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The mystery of 2-dimensional ideal incompressible fluid

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I. Setup: the Euler equations

$M \subset \mathbf{R}^n$ – the flow domain filled with ideal incompressible fluid;
$u(x,t)$ – velocity field; $p(x,t)$ – pressure.

Euler equations:

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0;$$
$$\nabla \cdot u = 0;$$
$$u_n|_{\partial M} = 0.$$

Initial conditions:

$$u(x,t)|_{t=0} = u_0(x).$$

Theorem (Lichtenstein, Giunter, Wolibner, Kato, Yudovich, ...)
(i) If $u_0 \in C^{1+\alpha}$, then there exists unique solution $u(x,t) \in C^{1+\alpha}$ for $|t| < T$, where T may depend on u_0;
(ii) If $n = 2$, $T = \infty$.

Question: If $n = 2$, what happens with the flow $u(x,t)$ as $t \to \infty$?

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II. Inverse cascade: 2-d flow at different moments of time

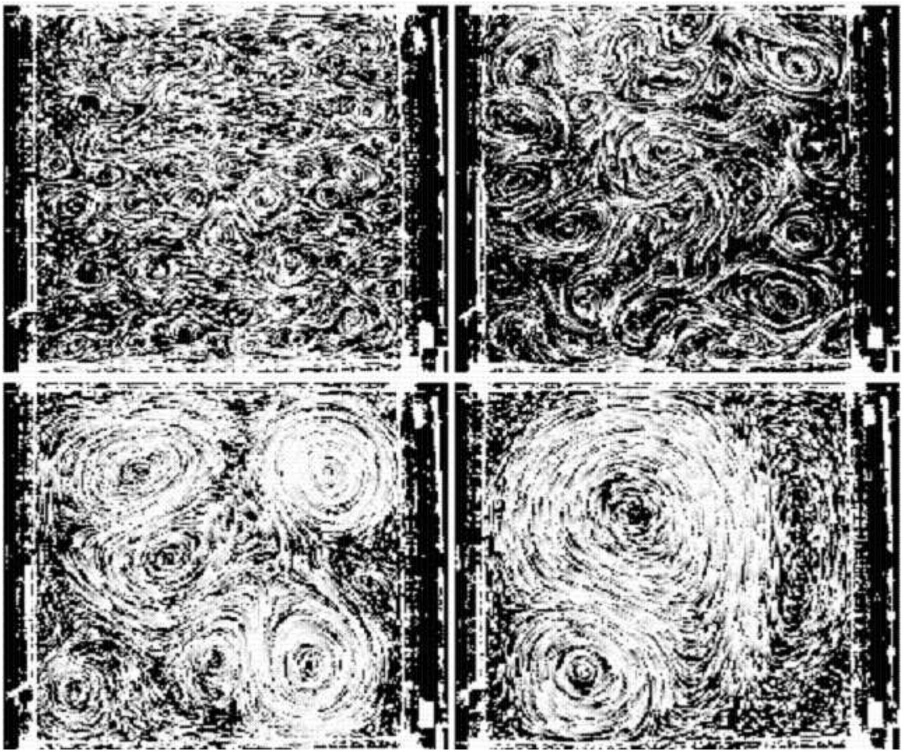

Figure 1: The velocity field of a 2-d flow at 4 consecutive moments; the scale of the flow is growing

How to explain this picture?
Is it possible to deduce this phenomenon from the Euler equations?

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III. Vorticity equation

$$\omega = \text{curl } u$$

For any $\omega(x)$ $\exists!$ $u = \text{curl}^{-1}(\omega)$, such that
$\omega = \text{curl } u$, $\nabla \cdot u = 0$, $u_n|_{\partial M} = 0$.

Vorticity transport equation:

$$\frac{\partial \omega}{\partial t} + (u, \nabla)\omega = 0;$$

Hence,

$$\omega(x, t) = \omega(g_t^{-1}(x), 0),$$

where $g_t : M \to M$ is the flow map.

Vorticity ω is transported by the flow, distorted and, generally, irreversibly mixed.

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IV. Mixing operators in $L^2(M)$

(bistochastic operators, polimorphisms (Vershik)).

$$K\varphi(x) = \int\limits_M K(x,y)\varphi(y)dy,$$

where the kernel $K(x,y)$ satisfies the following conditions:

$$\begin{aligned} &(1) \quad K(x,y) \geq 0; \\ &(2) \quad \int\limits_M K(x,y)dx \equiv 1; \\ &(3) \quad \int\limits_M K(x,y)dy \equiv 1. \end{aligned}$$

Examples: (1) $K(x,y) = \delta(y - g^{-1}(x)$, where $g : M \to M$ is a volume preserving diffeomorphism;
(2) $K(x,y) \equiv 1$.

