A Useful Renormalization Argument

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We define a collection of 'generic' population models for which we prove a survival criterion by using a renormalization argument. These models can be compared with other more familiar models, leading to simple proofs of various survival results. In particular, we prove a generalization of Toom's Theorem concerning survival in multidimensional probabilistic cellular automata. Our technique should also be applicable to a variety of other discrete and continuous time models.

0. Introduction. Consider the discrete time version of the basic asymmetric one-dimensional contact process. In this model, one imagines that a population of particles is located at the sites (points) of the integer lattice $\mathbf{Z}$, with at most one particle per site. A site containing a particle is called **occupied**, and a site with no particles is called **vacant**. During each discrete time step, the population changes according to a two step procedure: first, a new particle is placed at each vacant site x for which the neighboring site $x+1$ is occupied; second, occupied sites (including newly occupied sites) are vacated with probability ε, independently of one another. Thus, the probability that x is occupied at time t, given the past, is $1-\varepsilon$ if x or $x+1$ is occupied at time $t-1$, and 0 otherwise. The parameter ε is called the **death rate**. This describes a simple model for a population of individuals with geometrically distributed lifetimes and with births occurring only at vacant sites immediately to the left of (in 'contact' with) an occupied site. (The name 'contact process' was first applied to a continuous time version of this model, with the population under consideration being a collection of diseased cells. For simplicity, we will restrict this introductory discussion mostly to discrete time models. The basic asymmetric one-dimensional discrete time contact process is equivalent to oriented site percolation on $\mathbf{Z}^2$.)

If the death rate ε is close to 1, it is easy to show that even if all the sites are occupied at time 0, the population will die out, in the sense that the probability any given site is occupied will converge to 0 as $t \to \infty$. On the other hand, it was shown by Stavskaya and Pyatetskii-Shapiro [12] that if $\varepsilon > 0$ is sufficiently small, the population does not die out if it starts with all sites occupied. Instead, there is a non-trivial equilibrium for the process

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in which the probability that a given site is occupied is strictly positive. We describe this situation by saying that the contact process **survives** for all sufficiently small death rates.

A large proportion of the work done in the field of interacting particle systems has been concerned with the question of whether a given parametrized family of processes survives at certain parameter values. It is interesting that between 1968 and 1988, with the notable exception of the work of Holley and Liggett [10] concerning the one-dimensional contact process in continuous time, virtually all work concerned with showing survival relied on a common technique, the so-called contour method. This method was first introduced in a statistical mechanics setting by Peierls [11], much later developed to a very high level for discrete time systems by Toom [13], adapted to the continuous time setting for the contact process by Harris [9], and then developed for more general use in continuous time by Gray and Griffeath [8], Bramson and Gray [4], and Durrett and Gray [5].

In the work of Durrett and Griffeath [6], another idea borrowed from statistical mechanics (and percolation theory) made its way into the field of interacting particle systems, namely the block rescaling method. Durrett and Griffeath used this technique to get more detailed information about the survival probabilities of the contact process.

Block rescaling is a powerful tool for examining the survival of a large class of processes. Many processes can be compared with the contact process by using this method, thus allowing a relationship to be established between the survival of the contact process and the survival of these other processes. This technique was first applied in Bramson [1], Bramson and Durrett [2], and Bramson, Durrett, and Swindle [3]. In particular, this procedure allows one to prove survival for interesting models that were too complicated to yield to contour methods.

