A jump telegraph model for option pricing

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Markets and strategies

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Here $\varphi_k(t)$ is number of units of kth asset in the current portfolio.

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This strategy is self-financing, if the increments in the strategy value $F = F(t)$ are provoked only by the increments in assets' prices:

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Our main objective is to construct arbitrage-free and complete market models.

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Consider a contract which gives to its holder the payment X (*claim*) at some specified expiry date, T (*maturity time*). The value X depends on the terminal state of the market, $X = f(\mathbf{S}(T))$. This contract is called the *European option*.

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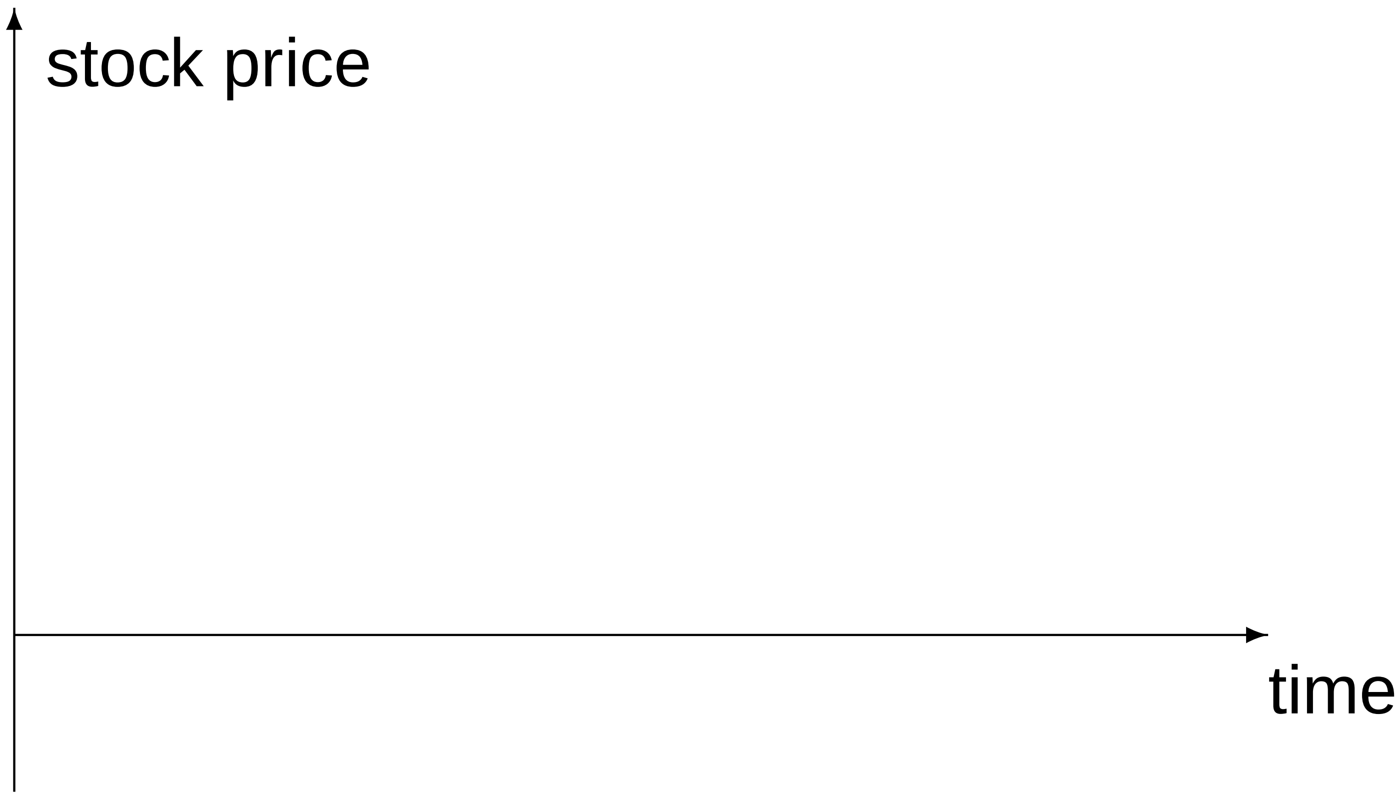

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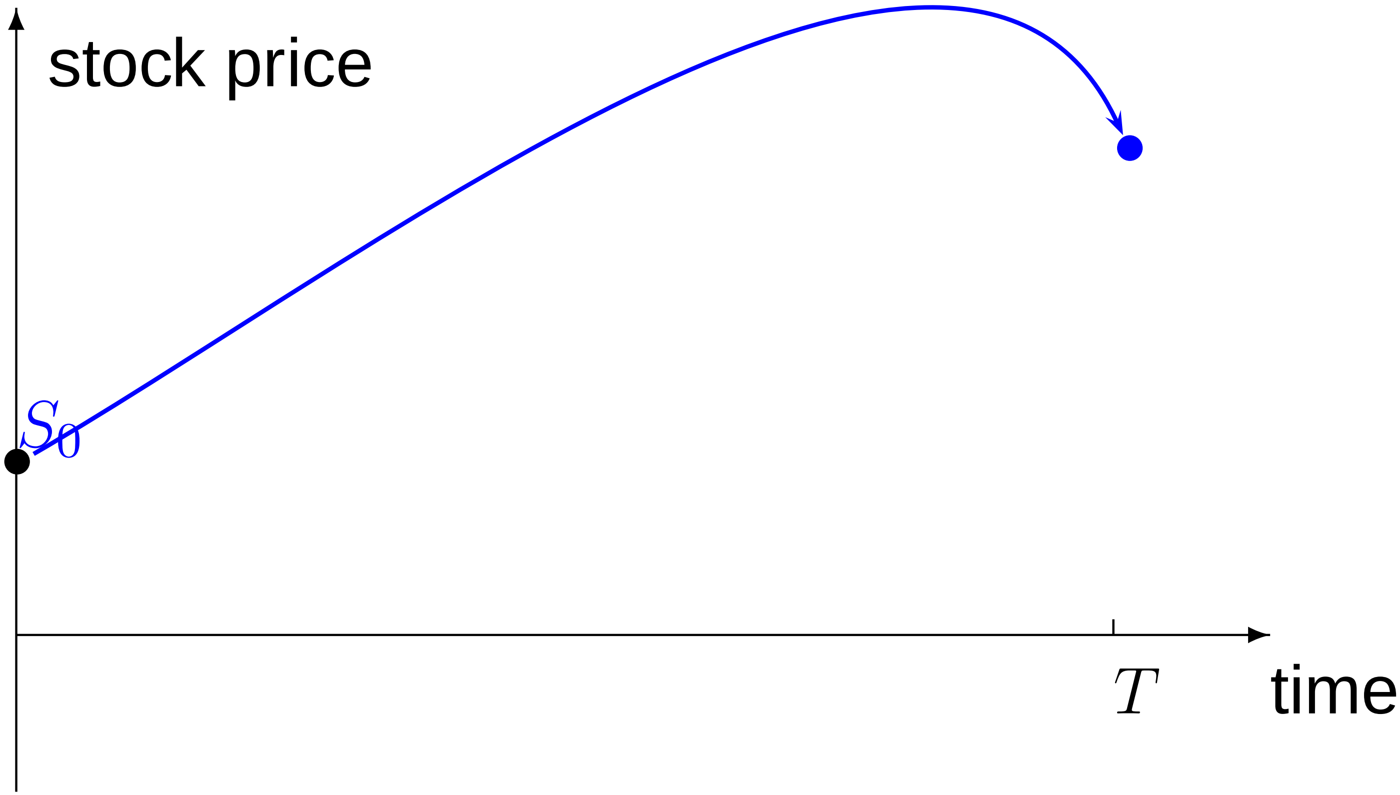

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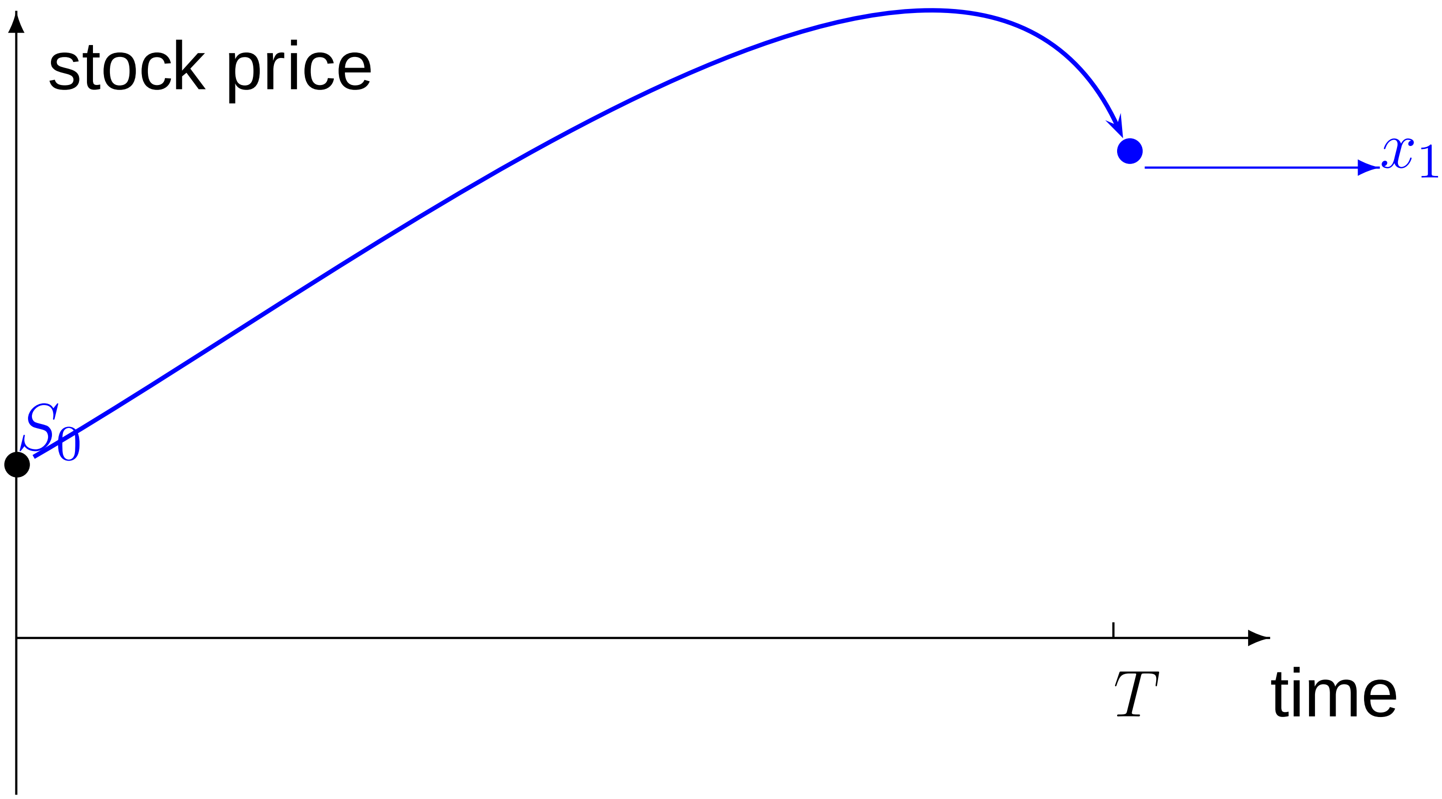