$\mathcal{K} = \{K\}$ is a convex, weakly compact semigroup of contractions in $L^2(M)$. Hence, it defines in $L^2(M)$ a partial order.

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$V^s = \{u \in H^s(m) | \nabla \cdot u = 0,\ u_n|_{\partial M} = 0\}.$

Definition. *(1) If $\varphi_1, \varphi_2 \in L^2$, then we say that $\varphi_1 \prec \varphi_2$, if there exists a mixing operator $K \in \mathcal{K}$ such that $\varphi_1 = K\varphi_2$;*

(2) If $u_1, u_2 \in V^1$, we say that $u_1 \prec u_2$, if curl $u_1 \prec$ curl u_2.

For any $u_0 \in V^1$ define

$$\Omega_{u_0} = \Big\{ u \in V^1 \ \Big| \ u \prec u_0,\ ||u||_{L^2} = ||u_0||_{L^2} \Big\}.$$

If $u(t)$ is a solution of the Euler equations,
$u(0) = u_0$, then $\overline{\{u(t)\}} \subset \Omega_{u_0}$ (closure in V^0).

Definition. *Minimal elements of Ω_{u_0} (w.r.t. the order relation $\prec$) are called* **minimal flows**.

Minimal flows exist by the Zorn Lemma.

Theorem. *(1) Every minimal flow $w(x)$ is a steady and stable solution of the Euler equations.*

(2) If $\psi(x)$ is the stream function for w, i.e. $w_1 = \partial\psi/\partial x_2$, $w_2 = -\partial\psi/\partial x_1$, then

$$\text{curl } w = \Delta\psi = F(\psi)$$

for some monotone function F.

Note. It is not known whether any steady and stable in V^1 flow is minimal. Hence the following hypotheses.

Conjecture 1. *The set $\mathcal{S} \subset V^1$ of steady stable flows is attracting for typical solutions $u(t)$ of (E).*

Conjecture 2. *The set $\mathcal{M} \subset V^1$ of minimal flows is attracting for typical solutions $u(t)$ of (E).*

Conjecture 3. *For a typical solution $u(t) \in V^1$ there exists $w \in \mathcal{M}$ s.t. $||u(t) - w||_{L^2} \to 0$ as $t \to \infty$.*

V. Existence of the inverse cascade solution of the Euler equations

We don't know even a single example of a classical solution of (E) behaving like the flow on the picture. But we can construct a **weak** solution of (E) having the inverse cascade property in the strongest possible sense: the initial scale of the flow is **zero**.

Definition. *Vector field* $u(x,t) \in L^2$ *is called a* **weak solution** *of (E), if for any test-field* $v(x,t) \in C_0^\infty$, $\nabla \cdot v = 0$, *and any scalar test-function* $\varphi(x,t) \in C_0^\infty$,

$$\int\int \Big[(u, \frac{\partial v}{\partial t} + (u, \nabla v \cdot u)\Big] dxdt = 0;$$

$$\int\int (u, \nabla_x \varphi) dxdt = 0.$$

Theorem. *There exists a weak solution* $u(x,t) \in L^2$ *of (E) such that* $u(x,t) \rightharpoonup 0$ *as* $t \to 0$ *weakly in* L^2, *while* $||u(\cdot,t)||_{L^2} = \text{const} > 0$.

VI. 2-dimensional fluid as a dynamical system

In fact, there are two dynamical systems: a system in the space V^s, defined by the Euler equations, and the dynamical system in the space $\mathcal{D}$ of volume-preserving diffeomorphisms of M, defined by the vorticity equations

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\begin{aligned}
&\dot{g}_t(x) = u(g_t(x));\\
&\operatorname{curl} u(y) = \omega(g_t^{-1}(y));\\
&\dot{\omega} = 0.
\end{aligned}
$$

We consider the last system.

Liapunov function. Consider a dynamical system in some phase space X, defined by the equation $\frac{dx}{dt} = f(x),\ x \in X$.