In the rescaling procedure used in [1], [2], and [3], the discrete time contact process (or at least a close relative) plays a special role. The point is that the contact process is a relatively simple process whose behavior is fairly well understood, and it is possible to find transformations for many other processes that allow them to be compared with the contact process. However, not every process that survives can be compared in this fashion with the contact process. An example is the discrete time process known as **Toom's model**. We describe one version of this model here. It is similar to the discrete time contact process, with two important differences. The first difference is that it is a two-dimensional model, that is, the sites are the points of the two-dimensional integer lattice $\mathbf{Z}^2$. Again, there is a two stage procedure for changing the state of the system from one time to the next. The second stage of this procedure is the same as before (vacate occupied sites with probability ε), but during the first stage, a new particle appears at a vacant site (x, y) if and only if *both* of the two neighboring sites $(x+1, y)$ and $(x, y+1)$ are occupied by particles. (As before, a given particle disappears with probability ε during any time step.) Thus it takes at least two particles, properly positioned, to produce a new particle at a vacant site, which is why the continuous time version of this model is sometimes said to have sexual reproduction, to distinguish it from the contact process in which reproduction is asexual. There is no known way to prove that Toom's model survives by comparing it to a basic contact process. Note that if we change the rule to 'a vacant site is occupied if and only if *at least one* of the two neighboring sites $(x+1, y)$ or $(x, y+1)$ is occupied', then it is easy to make a direct comparison to the asymmetric contact process in one dimension to prove survival for small death rates. This 'at least one' model is a two-dimensional contact process, with asexual reproduction. Like the one-dimensional contact process, it can be used as a comparison to many models, but not to Toom's model. Toom [13] used contour methods to prove survival for Toom's model, and Durrett and Gray [5] used contour methods to prove this and other interesting facts about the continuous time version. There are several variations on Toom's model in which more symmetry exists, but they all behave similarly, and none of them seems to be comparable in any useful way to basic contact processes.

One can imagine that Toom's model could be used as a comparison process in the same way as the basic contact process. That is, one should be able to find other processes whose survival can be proved by transforming them into Toom's model, using some version of the block rescaling technique. There are still other processes which survive, but which can neither be transformed into Toom's model, nor into the basic contact process. In fact many (but not all!) of the higher dimensional processes in the general class studied by Toom appear to be not comparable with the one- or two-dimensional models.

In this paper, we introduce a class of **generic population models** which can be used as comparison processes via the block rescaling technique. These models have two desirable properties. The first is that they form a sufficiently general class so that many interesting attractive systems that survive can be compared to them by an appropriate application of the block rescaling technique. In particular, we will be able to use them to give a much simpler proof of Toom's Theorem [13] concerning the survival of multidimensional discrete time systems. The second property of these generic models is that they lend themselves to *renormalization*, which is to say that when the block rescaling argument is applied to them in a natural way, they are transformed into themselves. Once this is done, an induction argument then leads to a necessary and sufficient criterion for the survival of generic models. Thus, the block rescaling technique accomplishes two purposes: it transforms models of interest into generic processes, and it then tells us which generic processes survive. In previous work, this second step has usually required a (sometimes quite difficult) contour argument.

Here is an overview of the rest of the paper. In Section 1 we construct the generic population models and indicate briefly through some examples why they work well as comparison processes. It will be seen that they are temporally *and spatially* continuous analogues of Toom's general class of probabilistic cellular automata. In Section 2, we carry out the renormalization procedure and establish necessary and sufficient conditions for the survival of generic processes (Theorem 1). In Section 3, we prove that a general class of processes, which includes the class studied by Toom, can be compared to generic processes (Theorem 2). As a consequence, we immediately get a generalization of Toom's Theorem (the Corollary to Theorem 2). This generalization does not seem to be amenable to contour methods. We believe that the Corollary to Theorem 2 is only one of many potential applications of generic processes. For the reader who wants to first look at the application discussed in Section 3, we remark that it is not necessary to read the proof of Theorem 1 in Section 2 before reading Section 3.

1. Construction of generic population models. We will denote the state at time t of a generic process by $\mathcal{A}_t$. In general, $\mathcal{A}_t$ will be a collection of compact convex $(d+1)$-sided polyhedral regions in $\mathbf{R}^d$. Each such region will correspond to some point $(\mathbf{x}, s)$ in space-time $\mathbf{R}^d \times [0, \infty)$, and for $t \geq s$, we will denote the state of this region by $D(\mathbf{x}, s; t)$. The space-time points $(\mathbf{x}, s)$ will be contained in some random subset of $\mathbf{R}^d \times [0, \infty)$. We will be interested in the union

$$A_t = \bigcup_{(\mathbf{x},s):s\leq t} D(\mathbf{x}, s; t).$$

When we make comparisons to population models like the contact process, the set A_t will correspond to the set of vacant sites, so we call A_t the **vacant region**. We will always take

$$\mathcal{A}_0 = \emptyset$$

as the initial state of our generic process. This corresponds to the state in which everything is occupied.