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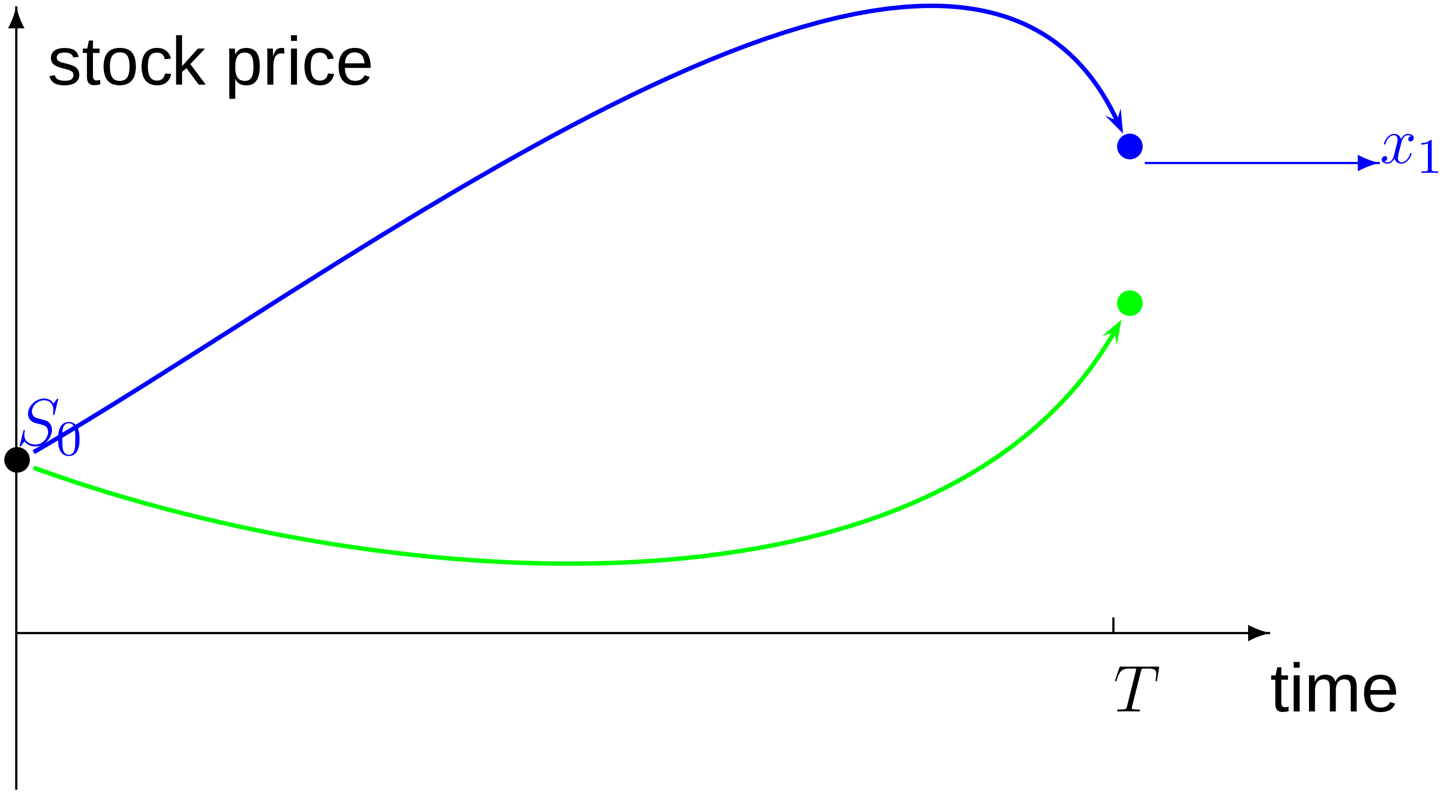

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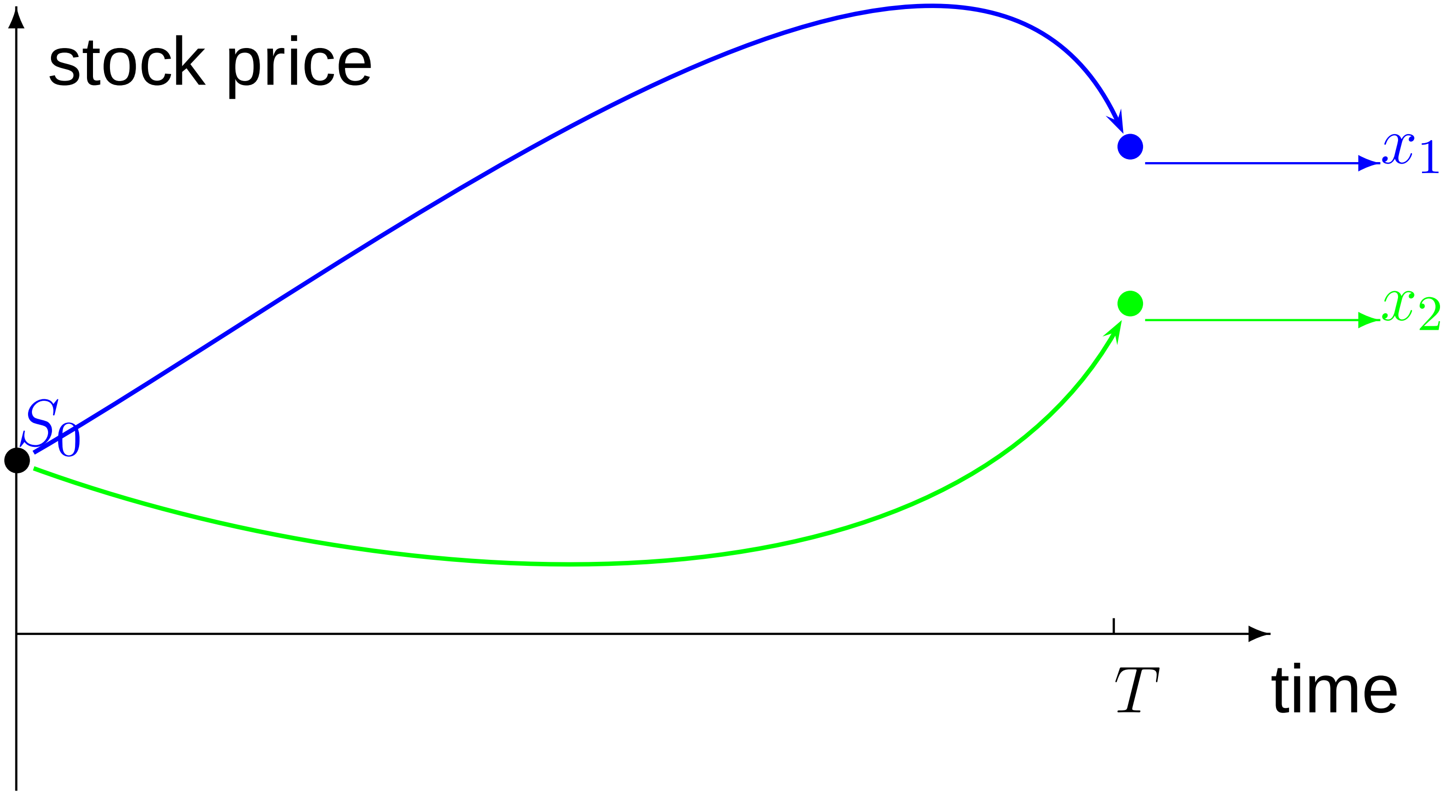

