|\Gamma^0|}{V(\Gamma^0)} \to 0. \tag{19.16}$$

Proof. The equality

$$\exp(aV(\Gamma^0))\Xi(\Gamma^0 | F_0) = \Xi(\Gamma^0 | F_0^a) \tag{19.17}$$

can be written as

$$\exp(-F_0^a(\Gamma^0)) = \frac{\exp(aV(\Gamma^0))\Xi(\Gamma^0 | \mathrm{F}_0)}{\Sigma\Pi \exp(aV(\Gamma_b^0))\Xi(\Gamma_b^0 | F_0)} \tag{19.18}$$

where the sum is taken over all combinations of external contours (Γ_b^0) inside Γ^0. Now the right-hand side of (19.18) does not exceed

$$\frac{\Xi(\Gamma^0 | F_0)}{\Sigma\Pi\Xi(\Gamma_b^0 | F_0)} = \exp(-F_0(\Gamma^0)),$$

since $a < 0$ and

$$\Sigma V(\Gamma_b^0) \leqslant V(\Gamma^0).$$

Thus

$$F_0^a(\Gamma^0) \geqslant F_0(\Gamma^0),$$

whence F_0^a is a τ-functional and its 'volume part' is non-negative.

Now to prove that $\varphi(a)$ is positive. In fact, if $\varphi(a) = 0$, (19.17) infers

$$a + S(F_0) = S(F_0^a),$$
$$-F_0(\Gamma^0) + \nabla(\Gamma^0 | F_0) = -F_0^a(\Gamma^0) + \nabla(\Gamma^0 | F_0^a),$$

whence $F_0 = F_0^a$ and $a = 0$, which contradicts the lemma's assumption. Thus $\varphi(a) > 0$. Lemma 19.12 is proved.

Now to finish the proof of Lemma 19.11, we assume

$$\Xi(\Gamma^0 | H) = \Xi(\Gamma^0 | F_0^a), \ \varphi(a) > 0$$

for the original Stavskaya system and are going to come to a contradiction. We know that for this system:

$$\lim \frac{\ln \Xi(\Gamma^0 | H)}{V(\Gamma^0)} = -b = -\ln(1 - \theta). \tag{19.19}$$

Just from the definition

$$\varphi(a) = \lim \frac{F_0^a(\Gamma^0)}{V(\Gamma^0)} = \lim \left(\frac{\ln \Sigma\Pi\Xi(\Gamma_b^0 | H)}{V(\Gamma^0)} - \frac{\ln \Xi(\Gamma^0 | H)}{V(\Gamma^0)} \right). \tag{19.20}$$

So it is sufficient to show that

$$\lim \frac{\ln \Sigma\Pi\Xi(\Gamma_b^0 | H)}{V(\Gamma^0)} \leqslant -\ln(1-\theta). \tag{19.21}$$

But $\Sigma\Pi\Xi(\Gamma_b^0 | H)$ does not exceed the sum of statistical weights of all the configurations in the volume $V(\Gamma^0)$ normed in such a way that makes the weight of the 'all zeros' configuration equal to 1.

Before norming, the sum of statistical weights (that is, the sum of products of transitional probabilities) was equal to 1. After norming we have the desired estimation

$$\Sigma\Pi\Xi(\Gamma_b^0 | H) \leqslant \frac{1}{(1-\theta)^{V(\Gamma^0)}}. \tag{19.22}$$

Lemma 19.11 is proved.

Now to prove Theorem 19.10 we have only to repeat the argument of [50]. From (19.11) we infer the 'instability' of the boundary condition 'all zeros'. Moreover, the only limit distribution is achieved with the boundary condition 'all ones', whence it is 'all ones' too.

Chapter 20

From discrete to continuous time

This chapter is exceptional in our survey, for here we consider continuous time. We cannot even mention all the papers that treat multicomponent systems with continuous time. Just definitions of such systems are not trivial in the continuous-time case – we require proofs and auxiliary constructions [12].

The cluster expansions like those of Chapter 5 are among such constructions. Let us apply them here. We shall build the Markov process on $X = X_0^V$ (for simplicity we assume $V = \mathbb{Z}$) with continuous time, having built some Markov process with discrete time step Δt and tending $\Delta t \to 0$. First fix $\Delta t > 0$ and examine the interacting automata system with transitional probabilities

$$\mathbb{P}(x_h^t = a \,|\, x^{t-\Delta t}) = \delta(a, x_h^{t-\Delta t}) + \lambda_h(a, x^{t-\Delta t})\Delta t, \tag{20.1}$$

where $t \in \{h\Delta t, n \in \mathbb{Z}\}$, δ is the Kronecker symbol and λ_h are the 'infinitesimal transition probabilities' which depend on parameter a and on the values of