Let us fix unit vectors $\mathbf{n}_1, \dots, \mathbf{n}_{d+1} \in \mathbf{R}^d$. We call these vectors **orientation vectors**. In general, each of the regions $D(\mathbf{x}, s; t)$ will be a convex polyhedron whose faces are perpendicular to the vectors $\mathbf{n}_i$. Let us introduce some notation that will be helpful in describing such sets. Let $\mathbf{p}$ be a point and $\mathbf{n}$ a unit vector, both in $\mathbf{R}^d$, and define

$$H(\mathbf{n}, \mathbf{p}) = \{\mathbf{x} \in \mathbf{R}^d : \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle \geq 0\},$$

where $\langle\ ,\ \rangle$ denotes the usual Euclidean inner product in $\mathbf{R}^d$. In more descriptive terms, $H(\mathbf{n}, \mathbf{p})$ is one of the two closed half-spaces whose boundary contains the point $\mathbf{p}$ and is perpendicular to the vector $\mathbf{n}$. The vector $\mathbf{n}$ points into the set $H(\mathbf{n}, \mathbf{p})$. In a generic process, all of the polyhedral regions will be of the form $\mathbf{p}+R(r)$, where $\mathbf{p}\in\mathbf{R}^d$, $r\geq 0$, and

$$R(r)=\bigcap_{i=1}^{d+1} H(\mathbf{n}_i,-r\mathbf{n}_i)$$

for some fixed unit vectors $\mathbf{n}_1,\ldots,\mathbf{n}_{d+1}$. Note that according to this definition, $\mathbf{p}+R(r)$ contains the closed ball of radius r centered at $\mathbf{p}$, and that r is the distance from $\mathbf{p}$ to each of the faces of $\mathbf{p}+R(r)$. We call $\mathbf{p}$ the **base** and r the **size** of $\mathbf{p}+R(r)$. The following proposition collects some elementary facts about regions of the form $\mathbf{p}+R(r)$.

Proposition 1. *If the set*

$$R=\bigcap_{i=1}^{d+1} H(\mathbf{n}_i,\mathbf{p}_i)$$

is nonempty, then it is bounded if and only if $\mathbf{n}_1,\ldots,\mathbf{n}_{d+1}$ span $\mathbf{R}^d$ and there exist strictly positive scalars $a_1,\ldots,a_{d+1}$ such that

$$a_1\mathbf{n}_1+\cdots+a_{d+1}\mathbf{n}_{d+1}=\mathbf{0},$$

where $\mathbf{0}$ is the origin in $\mathbf{R}^d$. In this case, there exists a unique real number $r\geq 0$ and a unique vector $\mathbf{p}$ such that $R=\mathbf{p}+R(r)$, where $R(r)$ is defined above. The quantity r and the vector $\mathbf{p}$ are linear functions of the vector

$$\mathbf{y}=(\langle\mathbf{p}_1,\mathbf{n}_1\rangle,\ldots,\langle\mathbf{p}_{d+1},\mathbf{n}_{d+1}\rangle).$$

Proof. Without loss of generality, assume that R contains the origin. Under this assumption, the convexity of R implies that R is unbounded if and only if there exists a nonzero vector $\mathbf{x}$ such that for all $a\geq 0$,

$$\langle a\mathbf{x}-\mathbf{p}_i,\mathbf{n}_i\rangle\geq 0 \quad \text{for} \quad i=1,\ldots,d+1.$$

Since R contains the origin, the quantities $\langle-\mathbf{p}_i,\mathbf{n}_i\rangle$ are all nonnegative, so R is unbounded if and only if there exists a nonzero vector $\mathbf{x}$ such that for all $a\geq 0$,

$$\langle a\mathbf{x},\mathbf{n}_i\rangle\geq 0 \quad \text{for} \quad i=1,\ldots,d+1.$$

It is now an easy matter to finish the proof of the first part of the proposition.