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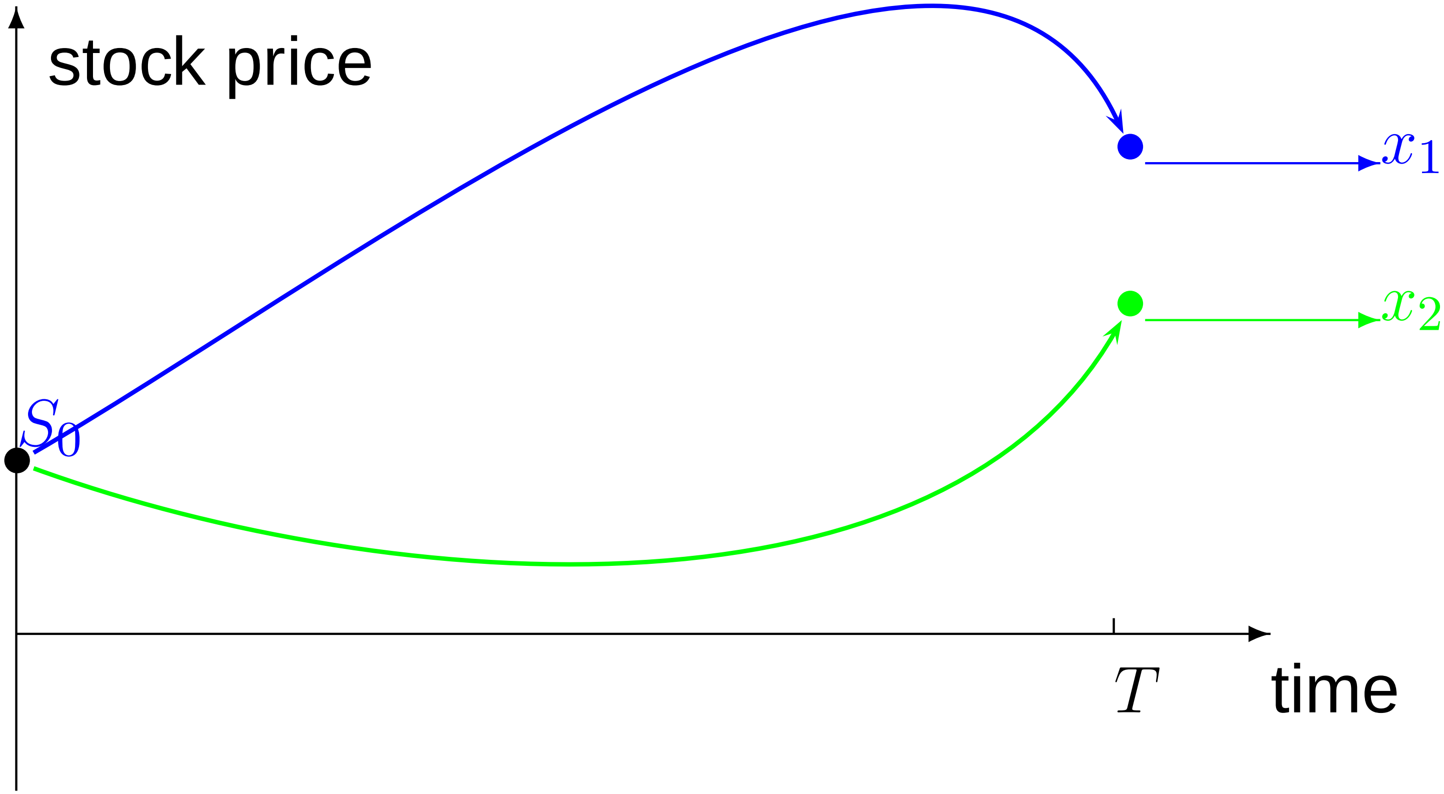

Denote its price at time $t = 0$ by $\mathbb{c}$.

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The main questions of the theory are

1) What is the fair option price?
2) What is the respective trading strategy?

Standard European options

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$$X = (S(T) - K)^+ = \begin{cases} S(T) - K, & \text{if } S(T) > K, \\ 0, & \text{if } S(T) < K \end{cases}$$ defines

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The claim

$$X = (K - S(T))^+ = \begin{cases} 0, & \text{if } S(T) > K, \\ K - S(T), & \text{if } S(T) < K \end{cases}$$ defines

the put option with the strike K.

Option pricing & risk-neutral measure

The measure $\mathbb{P}^*$ is named the risk-neutral measure, if $B(t)^{-1}\mathbf{S}(t)$ is a martingale:

$$\mathbb{E}_{\mathbb{P}^*}\{B(t)^{-1}\mathbf{S}(t)|\mathcal{F}_\tau\} = B(\tau)^{-1}\mathbf{S}(\tau),\ \tau < t.$$

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$$\mathbb{c} = \mathbb{E}_{\mathbb{P}^*}\{B(T)^{-1}X\}$$

and

$$F(t) = \mathbb{E}_{\mathbb{P}^*}\{B(T)^{-1}X|\mathcal{F}_t\}.$$

Black-Scholes model

Assume that the stock price moves according to geometric Brownian motion:

$$S(t) = S(0)\mathrm{e}^{vw(t)+at},\ B(t) = B(0)\mathrm{e}^{rt},\ t \geq 0,$$

where $v, r > 0,\ a \in (-\infty, \infty)$ are constants.

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It is proved that the risk-neutral measure $\mathbb{P}^*$ exists and it is unique.
The option prices $\mathbb{c} = \mathbb{c}(X)$ for many particular claims X can be expressed exactly. For example, the standard call option with strike K has the following expression for its price:

$$\mathbb{c} = S_0 \cdot \Phi(z_+) - \mathrm{e}^{-rT} K \cdot \Phi(z_-),$$

where $z_\pm = \frac{\ln(S_0/K)+(r\pm v^2/2)T}{v\sqrt{T}}$ and $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z} \mathrm{e}^{-x^2/2}\mathrm{d}x$.

Shortages of Black-Scholes model

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The main goal of this work is to create the financial model which is free from the shortages of the parabolic world.

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Let $\sigma_i(t), t \geq 0$ be a Markov chain with values in the finite set $I, |I| = d$ and with initial state i: $\sigma_i(0) = i, i \in I$.
For given numbers $c_i, i \in I$ we define the processes

$$X_i(t) = \int_0^t c_{\sigma_i(\tau)} \mathrm{d}\tau, \ t \geq 0 \tag{1}$$

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Consider the asset with the price dynamic $S = S(t), t \geq 0$

$$\mathrm{d}S(t) = S(t)\mathrm{d}X_i(t) \Leftrightarrow \mathrm{d}S = S(t)c_{\sigma_i(t)}\mathrm{d}t \Leftrightarrow S(t) = S(0)\exp(X_i(t)),$$

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$i \in I$. This model admits arbitrage opportunities.

Jump-telegraph model

For given numbers $h_i > -1$ we define jump process

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and consider the stock price dynamics of the form

$$\mathrm{d}S(t) = S(t-)[\mathrm{d}X_i(t) + \mathrm{d}J_i(t)], \qquad (4)$$

where X_i is defined in (1) and $N_i(t), t \geq 0$ is the Poisson process which counts the number of switchings of σ_i.

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Integrating we have

$$S(t) = S(0)\mathcal{E}_t(X_i + J_i) = S(0)\exp(X_i(t)) \cdot \prod_{j=1}^{N_i(t)} (1 + h_{\sigma_i(\tau_j -)}).$$

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Seeking for simplicity, we consider two-state Markov chain, $|I| = 2$.

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Sample path of $X + J$

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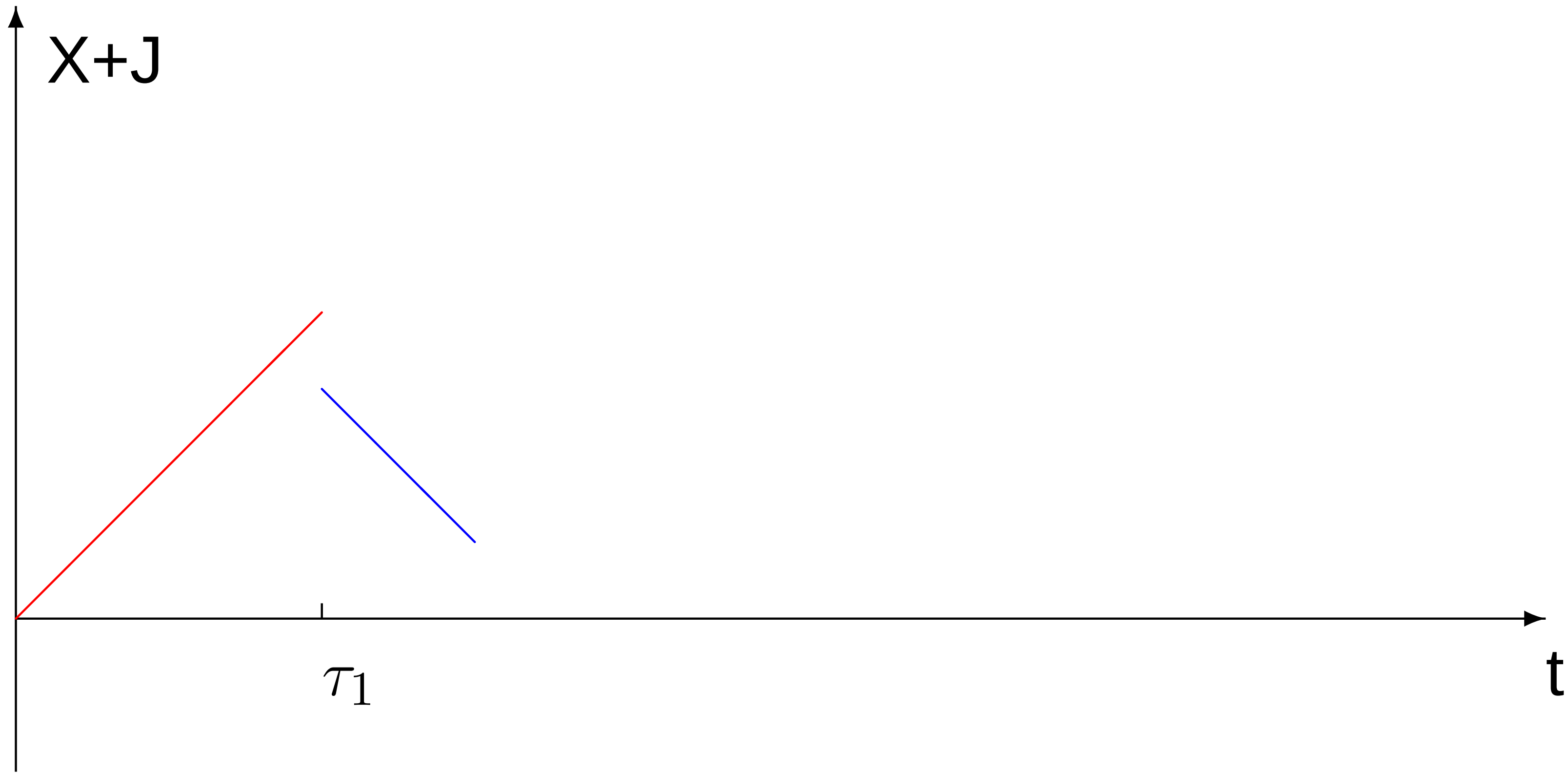