$$x_j^{t-\Delta t}, |j - h| \leqslant R$$

with

$$\sum_a \lambda_h(a, x^{t-\Delta t}) = 0.$$

Thus the following conditional probabilities are defined:

$$\mathbb{P}(x_h^t = a_h, h \in K \,|\, x^{t-\Delta t}) = \prod_{h \in K} \mathbb{P}(x_h^t = a_h \,|\, x^{t-\Delta t}) \tag{20.2}$$

for any $K \subset V$. Hence the more general conditional probabilities

$$\mathbb{P}(x_h^{n\Delta t} = a_h, h \in K \,|\, x^0)$$

are also defined by induction. It remains only to prove the existence of the limit

$$\lim_{\substack{\Delta t \to 0 \\ n\Delta t \to t}} \mathbb{P}(x_h^{n\Delta t} = a_h, h \in K \,|\, x^0) \tag{20.3}$$

for any $K \subset V$. For simplicity let $|K| = 1$. The following definition suits the discrete-time case $\Delta t > 0$ and recalls definition 5.1.

Definition 20.1

(1) For any point (h,t) of the evolution space $\mathbb{Z}^2$ there is a branch $B_{ht} \subset V^{\mathbb{Z}}$:

$$B_{ht} = \{(h,t);(j,t-\Delta t): |j-h| \leqslant R\}.$$

The point (h,t) is the root of B_{ht}, the other points of B_{ht} are its ends.

(2) The branch B_{ju} is said to be following the point (j,t) if $u \leqslant t$. The branch B_{ju} is said to be following the branch B_{it} if it follows one of its ends.

(3) A crown with the root (i,t) stands for any finite set of branches which is connected in the following sense: one of its branches follows the point (i,t) and every other one of its branches follows another branch of the crown. Ends of the crown are those ends (j,u) of its branches for which no point (j,v), $v < u$ belongs to a branch of the crown. Any point (i,t) is the root and the only end of the corresponding void crown without a branch.

(4) A crown with several roots $(i_1,t), \ldots, (i_r,t)$ is a finite set of branches, every one of which follows either one of the roots or another branch of the crown.

(5) A filled crown or a diagram (C,x_C) is a state $x_C \in X_C$ where $X_C = \prod_{h \in C} X_h$ and C is the union of the branches and roots.

Every diagram x_C has a weight defined as follows:

(a) The state of a branch has a function $\lambda(.,.)\Delta t$ of this state.

(b) As about any vertical segment

$$(j,v),\ (j,v+\Delta t),\ \ldots,\ (j,u)$$

where (j,v) belongs to a branch of the crown and (j,u) either belongs to a branch or is the root of the crown and all the other points

$$(j,v+\Delta t),\ \ldots,\ (j,u-\Delta t)$$

belong to no branch of the crown, such a segment has the factor $\delta(x_j^v, x_j^u)$ which is the Kronecker symbol.

(c) Every end (j,u) of the diagram (C,x_C) has the factor $\delta(x_j^0, x_j^u)$.

(d) The weight of the diagram is the product of all the factors mentioned in the items (a), (b) and (c).

Lemma 20.2 The probability

$$\mathbb{P}(x_h^{n\Delta t} = a \,|\, x^0) \tag{20.4}$$

is equal to the sum of the weights of all the filled crowns that have the root $(h,n\Delta t)$, $x_h^{n\Delta t} = a$ and belongs to the evolution space band $t \in [0,n\Delta t]$.

Proof is obvious. It remains only to prove the convergence of these sums of weights with $\Delta t \to 0$, $n\Delta t \to t$. For that we expand the sum (20.4) into a series $G_0 + G_1 + G_2 + \ldots$ where G_k is the sum of weights of filled crowns having k branches. Every G_n has a limit with $\Delta t \to 0$, $n\Delta t \to t$, because it is the Riemann sum of the n-fold integral of a constant. We must estimate this constant. Let us have two filled diagrams whose branches have ends

$$(j_1{=}i, u_1),\ (j_2, u_2),\ \ldots,\ (j_n, u_n)$$

and

$$(j'_1{=}i, u'_1),\ (j'_2, u'_2),\ \ldots,\ (j'_n, u'_n).$$

We term these diagrams 'combinatorial-equivalent' if

$$j'_2 = j_2,\ \ldots,\ j'_n = j_n$$

and the corresponding branches have equal states. The number of the resulting equivalence classes is no more than A^n where A = const. Since every class corresponds to one item in the integrand (remember that the integral is over the variables $u_1, \ldots, u_n$), this integrand does not exceed A^nC^n where $C = \max|\lambda(.,.)|$. The integral being taken over the volume t^n, the value of $|G_n|$ has en estimator of the order B^n uniformly in Δt. So the series converges uniformly for t small enough.