For the second part of the proposition, consider the following system of d equations:

$$\langle\mathbf{p}-\mathbf{p}_{d+1},\mathbf{n}_{d+1}\rangle=\langle\mathbf{p}-\mathbf{p}_i,\mathbf{n}_i\rangle \quad i=1,\ldots,d.$$

The conditions on the vectors $\mathbf{n}_i$ imply that any d of them are linearly independent. Hence this system has a unique solution $\mathbf{p}$. This solution lies equidistant from the hyperplanes that bound the half-spaces that determine R. Let r be the common value of the quantities $\langle \mathbf{p} - \mathbf{p}_i, \mathbf{n}_i \rangle$. Note that both $\mathbf{p}$ and r are linear functions of the vector $\mathbf{y}$. If R is nonempty, it is straightforward to check that $R = \mathbf{p} + R(r)$ as claimed. (If R is empty, it turns out that $r < 0$.) ■

From now on, we assume that the vectors $\mathbf{n}_i$ satisfy the conditions of Proposition 1. Note that the intersection of any collection of n regions of the form $\mathbf{p} + R(r)$ can be written as an intersection of half-spaces that are oriented by the vectors $\mathbf{n}_i$. By grouping the half-spaces according to orientation, such an intersection may be rewritten as the intersection of $d+1$ half-spaces, each with a different orientation $\mathbf{n}_i$. Thus Proposition 1 has the following consequence.

Corollary. *Let $r_1, \dots, r_n$ be nonnegative numbers and let $\mathbf{p}_1, \dots, \mathbf{p}_n$ be vectors in $\mathbf{R}^d$. If the set*

$$R = \bigcap_{j=1}^{n} [\mathbf{p}_j + R(r_j)]$$

is nonempty, then there is exists a unique number $r \geq 0$ and a unique vector $\mathbf{p}$ such that $R = \mathbf{p} + R(r)$.

We wish to describe the dynamics of a generic process. We will give three different ways in which new polyhedral regions enter the collection $\mathcal{A}_t$, namely by death, overlap, and collision, and we will say how each type of polyhedral region evolves. Afterwards we will explain why there exists a process that corresponds to our description.

Deaths. In order to describe the first way in which new polyhedral regions enter the collection $\mathcal{A}_t$, we need to define certain point processes. Fix a parameter $\varepsilon > 0$, called the **death rate**, and a probability density μ on the positive integers, called the **size distribution**. For $k \geq 1$, let $\mathcal{P}_k$ be independent Poisson point processes in $\mathbf{R}^d \times [0, \infty)$ with intensity measure equal to $\varepsilon\mu(k)\lambda$, where λ is Lebesgue measure on $\mathbf{R}^d \times [0, \infty)$. In other words, the $\mathcal{P}_k$'s are random subsets of $\mathbf{R}^d \times [0, \infty)$ such that the random variables

$$\sharp(A \cap \mathcal{P}_k), \quad A \text{ Borel},$$

satisfy two properties: (i) $\sharp(A_i \cap \mathcal{P}_k)$, $i = 1, \dots, n, k = 1, 2, \dots$, are independent if the sets A_i are pairwise disjoint; and (ii) $\sharp(A \cap \mathcal{P}_k)$ is a Poisson random variable with mean $\varepsilon\mu(k)\lambda(A)$. (Here and throughout the paper, we use $\sharp S$ to denote the cardinality of a set S.)

We combine the sets $\mathcal{P}_k$ into a point process

$$\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k.$$

Since the $\mathcal{P}_k$'s are independent, $\mathcal{P}$ is itself a Poisson point process with intensity measure $\varepsilon\lambda$. We will label a typical point in $\mathcal{P}$ by $(\mathbf{x}, s)$, where $\mathbf{x} \in \mathbf{R}^d$ and $s \geq 0$. For $(\mathbf{x}, s) \in \mathcal{P}_k$, let

$$\Delta(\mathbf{x}, s) = k.$$

The quantities $\Delta(\mathbf{x}, s)$, $(\mathbf{x}, s) \in \mathcal{P}$, are independent and identically distributed random variables, with probability density μ. For the main part of our work, we will often assume that the size distribution μ satisfies the following condition:

$$\exists\, \theta > 0 : \sup_k \left[\exp(\theta k)\, \mu(k)\right] < \infty. \tag{1-1}$$

It is possible to get by with less, but the extra work required does not seem to be worth it.