Definition. *A function $\lambda(x)$, defined and continuous everywhere in X, is called a Liapunov function, if it is growing along any trajectory, i.e. $\frac{d\lambda}{dt} \geq 0$, and $\frac{d\lambda}{dt} > 0$ "almost everywhere" in X.*

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Examples. (1) There exist no Liapunov function for the harmonic oscillator.
(2) For a free particle in $\mathbf{R}^3$ there exists a Liapunov function $\lambda(x,v) = (x,v)$, because $\frac{d\lambda}{dt} = (v,v) \geq 0$.

Existence of a Liapunov function for a conservative mechanical system means that this system cannot be in the statistical equilibrium.

Theorem. *There exists a Liapunov function $\lambda(\omega, g)$ for the 2-dimensional ideal incompressible fluid described by the vorticity equations.*

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Construction of the Liapunov function. If $M = \mathbf{T}^2$, then

$$\lambda(\omega, g) = \varphi\Big(\Big[(\mathrm{curl}\, T_{g'^{-1}}(g - Id), \omega)_{ml}\Big]\Big),$$

where

$T_f h$ – paraproduct;

$(\cdot, \cdot)_{ml}$ – microglobal scalar product (measure on T^*M);

$[\nu]$ – germ of the measure $\nu(dx d\xi)$ on T^*M at infinity;

$\varphi(\cdot)$ – linear functional on germs of measures at infinity, positive on positive measures.

This Liapunov function describes the monotone growth of weak singularities of the flow map g_t. This is a rudimentary form of the mixing of vorticity.

Problem: Does there exist a more "physical" Liapunov function for the fluid?

VII. The entropy problem.

The picture of the inverse cascade in 2-d fluid reminds a machine containing countable number of wheels, connected with gears, chains, springs, etc., without any friction, having the following property. If at $t = 0$ any finite number of wheels is set into motion, then all other wheels begin to move, but eventually, as $t \to \infty$, all the energy accumulates in the first few wheels, while all other parts come to rest. (Such Machine can be really "constructed"; the difficult part is to prove that it works as intended.)

Consider an ensemble of initial velocities of the wheels defined by the measure $\mu_0(dv_1 dv_2 \ldots)$, where v_i is the rotation speed of the i-th wheel. Let H_0 be its entropy. The entropy of the final distribution H_∞ is clearly less than H_0, because the final distribution is concentrated on a small set. **Where does the entropy go?**

Answer: There is the second set of variables, the angles, and the entropy of velocities is transformed into the entropy of angles (the configurational entropy).

It'd be interesting to imagine the above Machine in more detail. Here is the sketch of a possible design of the Machine. We use elastic balls of different size instead of wheels. In fact, the Machine is a sequence of heat engines H_n; for any engine H_n, the balls in the previous engine H_{n-1} play the role of a load, while the next engine H_{n+1} is a cooler.

Another system with the velocity entropy transforming into the configuration entropy is the ideal gas of noninteracting particles expanding freely into the vacuum. As a consequence its temperature decreases:
Again, the volume in the configuration space, occupied by the system, grows, while the volume in the velocity space decreases.

For the 2-d fluid, the configuration space is the group $\mathcal{D}(M)$ of area-preserving diffeomorphisms of M. This is an infinite-dimensional Riemannian manifold: If g_t, $0 \le t \le 1$ is a curve in $\mathcal{D}(M)$, then its length

$$L\{g_t\}_0^1 = \int_0^1 \Big(\int_M (\dot{g}_t(x), \dot{g}_t(x)) dx \Big)^{1/2} dt;$$

For any $g,\ h \in \mathcal{D}(M)$, the distance

$$\operatorname{dist}(g,h) = \inf_{\substack{g_0=g \\ g_1=h}} L\{g_t\}_0^1;$$

Diameter

$$\operatorname{diam}(\mathcal{D}(M)) = \sup_{g,h\in\mathcal{D}(M)} \operatorname{dist}(g,h).$$

Theorem (Eliashberg-Ratiu). *If* $\dim M = 2$, *then* $\operatorname{diam}(\mathcal{D}(M)) = \infty$.
So, there is enough space in $\mathcal{D}(M)$ to absorb any amount of entropy. However, how it is done, remains a mystery.

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VIII. Conclusions

1. 2-dimensional ideal incompressible fluid is a system extremely far from equilibrium.

2. The basic property of its motion is the irreversible transformation of the velocity entropy into the configuration entropy, which is the feature of different strongly nonequilibrium systems.

3. The inverse cascade is a visible manifestation of the above transformation.