Thus the transitional function of the continuous-time Markov process is the limit of transitional functions of the discrete-time processes and has a simple cluster expansion which we are going to describe.

Let us double the t-axis:

$$t \in \{n\Delta t, (n+\tfrac{1}{2})\Delta t : n \in \mathbb{Z}\}$$

and define a branch in the new way:

Definition 20.3

(1) *A branch* is a set

$$B_{ht} = \{(h, t+\tfrac{1}{2}); (j,t) : |j-h| \leqslant R\},$$

the point $(h,t+\frac{1}{2})$ being its root, the other points being its ends.
(2) A branch B_{ju} is termed following the point (j,t) if $u \leqslant t$. A branch is termed following another branch if it follows one of its ends.

State of a branch and state of a crown are the states of the corresponding subsets of the evolution space.

Any state x_C of a crown C has a weight defined as follows:

(a) Every branch has a factor $\lambda(.,.)$ which is a function of its state.
(b) Every vertical segment

$$[(j,v),(j,u)] \quad \text{or} \quad [(j,v+\tfrac{1}{2}),(j,u)]$$

of which ends belong to C, (or (j,u) is the root) and the other points do not, has a factor

$$\delta(x_j^v,x_j^u) \quad \text{or} \quad \delta(x_j^{v+1/2},x_j^u).$$

(c) Every end (j,u) of the crown has a factor $\delta(x_j^0,x_j^u)$.
(d) The diagram's weight is the product of the factors belonging to its branches, vertical segments, and ends.

Now we are ready to describe the cluster expansion: $\mathbb{P}(x_h^t = a|x^0)$ equals the sum of weights of all the filled crowns with the root (h,t) with $x_h^t = a$, which belong to the band $(0,t)$. Since u is continuous, the sum over it means an integral.

The cluster expansion for

$$\mathbb{P}(x_{h_1}^t=a_1,\ldots,x_{h_r}^t=a_r|x^0)$$

is defined in the analogous way. The cluster expansion can be used to check the semi-group property of the transitional function.

Unfortunately, our cluster expansion is not suitable to go to the limit $t \to \infty$. But there is another cluster expansion which allows us to make $t \to \infty$ and hence to prove ergodicity of the system. This results from an extra assumption about the infinitesimal transitional probabilities. We put

$$\mathbb{P}(x_h^t = a|x^{t-\Delta t}) = \delta(a,x_h^{t-\Delta t}) + \lambda_0(a,x_h^{t-\Delta t})\Delta t + w_h(a,x^{t-\Delta t})\Delta t. \quad (20.5)$$

Here λ_0 is the matrix of the infinitesimal transitional probabilities of an irreducible continuous-time Markov chain with X_0 set of states. Now, $w_h(a,x^{t-\Delta t})$ is some 'small perturbation' of these infinitesimal probabilities. It depends on the values of

$$x_j^{t-\Delta t},\ |j-h| \leqslant R$$

and is subject to

$$\sum_a w_h(a,.) = 0.$$

The Markov chain having

$$(P_0^t = \exp(\lambda_0 t))$$

as the matrix of the infinitesimal transitional probabilities is contractive in the following sense:

$$\|P_0^t\mu - P_0^t\nu\| \leqslant \exp(-rt)\|\mu - \nu\| \tag{20.6}$$

where P_0^t is the chain's transitional function, λ_0, $r > 0$, μ and ν are any normed measures on X_0, and $\|.\|$ is the full variation. Hence $P_0^t\mu$ for any initial μ tends to the chain's invariant measure μ_0 and

$$|P_0^t(x|y) - \mu_0(x)| \leqslant 2\exp(-rt), \tag{20.7}$$

$$\int_0^\infty |P_0^t(x|y) - \mu_0(x)| \leqslant \frac{2}{r}. \tag{20.8}$$

Our cluster expansion converges if $\max|w(.,.)|/r$ is small.

Let us reword (20.5) as follows:

$$\mathbb{P}(x_h^t = a|x^{t-\Delta t}) = P_0^{\Delta t}(a|x_h^{t-\Delta t}) + \overline{w}_h(a,x^{t-\Delta t})\Delta t \tag{20.9}$$

where $|\overline{w} - w| = O(\Delta t)$.

The last formula (20.9) allows us to formulate the following analogue of Lemma 5.2. The corresponding diagrams are the same as in (20.1) but their weights are different: the branches have factors $\overline{w}(.,.)\Delta t$ instead of $\lambda(.,.)\Delta t$, the vertical segments have factors $P_0^{u-v}(.|.)$, the ends have factors $\mu_0(.)$.