For each point $(\mathbf{x}, s) \in \mathcal{P}$, we say that a **death** occurs at $(\mathbf{x}, s)$. If a death occurs at $(\mathbf{x}, s)$, a new region is added to the collection $\mathcal{A}_s$. It is defined by

$$D(\mathbf{x}, s; s) = \mathbf{x} + R(\Delta(\mathbf{x}, s)).$$

We call any region produced in this way a **death region**, and say that it arises due to the death at $(\mathbf{x}, s)$. The evolution of a death region after time s is determined by fixed real numbers $\alpha_1, \ldots, \alpha_{d+1}$, called **occupation rates**. For each $(\mathbf{x}, s) \in \mathcal{P}$ and $t \geq s$, the corresponding death region $D(\mathbf{x}, s; t)$ at time t is given by

$$D(\mathbf{x}, s; t) = \bigcap_{i=1}^{d+1} H(\mathbf{n}_i, \mathbf{x} + ((t - s)\alpha_i - \Delta(\mathbf{x}, s))\mathbf{n}_i).$$

Thus, $D(\mathbf{x}, s; t)$ starts at time s as the set $\mathbf{x} + R(\Delta(\mathbf{x}, s))$, and as time progresses, its i^{th} face moves inward at rate α_i (if α_i is negative, the movement is actually outward). To picture this, the reader might want to think about the 2-dimensional case, in which each of the regions is a triangle with moving edges. Also, it is helpful to think about the case in which all of the α_i's are all equal to some positive number α, for which it is easy to see that the region $D(\mathbf{x}, s; t)$ vanishes after time $t = s + (\Delta(\mathbf{x}, s)/\alpha)$.

We now define a process

$$N_t = \bigcup_{(\mathbf{x}, s) \in \mathcal{P} : s \leq t} D(\mathbf{x}, s; t).$$

We call N_t the **noninteractive region** at time t. It is merely the union of all the death regions that exist at time t and does not involve any of the other two types of regions yet to be defined. When we complete our construction, it will be apparent that N_t is a subset of the vacant region A_t.

Before going on to describe the other two types of regions, we prove a result concerning the behavior of N_t. This result implies the 'easy half' of our main theorem, Theorem 1 in Section 2. Consider the following condition on the orientation vectors $\mathbf{n}_i$ and occupation rates α_i:

Generic eroder condition. *There exists a time* $t > 0$ *such that*

$$\bigcap_{i=1}^{d+1} H(\mathbf{n}_i, (\alpha_i t - 1)\mathbf{n}_i) = \emptyset.$$

Note that if all of the α_i's are positive, the generic eroder condition is satisfied, whereas it fails if they are all less than or equal to 0. Other cases depend on the orientation vectors. The following result says that if the generic eroder condition fails, then the noninteractive vacant region 'takes over'.

Proposition 2. *If the generic eroder condition does not hold, then for all* $\varepsilon > 0$,

$$P(\mathbf{y} \in N_t) \to 1 \text{ as } t \to \infty,$$

uniformly in $\mathbf{y} \in \mathbf{R}^d$.

Proof. The proof is based on the following claim:

CLAIM. If the eroder condition does not hold, then there exists a nonzero vector $\mathbf{v} \in \mathbf{R}^d$ such that for all $(\mathbf{x}, s)$ in $\mathcal{P}$ and $t \geq s$,

$$\mathbf{x} + B_1((t-s)\mathbf{v}) \subseteq D(\mathbf{x}, s; t),$$

where $D(\mathbf{x}, s; t)$ is the death region corresponding to $(\mathbf{x}, s)$, and where for any $\mathbf{z} \in \mathbf{R}^d$ and $\delta > 0$, $B_\delta(\mathbf{z})$ is the closed ball of radius δ centered at $\mathbf{y}$.

Let us show how to finish the proof of the proposition by applying the claim. Fix $\mathbf{x} \in \mathbf{R}^d$ and $t \geq 0$. Note that the condition $\mathbf{y} \in \mathbf{x} + B_1((t-s))\mathbf{v})$ is equivalent to the condition $(\mathbf{x}, s) \in B_1(\mathbf{y} - (t-s)\mathbf{v}) \times \{s\}$. It follows from the claim and our construction of the set N_t that $\mathbf{y} \in N_t$ if

$$\mathcal{P} \cap \left[\bigcup_{s \leq t} (B_1(\mathbf{y} - (t-s)\mathbf{v}) \times \{s\}) \right] \neq \emptyset.$$

The cardinality of the set on the left side of the above expression is a Poisson random variable with mean $\varepsilon t \lambda(B_1(\mathbf{0}))$. Since this quantity goes to ∞ as $t \to \infty$, the proposition now follows.