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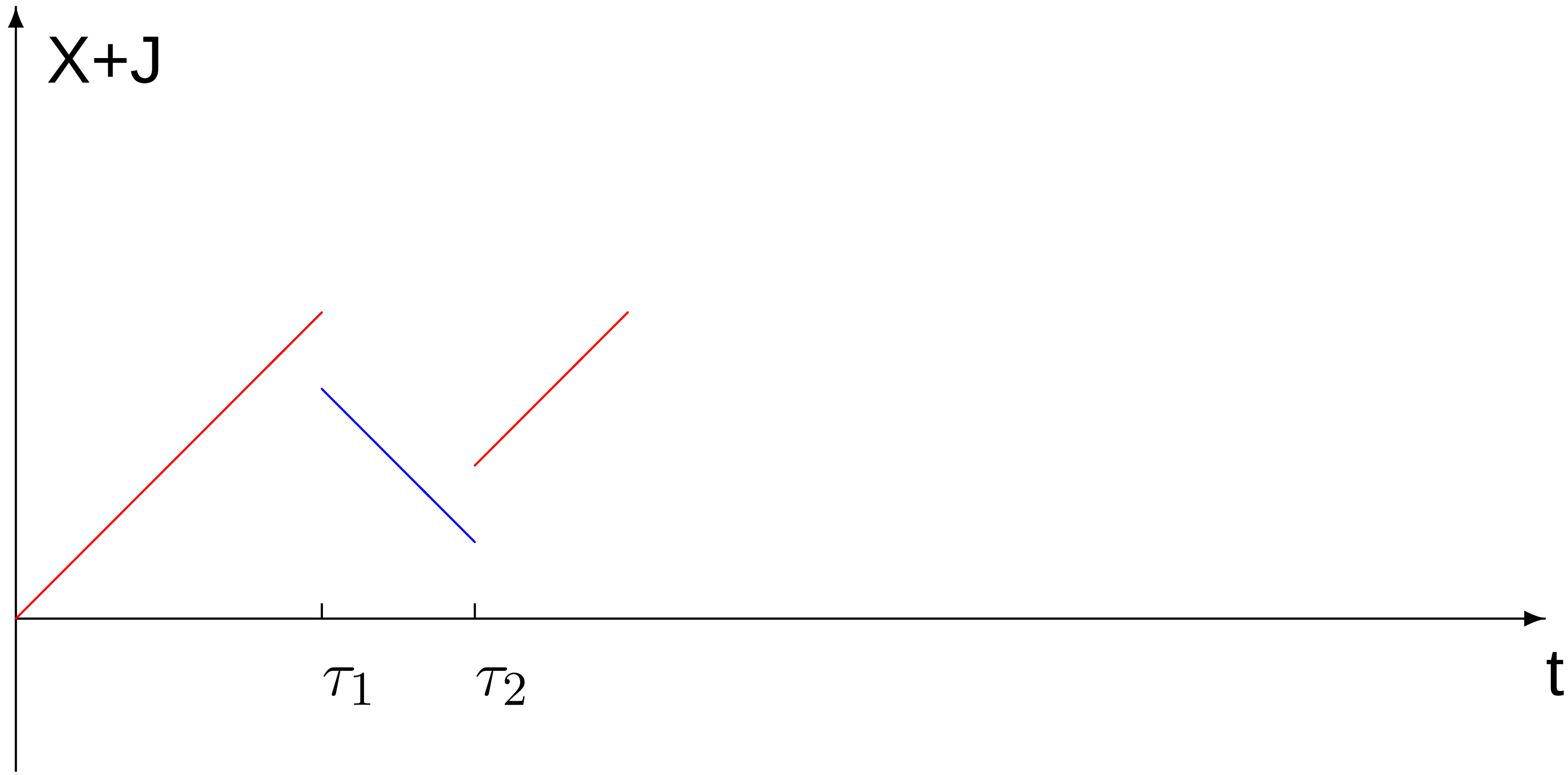

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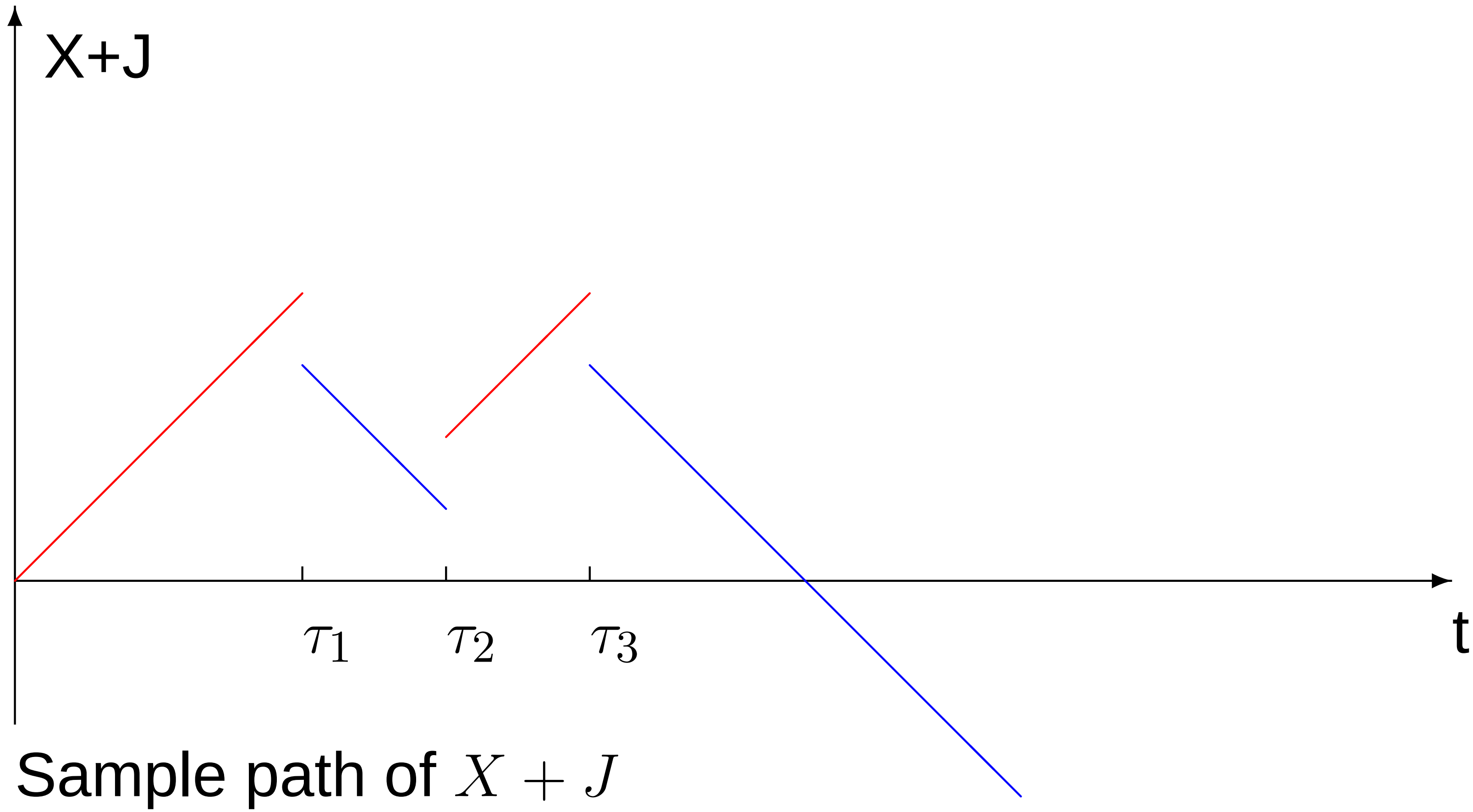

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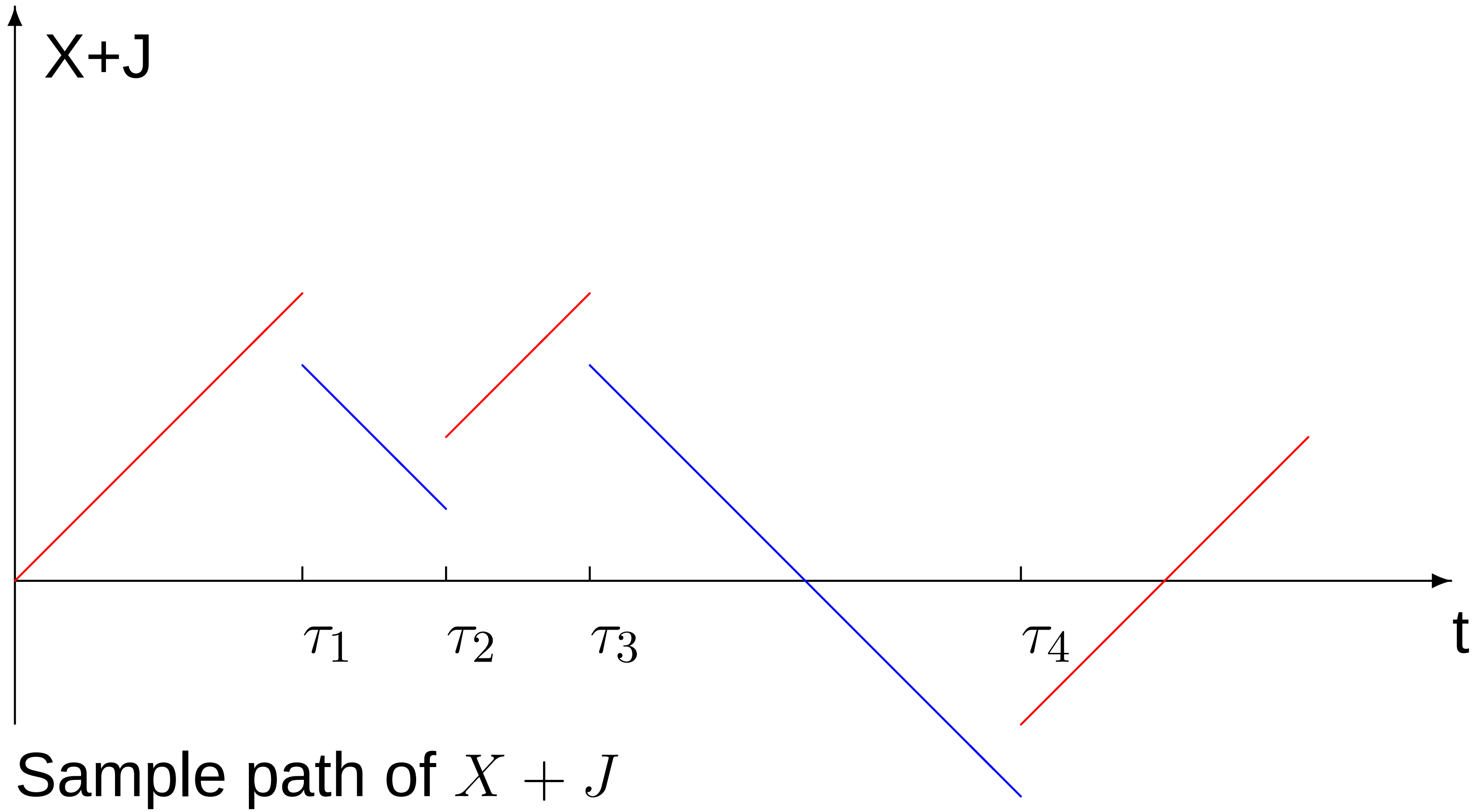