Lemma 20.4 Let the initial measure be independent with every automaton distributed in μ_0. Then the probability $\mathbb{P}(x_h^{n\Delta t} = a)$ equals the sum of weights of all the filled crowns with the root $(h,n\Delta t)$ with $x_h^{n\Delta t} = a$ belonging to the band $[0,t]$.

Proof is obvious.

Going to the limit $\Delta t \to 0$ we obtain the corresponding expansion for the continuous-time system. The diagrams remain as before (for the continuous time) but now we can take $w(.,.)$ as the branches' factors (because the difference between $\overline{w}(.,.)$ and $w(.,.)$ vanishes when $\Delta t \to 0$). The vertical segments have $P_0^{u-v}(.,.)$ and ends have $\mu_0(.)$ as factors now. To make the limit transitions $t \to \infty$ it remains only to obtain an estimation, uniform in t, for the members G_n of our cluster expansion. When t is finite, G_n is an n-multiple integral (over the simplex $0 < u_n < \ldots < u_2 < u_1 < t$ or the shifted one $-t < u_n < \ldots < u_2 < u_1 < 0$). Thus we have to investigate the convergence of the corresponding improper integral. We can go to new variables u_1, $u_2 - u_1$, $u_3 - u_2$, ..., $u_n - u_{n-1}$. Note that the sub-integral expression includes the product

$$P_0^{u_1}(a_1|x_1)P_0^{u_2-u_1}(y_2|x_2) \ \ldots \ P_0^{u_n-u_{n-1}}(y_n|x_n) \tag{20.10}$$

which is due to the vertical segments starting from some ends of the crown's branches. This expression alone is not sufficient for the integral to converge because $P_0^n(.|.)$ does not tend to 0 as $u \to \infty$. But this expression multiplies by the branches' factors and is summed over x_1, ..., x_n. Remember that $\sum_x w(x,.) = 0$. Thus the integral will not change if we substract a constant not depending on x_1, ..., x_n from every factor of (20.10). Thus (20.10) equals

$$(P_0^{u_1}(a|x_1) - \mu_0(a)) \ \ldots \ (P_0^{u_n-u_{n-1}}(y_n|x_n) - \mu_0(y_n)) \tag{20.11}$$

where every factor tends to 0 in the exponential way with $u_1 \to \infty$, $u_2 - u_1 \to \infty$ and so on. Thus the improper integral converges and the n factors of (20.11) provide an upper estimator of the integral:

$$|G_n| \leqslant \left(\frac{2}{r}\right)^n \Sigma(\max \Pi P^s(.|.)\mu_0(.)|w(.,.)|), \tag{20.12}$$

where all the factors of ends of branches and all the other vertical segments are substituted by their maxima and the sum over all classes of diagrams having n branches is taken. But all P^s and μ_0 do not exceed 1, the number of branches is n, and the number of classes of diagrams does not exceed A^n. Thus

$$|G_n| \leqslant \left(\frac{2}{r}\right)^n A^n C^n \tag{20.13}$$

where $C = \max|w(.,.)|$.

Thus, with C/r small enough, the cluster expansion converges absolutely and uniformly in t. So we can go to the limit $t \to \infty$. To prove the independence of the limit distribution of the initial one (which is ergodicity) one has to examine how change of the initial distribution changes our cluster expansion. It is sufficient to speak only of initial states $x^0 \in X$. The only change of the cluster expansion caused by it is the following: at any end (j,u) of a diagram the new factor $P_0^u(x_j^u|x_j^0)$ substitutes $\mu_0(x_j^u)$. After shift of time variable the new factor can be rewritten as $P_0^{u+t}(x_j^u|x_j^0)$. The new cluster expansion for t finite converges uniformly in t for the same reasons as the former one and allows the limit transition $t \to \infty$ of every member. But $P_0^t(y|x)$ tends to $\mu_0(y)$ as $t \to \infty$. So every member tends to that one of the former cluster expansion. Thus we have proved the following theorem.

Theorem 20.5 With C/r small enough, the limits

$$\lim_{t\to\infty} P(x_h^t = a) \quad \text{and} \quad \lim_{t\to\infty} P(x_{h_1}^t = a_1, \ldots, x_{h_r}^t = a_r)$$

exist, do not depend on the initial measure and equal the sum of weights of all the filled crowns with the root $(h,0)$ with $x_h^0 = a$, and the roots $(h_1,0)$, …, $(h_r,0)$ with $x_{h_1} = a_1$, …, $x_{h_r} = a_r$ correspondingly.