It remains to prove the claim. Fix $(\mathbf{x}, s) \in \mathcal{P}$. By Proposition 1, for all $t \geq 0$ such that $D(\mathbf{x}, s; t)$ is nonempty, there exists a point $\mathbf{p}(t)$ and a nonnegative real number $r(t)$ such that

$$D(\mathbf{x}, s; t) = \mathbf{p}(t) + R(r(t)).$$

The hyperplanes determining $D(\mathbf{x}, s; t)$ move at constant linear rates which do not depend on $(\mathbf{x}, s)$ or $\Delta(\mathbf{x}, s)$, so it also follows from the last part of Proposition 1 that $\mathbf{p}(t)$ and $r(t)$ are also changing linearly: $\mathbf{p}(t) = \mathbf{x} + (t - s)\mathbf{v}$ and $r(t) = -\alpha(t - s) + \Delta(\mathbf{x}, s)$, for some vector $\mathbf{v}$ and real constant α, neither of which depends on $(\mathbf{x}, s)$ or $\Delta(\mathbf{x}, s)$. That is

$$D(\mathbf{x}, s; t) = \mathbf{x} + (t - s)\mathbf{v} + R(-\alpha(t - s) + \Delta(\mathbf{x}, s))$$

for all $t \geq 0$ such that $D(\mathbf{x}, s; t)$ is nonempty. In particular, this statement holds for sufficiently small $t > 0$. We may apply this same argument to the set

$$\bigcap_{i=1}^{d+1} H(\mathbf{n}_i, (\alpha_i t - 1)\mathbf{n}_i)$$

to conclude that it equals $t\mathbf{v} + R(-\alpha t + 1)$ for sufficiently small $t > 0$. From this we see that α is nonpositive, since the eroder condition fails. It follows that $D(\mathbf{x}, s; t)$ is nonempty for all $t \geq 0$, and that

$$\mathbf{x} + (t - s)\mathbf{v} + R(\Delta(\mathbf{x}, s)) \subseteq D(\mathbf{x}, s; t).$$

Since $\Delta(\mathbf{x}, s) \geq 1$, $B_1(\mathbf{0}) \subseteq R(\Delta(\mathbf{x}, s))$ and the claim follows. ■

The generic eroder condition is the continuum analogue of the eroder condition of Toom [13], and the proof just given is essentially the same as Toom's proof of the corresponding fact for discrete time lattice models (except for the part concerning the claim, which is a technical feature that becomes necessary in our continuous setting). In Section 3 we present a description of Toom's results. We now return to the construction of generic processes.

Overlap interactions. Whenever a new polyhedral region appears as the result of a death, it may be that it overlaps one or more already existing polyhedral regions. These already existing polyhedral regions may be any of the three types of regions that we are in the process of defining. When such an overlap occurs, further new regions are produced in a manner that we now describe. Suppose that there is a death at a space-time point $(\mathbf{y}_1, u_1)$, and let $D(\mathbf{y}_1, u_1; u_1)$ be the corresponding new region. Write $s = u_1$, and let $D(\mathbf{y}_2, u_2; s), \ldots, D(\mathbf{y}_n, u_n; s)$ be any collection of regions in $\mathcal{A}_s$ such that $u_k < s$ for $k = 2, \ldots, n$, and

$$\bigcap_{k=1}^{n} D(\mathbf{y}_k, u_k; s) \neq \emptyset.$$

By the Corollary to Proposition 1, this set may be written as $\mathbf{x} + R(r)$ for some $r \geq 0$ and $\mathbf{x} \in \mathbf{R}^d$. We assume that the collection of regions $D(\mathbf{y}_k, u_k; s)$ is the maximal collection satisfying the above description with intersection equal to $\mathbf{x}+R(r)$ (there will be only one such maximal collection for each different set $\mathbf{x}+R(r)$ formed in this way). We write $D(\mathbf{x}, s; s) = \mathbf{x} + R(r)$, and add this new region to our collection $\mathcal{A}_s$. We say that it appears due to an **overlap interaction**. Any region that appears due to an overlap interaction is called an **overlap region**.