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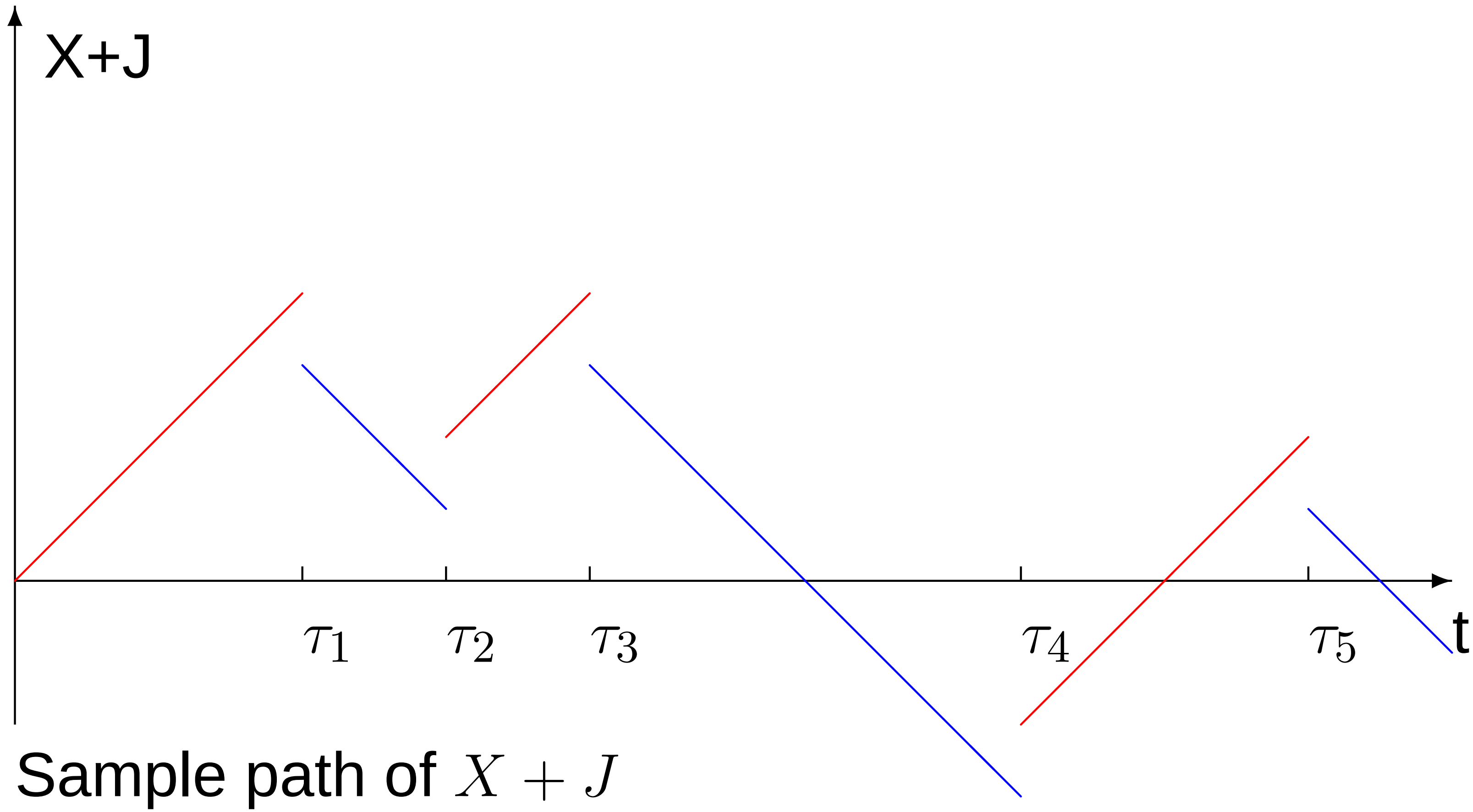

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Moreover this model reflects oversold/overbought market situations: changes are gradually building up before crashes and spikes.

Technical background

We study the model that is both realistic and general enough to enable us to incorporate different trends and extreme events. This is the complete market model and hedging is perfect. Jump component here is supplied not only by reasons of adequacy. Jumps serve as the unique tool to avoid arbitrage opportunities.

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Under suitable rescaling the jump telegraph model converges to classic Black-Scholes model, which permits to interpret volatility.

Telegraph martingales

The next theorem could be considered as a version of the Doob-Meyer decomposition for telegraph processes with alternating intensities.

Theorem 1. *Let* $X_i, i = 0, 1$ *be the telegraph process with velocities* c_0 *and* c_1*, and* J_i *be the jump process with jump values* $h_0, h_1 > -1$*, which is defined in (3). Then* $X_i + J_i$ *is a martingale if and only if*

$$\lambda_0 h_0 = -c_0, \qquad \lambda_1 h_1 = -c_1.$$

Here λ_i is the intensity of leaving of the state i.

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Remark. *In particular, it means that any (nontrivial) telegraph process without jumps (i.e. if $h_i = 0$) never possess a martingale measure. So Markov-modulated persistent dynamics is the arbitrage model.*

Change of measure

Let X_i^* be the telegraph process with the velocities c_i^*, and $J_i^* = -\sum_{j=1}^{N_i(t)} c^*_{\sigma_i(\tau_j-)}/\lambda_{\sigma_i(\tau_j-)}$ be the jump process with jump values $h_i^* = -c_i^*/\lambda_i > -1$.

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$$Z_i(t) = \frac{\mathrm{d}\mathbb{P}_i^*}{\mathrm{d}\mathbb{P}_i}|_t = \mathcal{E}_t(X_i^* + J_i^*), \qquad 0 \le t \le T.$$

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Using properties of stochastic exponentials, we obtain

$$Z_i(t) = \mathrm{e}^{X_i^*(t)} \cdot \prod_{j=1}^{N_i(t)} \left(1 + h^*_{\sigma_i(\tau_j-)}\right).$$

Girsanov theorem

Theorem 2. *Under the probability measure* $\mathbb{P}_i^*$,

- *process* $N_i = N_i(t),\ 0 \le t \le T$ *is the Poisson process with intensities* $\lambda_0^* = \lambda_0 - c_0^* = \lambda_0(1 + h_0^*)$ *and* $\lambda_1^* = \lambda_1 - c_1^* = \lambda_1(1 + h_1^*)$.
- *process* $X_i = X_i(t),\ 0 \le t \le T$ *is the telegraph process with states* (c_0, λ_0^*) *and* (c_1, λ_1^*).

Girsanov theorem

Theorem 2. *Under the probability measure* $\mathbb{P}_i^*$,

- *process* $N_i = N_i(t),\ 0 \le t \le T$ *is the Poisson process with intensities* $\lambda_0^* = \lambda_0 - c_0^* = \lambda_0(1 + h_0^*)$ *and* $\lambda_1^* = \lambda_1 - c_1^* = \lambda_1(1 + h_1^*)$.
- *process* $X_i = X_i(t),\ 0 \le t \le T$ *is the telegraph process with states* (c_0, λ_0^*) *and* (c_1, λ_1^*).

Theorems 1 and 2 allows us to give a following characterization of the martingale measure.

Martingale measure

Theorem 3. *Measure* $\mathbb{P}_i^*$ *is the martingale measure for the process* $B_i(t)^{-1}S_i(t),\ t \geq 0$ *if and only if*

$$c_0^* = \lambda_0 - \frac{r_0 - c_0}{h_0}, \quad c_1^* = \lambda_1 - \frac{r_1 - c_1}{h_1}$$

Moreover, under the probability measure $\mathbb{P}_i^*$*, process* N_i *is the Poisson process with alternating intensities* $\lambda_0^* = \frac{r_0 - c_0}{h_0}$ *and* $\lambda_1^* = \frac{r_1 - c_1}{h_1}$*.*

Fundamental equation (1)

Consider a European option with maturity time T and payoff function $f(S(T))$. We assume f is a continuous and piecewise smooth function. To price these options, we need to study the function

$$F(t,\ x,\ i) = \mathbb{E}_i^* \left[\mathrm{e}^{-Y_i(T-t)} f(x\mathrm{e}^{X_i(T-t)} \kappa_i(T-t)) \right], \qquad (7)$$

$$i = 0, 1,\ 0 \le t \le T,$$

where $\mathbb{E}_i^*$ denotes the expectation with respect to the martingale measure $\mathbb{P}_i^*$.

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where $\mathbb{E}_i^*$ denotes the expectation with respect to the martingale measure $\mathbb{P}_i^*$.
$F_t := F(t,\ S_i(t),\ \sigma_i(t))$ is the strategy value at time $t,\ 0 \le t \le T$ of the option with claim $f(S_i(T))$ at the maturity time T.

Fundamental equation (2)

Theorem 4. *Function F is a solution of the following hyperbolic system: for $0 < t < T$,*

$$\frac{\partial F}{\partial t}(t,x,i) + c_i x \frac{\partial F}{\partial x}(t,x,i)$$

$$= (r_i + \lambda_i^*)F(t,x,i) - \lambda_i^* F(t, x(1+h_i), 1-i), \; i = 0,1 \qquad (8)$$

with the terminal condition $F(T,\ x,\ i) = f(x)$. Here $\lambda_i^* = (r_i - c_i)/h_i$.