Note that if the death region $D(\mathbf{y}_1, u_1; u_1)$ overlaps m already existing regions when it appears, then as many as $2^m - 1$ new overlap regions may be produced as a result. We have avoided one possible ambiguity that may appear in the labelling of these regions (when two different sub-collections of these m regions have the same intersection with $D(\mathbf{y}_1, u_1; u_1)$) by requiring that the collections of regions $D(\mathbf{y}_k, u_k; s)$ be maximal, as described in the preceding paragraph. There is another possible ambiguity that can arise when two different overlap regions are produced with the same base point. This second possibility has probability zero (see the discussion of existence given later in this section). The possibility that $m = \infty$ is not excluded. It can be shown that m is finite almost surely if (1-1) is satisfied, but as we don't really need this fact, we omit the proof.

The evolution of an overlap region is determined by the occupation rates α_i and a fixed quantity $\beta < 0$ called the **interaction rate** (as we will see, specifying that β be negative means that the corresponding movement of faces will be outward). We assume that

$$\beta \leq \min_i \alpha_i.$$

For $(\mathbf{x}, s)$ and $(\mathbf{y}_k, u_k)$ as in the preceding paragraph, and $i = 1, \ldots, d+1$, define

$$H_i(\mathbf{y}_k, u_k; t) = \text{the half-space corresponding to the } i^{\text{th}} \text{ face of } D(\mathbf{y}_k, u_k; t).$$

Let

$$\tau_i = \inf\left\{ t \geq s : H(\mathbf{n}_i, \mathbf{x} + (\beta t - r)\mathbf{n}_i) \supseteq \bigcup_{k=1}^{n} H_i(\mathbf{y}_k, u_k; t) \right\},$$

and define a function $\gamma_i(\mathbf{x}, s; t)$ which is continuous and piecewise linear in $t \geq s$ and determined by the conditions

$$\gamma_i(\mathbf{x}, s; s) = 0 \quad \text{and} \quad \frac{d\gamma_i}{dt}(\mathbf{x}, s; t) = \begin{cases} \beta & t < \tau_i \\ \alpha_i & t > \tau_i. \end{cases} \tag{1-2}$$

We allow the possibility that $\tau_i = \infty$, which can happen when $\alpha_i = \beta$. The overlap region $D(\mathbf{x}, s; t)$ is now defined as

$$D(\mathbf{x}, s; t) = \bigcap_{i=1}^{d+1} H(\mathbf{n}_i, \mathbf{x} + (\gamma_i(\mathbf{x}, s; t) - r)\mathbf{n}_i)$$

for all $t \geq s$. In descriptive terms, this region starts at time s as $\mathbf{x} + R(r)$, after which its i^{th} face moves outward at speed $-\beta$ until time τ_i, at which time it has caught all of the corresponding faces of the regions $D(\mathbf{y}_k, u_k; t)$. The time τ_i is defined in such a way that at time τ_i, the i^{th} face of each of the regions $D(\mathbf{y}_k, u_k; \tau_i)$ is moving at rate α_i. From time τ_i on, the i^{th} face of $D(\mathbf{x}, s; t)$ also moves at rate α_i.

Collision interactions. The third way in which a new region appears is similar to the second. Fix s and suppose that $D(\mathbf{y}_1, u_1; s), \ldots, D(\mathbf{y}_n, u_n; s)$ are regions in the collection $\mathcal{A}_s$ such that $u_k < s$ for $k = 1, \ldots, n$ and

$$\bigcap_{k=1}^{n} D(\mathbf{y}_k, u_k; t) = \emptyset \quad \text{for} \quad t < s.$$

(In this formula and elsewhere, we interpret $D(\mathbf{y}_k, u_k; t)$ as the empty set for $t < u_k$.) Thus we have n regions that existed and had empty intersection prior to time s. Suppose that their intersection is not empty at time s. By the Corollary to Proposition 1, this intersection is of the form $\mathbf{x} + R(r)$ for some $\mathbf{x} \in \mathbf{R}^d$ and $r \geq 0$. All regions change continuously in our time evolution, so the intersection at time s cannot contain interior points of any of the regions. It follows that $r = 0$, so the intersection equals $\mathbf{x} + R(0) = \{\mathbf{x}\}$. We say that a **collision** occurs at $(\mathbf{x}, s)$. We assume that the collection of regions $D(\mathbf{y}_k, u_k; s)$, $k = 1, \ldots, n$, is maximal in the sense that it contains all regions that existed prior to s and have $\mathbf{x}$ as a boundary point. As with overlap interactions, we allow the possibility that $n = \infty$. Under condition (1-1), this occurs with probability 0.