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This system plays the same role for our model as the classical parabolic equation for Black-Scholes model:

$$\frac{1}{2}v^2x^2\frac{\partial^2 F}{\partial x^2} + rx\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} = rF \tag{9}$$

Fundamental equation (3)

In contrast with classical theory, system (8) is hyperbolic. In particular, it implies the finite velocity of propagation, which corresponds better to the intuitive understanding of financial markets and to the viewpoint of technical analysis.

Note that these equations do not depend on λ_i, just as the equation (9) in the Black-Scholes model does not depend on the drift parameter.

Convergence to Black-Scholes (1)

It is known that (homogeneous) telegraph process $X = X(t),\ t \geq 0$ converges to the standard Brownian motion $w(t),\ t \geq 0$, if $c,\ \lambda \to \infty$, $c/\sqrt{\lambda} \to 1$.

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The following theorem provides a similar connection (under respective scaling) between stock prices driven by geometric jump telegraph processes and geometric Brownian motion.

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The following theorem provides a similar connection (under respective scaling) between stock prices driven by geometric jump telegraph processes and geometric Brownian motion.
Let $c_1 - c_0 \to \infty$, $\lambda_0, \lambda_1 \to \infty,\ h_0, h_1 \to 0$ and

$$\frac{c_1 - c_0}{\sqrt{\lambda_1} + \sqrt{\lambda_0}} \to \sigma, \qquad \sqrt{\frac{\lambda_1}{\lambda_0}} \to \gamma, \qquad \sqrt{\lambda_i} h_i \to \alpha_i, \qquad (10)$$

$$\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_0}}(c_0 + \lambda_0 h_0) + \frac{\sqrt{\lambda_0}}{\sqrt{\lambda_1} + \sqrt{\lambda_0}}(c_1 + \lambda_1 h_1) \to \delta. \qquad (11)$$

Convergence to Black-Scholes (2)

Theorem 5. *Under the scaling conditions (10)-(11) model (4) converges to the Black-Scholes:*

$$S(t) \xrightarrow{D} S_0 \exp\{vw(t) + (\delta - \beta^2/2)t\},$$

where $v = \sqrt{(\sigma + (\gamma\alpha_1 - \alpha_0)/(1+\gamma))^2 + \beta^2}$ *and* $\beta^2 = \frac{\alpha_1^2 + \gamma\alpha_0^2}{1+\gamma}$.

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Remark. *Under the martingale measure* $\mathbb{P}^*$ *transition intensities take a form* $-c_i/h_i$ *(if* $r_i = 0$*). Thus the drift vanishes,* $\Delta = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}(c_0 + \lambda_0 h_0) + \frac{\sqrt{\lambda_0}}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}(c_1 + \lambda_1 h_1) = 0$*. Moreover, in this case* $\sigma = \lim \frac{c_1 - c_0}{\sqrt{\lambda_1}+\sqrt{\lambda_0}} = -\lim \frac{\lambda_1 h_1 - \lambda_0 h_0}{\sqrt{\lambda_1}+\sqrt{\lambda_0}} = -\frac{\gamma\alpha_1 - \alpha_0}{1+\gamma}$*. The limiting volatility* v *in this case coincides with* β*:* $v = \beta = \sqrt{\frac{\alpha_1^2+\gamma\alpha_0^2}{1+\gamma}}$.

Convergence to Black-Scholes (3)

Remark. *Condition (11) in this theorem means that the total drift* $\Delta \equiv A + \frac{\sqrt{\lambda_1\lambda_0}}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}(\sqrt{\lambda_1}h_1 + \sqrt{\lambda_0}h_0)$ *is asymptotically finite. Here* $A = \frac{\sqrt{\lambda_0}c_0+\sqrt{\lambda_1}c_1}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}$ *is generated by the velocities of the telegraph process, and the summand* $\frac{\sqrt{\lambda_1\lambda_0}}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}(\sqrt{\lambda_1}h_1 + \sqrt{\lambda_0}h_0)$ *represents the drift component (possibly with infinite asymptotics) that is motivated only by jumps. If here the limits of* $\lambda_i h_i$ *are finite, then* $A \to const$*, and* $\alpha_1 = \alpha_0 = 0$*. In this case the volatility of limit is* $v = \sigma = \lim(c_1 - c_0)/(\sqrt{\lambda_1} + \sqrt{\lambda_0})$*.*

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Hence in jump telegraph model value $(c_1 - c_0)/(\sqrt{\lambda_1} + \sqrt{\lambda_0})$ *can be interpreted as "telegraph" component of volatility, and* $\sqrt{\lambda_i}h_i$ *are volatility components engendered by jumps.*

Convergence to Black-Scholes (4)

In general, the limiting volatility $v = \sqrt{\left(\sigma + (\gamma\alpha_1 - \alpha_0)/(1+\gamma)\right)^2 + \beta^2}$ depends both on "telegraph" and jump components.

Convergence to Black-Scholes (4)

In general, the limiting volatility $v=\sqrt{(\sigma+(\gamma\alpha_1-\alpha_0)/(1+\gamma))^2+\beta^2}$ depends both on "telegraph" and jump components.

So it is natural to define volatility of jump telegraph market as

$$\mathrm{vol}^2=\left(\frac{c_1-c_0}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}\right)^2\left(1+\frac{\lambda_1 h_1-\lambda_0 h_0}{c_1-c_0}\right)^2 +\frac{\sqrt{\lambda_0}\lambda_1 h_1^2+\sqrt{\lambda_1}\lambda_0 h_0^2}{\sqrt{\lambda_1}+\sqrt{\lambda_0}}.$$

Pricing call options (1)

According to the theory on option pricing, we have

$$\mathbb{c}^i = \mathbb{E}_i^* \left[B_i(T)^{-1} (S_i(T) - K)^+ \right],$$

where K is the strike price and $\mathbb{E}_i^*(\cdot)$ is the expectation with respect to the martingale measure $\mathbb{P}_i^*$. In case of the model (4)-(5), one can rewrite $\mathbb{c}^i$ as

$$\mathbb{c}^i = S_0 U^{(i)}(y,T) - K u^{(i)}(y,T), \qquad i = 0, 1 \tag{12}$$

Pricing call options (2)

with

$$u^{(i)}(y,T) = \sum_{n=0}^{\infty} u_n^{(i)}(y - b_n^{(i)}, T),$$

$$U^{(i)}(y,T) = \sum_{n=0}^{\infty} U_n^{(i)}(y - b_n^{(i)}, T),$$

where $y = \ln K/S_i(0)$, $b_n^{(i)} = \sum_{j=1}^{n} \ln(1 + h_{\sigma_i(\tau_j -)})$, and functions $u_n^{(i)}, U_n^{(i)},\ n \geq 0, i = 0, 1$ can be directly calculated.

Pricing call options (3)

First, notice

$$u_n^{(i)}(y,t) = \mathbb{E}_i^* \left[B_i(t)^{-1} \mathbf{1}_{\{X_i(t)>y, N_i(t)=n\}} \right],$$

$$U_n^{(i)}(y,t) = \mathbb{E}_i^* \left[B_i(t)^{-1} \mathcal{E}_t(X_i + J_i) \mathbf{1}_{\{X_i(t)>y, N_i(t)=n\}} \right],$$

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and $U_n^{(i)}(y,\ t;\ \lambda_i^*,\ c_i,\ r_i) = u_n^{(i)}(y,\ t;\ \lambda_i^*(1+h_i),\ c_i,\ 0)$.

Pricing call options (4)

Finally, these functions can be calculated as

$$u_n^{(i)} = \begin{cases} 0, & y > c_1 t, \\ w_n^{(i)}(p,\ q), & c_0 t \le y \le c_1 t, \\ \rho_n^{(i)}(t), & y < c_0 t, \end{cases} \qquad i = 0, 1,$$

$p = \frac{c_1 t - y}{c_1 - c_0}$, $q = \frac{y - c_0 t}{c_1 - c_0}$, where

$w_n^{(i)} = \mathrm{e}^{-(\lambda_1^* + r_1)q - (\lambda_0^* + r_0)p} \Lambda_n^{(i)} v_n^{(i)}(p,\ q)$,
$\rho_n^{(i)}(t) = \mathrm{e}^{-(\lambda_0^* + r_0)t} \Lambda_n^{(i)} P_n^{(i)}(t)$ with
$\Lambda_n^{(i)} = (\lambda_i^*)^{[(n+1)/2]} (\lambda_{1-i}^*)^{[n/2]}$, $i = 0, 1,\ n \ge 0.$

Pricing call options (5)

Functions $P_n^{(i)}$ and $v_n^{(i)}$ are defined as follows:
$P_0^{(1)} = \mathrm{e}^{-at}, P_0^{(0)} \equiv 1,$
$P_n^{(i)} = P_n^{(i)}(t) = \frac{t^n}{n!}\left[1 + \sum_{k=1}^{\infty} \frac{(m_n^{(i)}+1)_k}{(n+1)_k} \cdot \frac{(-at)^k}{k!}\right], i = 0, 1,$ where
$m_n^{(1)} = [n/2]\, , m_n^{(0)} = [(n-1)/2]$,
$(m)_k = m(m+1)\ \ldots\ (m+k-1)$, $a = \lambda_1^* - \lambda_0^* + r_1 - r_0$;

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$(m)_k = m(m+1)\ \ldots\ (m+k-1)$, $a = \lambda_1^* - \lambda_0^* + r_1 - r_0$;

$v_0^{(0)} \equiv 0$, $v_0^{(1)} = \mathrm{e}^{-ap},\ \ v_1^{(i)} = P_1(p)$ and for $n \geq 1$

$$v_{2n+1}^{(i)} = v_{2n+1}^{(i)}(p,\ q) = P_{2n+1}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k,n}(p),$$

$$v_{2n}^{(0)} = v_{2n}^{(0)}(p,\ q) = P_{2n}^{(0)}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k+1,n}(p),$$

$$v_{2n}^{(1)} = v_{2n}^{(1)}(p,\ q) = P_{2n}^{(1)}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k-1,n-1}(p).$$

Pricing call options (6)

Here $\varphi_{0,n} = P_{2n+1}$,

$$\varphi_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P^{(-)}_{2n-j}, \; 1 \le k \le n,$$

where $\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}$.

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where $\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}$.
In particular case $\lambda_1^* = \lambda_0^* = \lambda, \; r_1 = r_0 = r$ these functions have a more simple form $\rho_n^{(i)}(t) = \mathrm{e}^{-(\lambda+r)t} \frac{(\lambda t)^n}{n!}$,
$w_n^{(i)} = \mathrm{e}^{-(\lambda+r)t} \frac{\lambda^n}{n!} \sum_0^{m_n^{(i)}} \begin{pmatrix} n \\ k \end{pmatrix} q^k p^{n-k}$. Here
$m_n^{(1)} = [n/2] \, , m_n^{(0)} = [(n-1)/2]$.

Memory effects and historical volatility

Historical volatility is defined as

$$\mathrm{HV}(t) = \sqrt{\frac{\mathrm{Var}\{\log S(t+\tau)/S(\tau)\}}{t}}. \qquad (13)$$

For classical Black-Scholes model $\log S(t+\tau)/S(\tau) \overset{D}{=} at + vw(t)$ the historical volatility is constant: $\mathrm{HV}_{\mathrm{BS}}(t) \equiv v$.

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In a moving-average type model

$$\log S(t)/S(0) = at + vw(t) - v\int\limits_0^t \mathrm{d}\tau \int\limits_{-\infty}^{\tau} p\mathrm{e}^{-(q+p)(\tau-u)}\mathrm{d}w(u),$$

where $v,\ q,\ q+p > 0$ the historical volatility is

Memory and historical volatility (2)

$$\mathrm{HV}(t) = \frac{\sigma}{2\lambda}\sqrt{q^2 + p(2q+p)\Phi_\lambda(t)}$$

with $2\lambda = q + p$ and $\Phi_\lambda(t) = \frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda t}$. Recently this type of models have been applied to capture memory effects of the market.

Memory and historical volatility (2)

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with $2\lambda = q + p$ and $\Phi_\lambda(t) = \frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda t}$. Recently this type of models have been applied to capture memory effects of the market.
Historical volatility in the jump telegraph model (in particular case $\lambda_0 = \lambda_1 = \lambda$) is

$$\mathrm{HV}_i(t) = \sqrt{\sigma^2 + \kappa^2\Phi_{2\lambda}(t)/\lambda + \gamma_i\Phi_\lambda(t) - 2B\kappa(-1)^i\mathrm{e}^{-2\lambda t}}$$

with $\sigma^2 = a^2/\lambda + \lambda B^2$, $\kappa = a + \lambda b$, $\gamma_i = -2a(\kappa - (-1)^i\lambda B)/\lambda$.
Here $b = \frac{1}{2}\ln\frac{1+h_1}{1+h_0}$, $B = \frac{1}{2}\ln(1+h_1)(1+h_0)$, $c = (c_1 - c_0)/2$, $a = (c_1 + c_0)/2$.

In general, we have the following limits

$$\lim_{t\to 0} \mathrm{HV}_i(t) = \sqrt{\lambda_i}\ln(1+h_i),$$

$$\lim_{t\to\infty} \mathrm{HV}_i(t) = \sqrt{\frac{\lambda_1\lambda_0}{2\Lambda^3}[(\lambda_0 B - c)^2 + (\lambda_1 B + c)^2]}$$

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These limits look reasonable: the limit at 0 is engendered by jumps only, the limit at ∞ contains both “velocity" component and a long term influence of jumps.

Memory and historical volatility(3)

The limits of historical volatility under a standard diffusion scaling are more complicated. Nevertheless, in the symmetric case $\lambda_1 = \lambda_0 = \lambda$, we have under the scaling conditions $\lambda,\ a \to \infty,\ h_i \to 0,\ a^2/\lambda \to \sigma^2,\ \sqrt{\lambda} h_i \to \alpha_i$ that the historical volatility $\mathrm{HV}_i(t)$ converges to $\sqrt{\sigma^2 + (\alpha_1 + \alpha_0)^2/4}$.

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Notice, that under the martingale measure $\mathbb{P}^*$, we have $\lambda = -c_i/h_i$, $\sigma = (-\alpha_1 + \alpha_0)/2$, and the diffusion limit of historical volatility equals to $v = \sqrt{(\alpha_1^2 + \alpha_0^2)/2}$, which coincides with the volatility expression for the diffusion scaling.

Implied volatility

Define the Black-Scholes call price function $f(\mu, v)$, $\mu = \log K$ by

$$f(\mu, v) = \begin{cases} F\left(\frac{-\mu}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - \mathrm{e}^{\mu} F\left(\frac{-\mu}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right), & \text{if } v > 0, \\ (1 - \mathrm{e}^{\mu})^{+}, & \text{if } v = 0. \end{cases}$$

The processes $V_i(\mu, t),\ t \geq 0,\ \mu \in \mathbb{R}$ defined by the equation

$$\mathbb{E}\left[(S(t+\tau)/S(\tau) - \mathrm{e}^{\mu})^{+} | \mathcal{F}_{\tau}^{(i)}\right] = f(\mu,\ V_{\sigma_i(\tau)}(\mu, t))$$

are referred to as implied variance processes.

Implied volatility is Markov-modulated

The implied volatilities $\mathbb{IV}_i(\mu, t)$ are defined as

$$\mathbb{IV}_i(\mu, \tau, t) = \sqrt{\frac{V_{\sigma_i(\tau)}(\mu, t)}{t}}.$$

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$$\mathbb{IV}_i(\mu, \tau, t) = \sqrt{\frac{V_{\sigma_i(\tau)}(\mu, t)}{t}}.$$

Notice that

$$\mathbb{IV}_i(\mu, \tau, t) = \mathbb{IV}_{\sigma_i(\tau)}(\mu, 0, t).$$

So the implied volatility is Markov-modulated, but it does not move in parallel shifts.

Numerical results

We performed the numerical valuation of the jump telegraph volatility and the historical volatility, which are compared with the implied volatilities with respect to different moneyness and to the initial market states.

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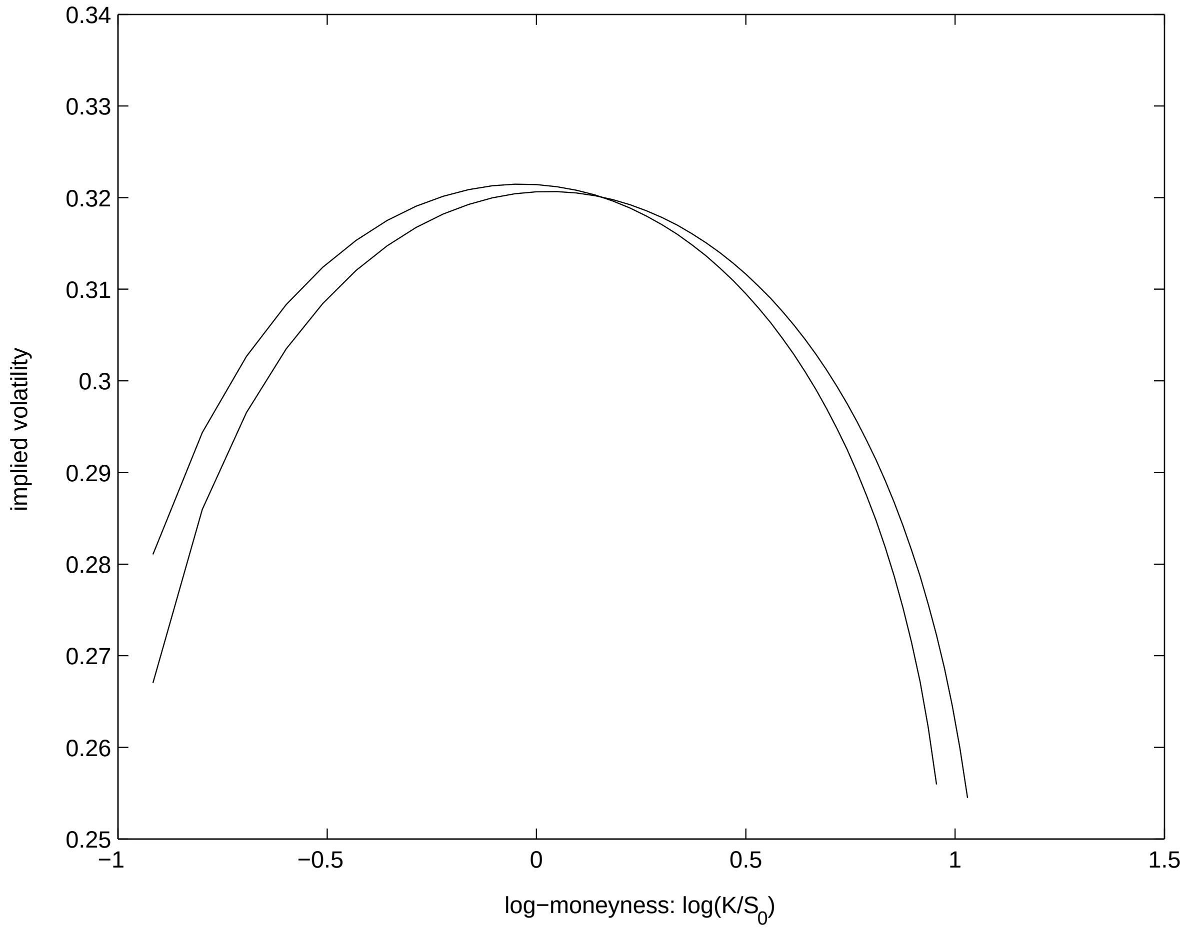
We performed the numerical valuation of the jump telegraph volatility and the historical volatility, which are compared with the implied volatilities with respect to different moneyness and to the initial market states.

We assume $S_0 = 100, T = 1$.

First, consider the symmetric case: $\lambda_i = 10, c_i = \pm 1$ and $h_i = \mp 0.1$.
The results are the following: $\mathrm{HV}_0 = \mathrm{HV}_1 = 0.3162$, jump telegraph volatility=0.3162.
Notice that these frowned smiles of implied volatilities IV_0 and IV_1 intersect at $K/S_0 = 1.17$.

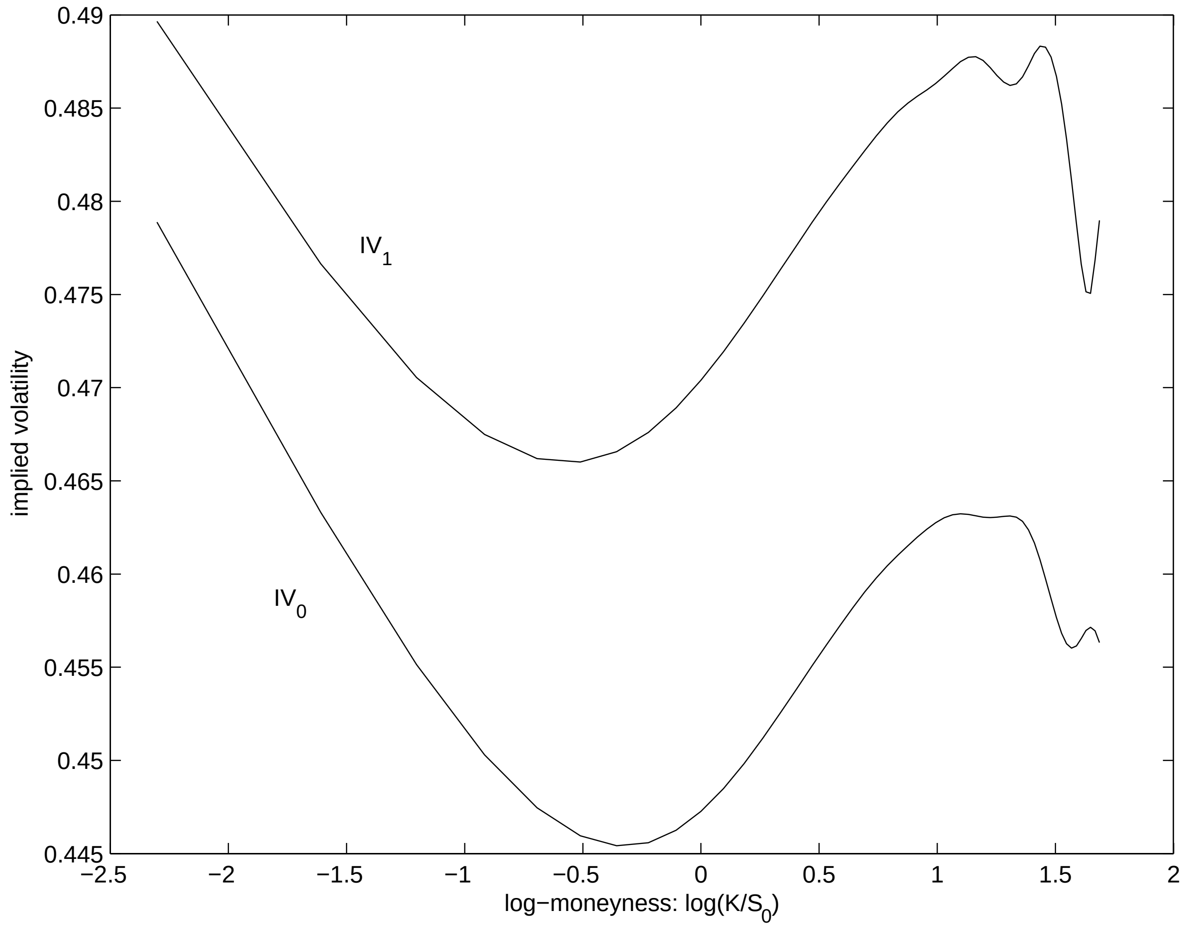
Volatility smile (1)



$$\lambda_i = 10, c_i = \pm 1, h_i = \mp 0.1$$

Volatility smile (2)

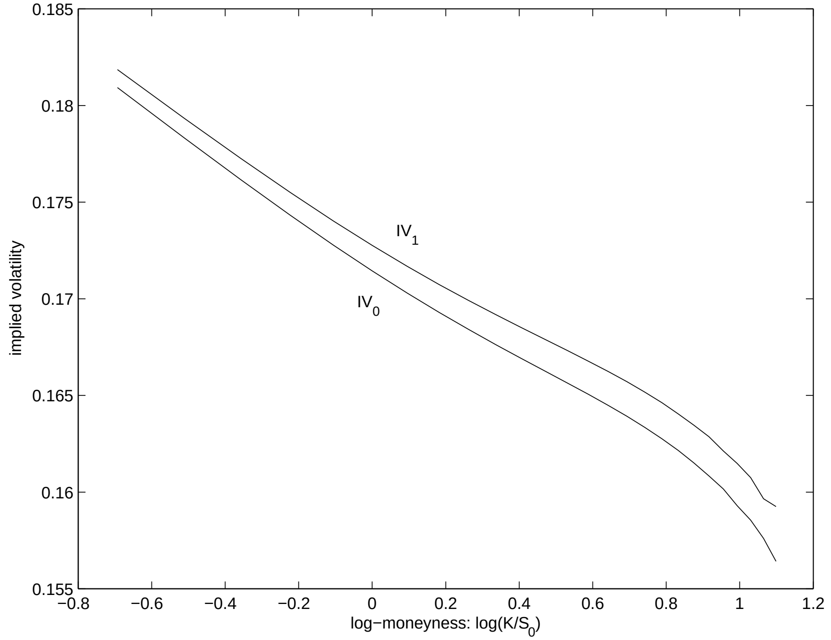
$\mathrm{HV}_0 = 0.4198, \mathrm{HV}_1 = 0.4402$; tel. volatility=0.4301



$r = 0,\ c_0 = 0.3,\ c_1 = 1.9,\ \lambda_i = 10, h_0 = -0.03, h_1 = -0.19$

Volatility smile (3)

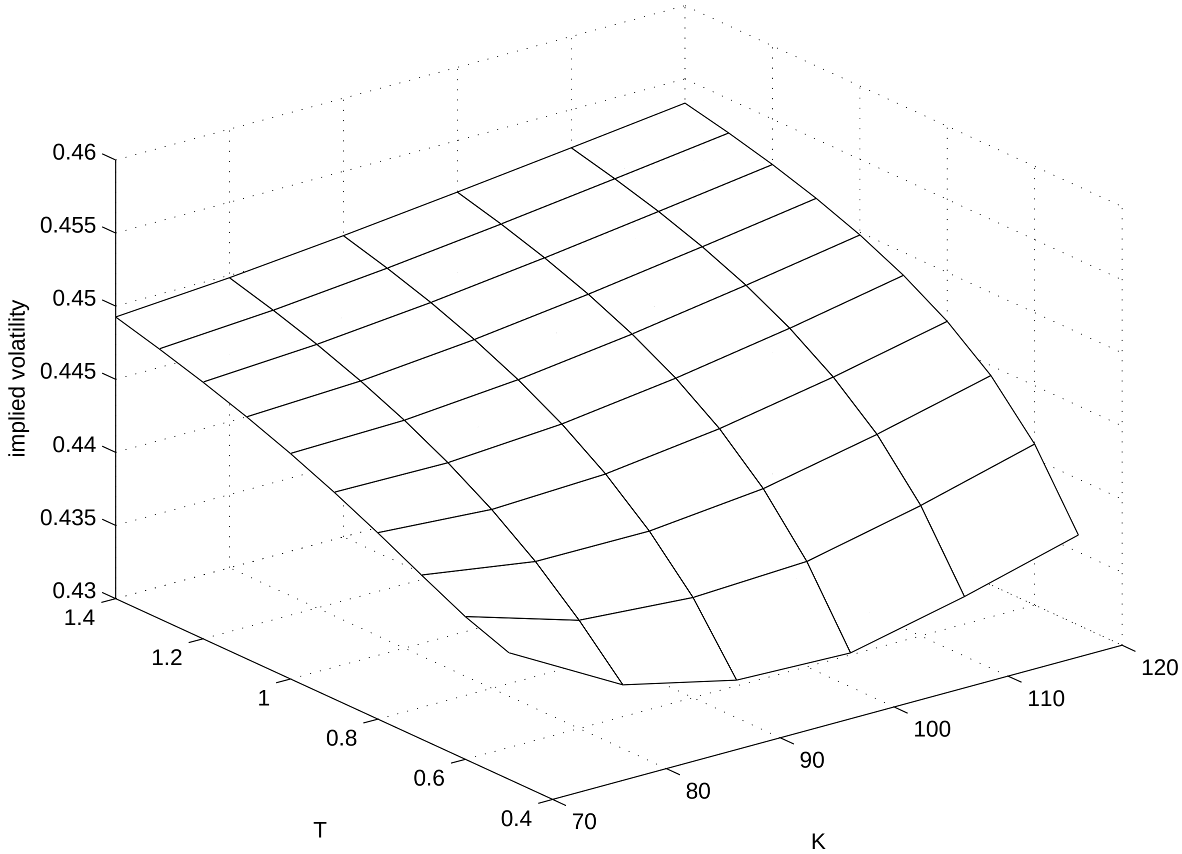
Dow-Jones industrial average July 1971-Aug 1974



$\lambda_0 = 48.53, \lambda_1 = 34.61, h_0 = -0.0126, h_1 = -0.0358$,
$c_0 = 0.61, c_1 = 1.24;\ \mathrm{HV}_0 = 0.1630, \mathrm{HV}_1 = 0.1642$;
jump telegraph volatility=0.1661

Volatility smile (4)

$\mathbb{IV}_0$



$$\lambda_i = 10, h_0 = -0.03, h_1 = -0.19, c_0 = 0.3, c_1 = 1.9$$

Problems and perspectives

- (a) Velocities move through a binary tree (as in CRR-model);
  (b) Calibration of the parameters of jump telegraph model according to real market data;
  (c) inhomogeneous case:
  $c_i = c_i(x,t),\ \lambda_i = \lambda_i(x,t),\ h_i = h_i(x,t)$

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  (c) inhomogeneous case:
  $c_i = c_i(x,t),\ \lambda_i = \lambda_i(x,t),\ h_i = h_i(x,t)$
- Jump and/or velocity values are *random*. In this case the model is incomplete

Problems and perspectives

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Thank you for your kind attention!

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