We start a new region $D(\mathbf{x}, s; s) = \{\mathbf{x}\}$ at time s, and say that this region arises due to a **collision interaction**. For $t \geq s$, this new region is of the form

$$D(\mathbf{x}, s; t) = \bigcap_{i=1}^{d+1} H(\mathbf{n}_i, \mathbf{x} + \gamma_i(\mathbf{x}, s; t)\mathbf{n}_i),$$

where $\gamma_i(\mathbf{x}, s; t)$ is defined as in (1-2).

Summary of the description of generic processes. The state of a generic process $\mathcal{A}_t$ is a collection of regions $D(\mathbf{x}, s; t)$ of three types. For all three types of regions, we have

$$D(\mathbf{x}, s; t) = \bigcap_{i=1}^{d+1} H_i(\mathbf{x}, s; t),$$

where

$$H_i(\mathbf{x}, s; t) = H(\mathbf{n}_i, \mathbf{x} + (\gamma_i(\mathbf{x}, s; t) - r(\mathbf{x}, s))\mathbf{n}_i) \tag{1-3}$$

for certain continuous piecewise linear functions $\gamma_i(\mathbf{x}, s; \cdot)$ and nonnegative numbers $r(\mathbf{x}, s)$. If the region arises due to a death, $r(\mathbf{x}, s) = \Delta(\mathbf{x}, s)$. In the case of a collision, $r(\mathbf{x}, s) = 0$. For overlap interactions, $r(\mathbf{x}, s)$ is the 'size' of the overlap. For regions arising due to a death, $\gamma_i(\mathbf{x}, s; t) = \alpha_i(t - s)$. In the two other cases, condition (1-2) says that $\gamma_i(\mathbf{x}, s; \cdot)$ starts at 0 at time $t = s$, then has slope β until some time τ_i, after which the slope is α_i. The vacant region A_t is the union of all of the regions in $\mathcal{A}_t$. As such, the vacant region involves regions of all three types, as opposed to the noninteractive region N_t which only involves death regions. It is clear that

$$N_t \subseteq A_t.$$

Our construction has been designed to ensure the following:

Proposition 3. *Suppose that $\mathcal{A}_t$ is a generic process defined in terms of orientation vectors $\mathbf{n}_i$, occupation rates α_i, interaction rate β, death rate ε, and size distribution μ satisfying (1-1). Let $D(\mathbf{y}_1, u_1; t), \dots, D(\mathbf{y}_n, u_n; t)$ be a collection of regions in $\mathcal{A}_t$ with nonempty intersection, and define*

$$s = \inf \left\{ u : \bigcap_{k=1}^{n} D(\mathbf{y}_k, u_k; u) \neq \emptyset \right\}.$$

Then there exists a region $D(\mathbf{x}, s; u), u \geq s$, with $(\mathbf{x}, s)$ possibly being one of the points $(\mathbf{y}_k, u_k)$, such that

$$\bigcap_{k=1}^{n} D(\mathbf{y}_k, u_k; u) \subseteq D(\mathbf{x}, s; u) \quad \text{for all} \quad u \in [s, t], \tag{1-4}$$

and such that for all $i = 1, \dots, d+1$, either

$$\frac{d\gamma_i}{du}(\mathbf{x}, s; u) = \beta \quad \text{for all} \quad u \in [s, t), \tag{1-5}$$

or

$$\bigcup_{k=1}^{n} H_i(\mathbf{y}_k, u_k; t) \subseteq H_i(\mathbf{x}, s; t). \tag{1-6}$$

This proposition says that an overlap or collision region grows at speed $-\beta$ in the direction $-\mathbf{n}_i$ until the half-space that determines the region in that direction contains all of the corresponding half-spaces of the regions that initiated the overlap or collision. The proposition is a straightforward consequence of our description of the appearance and evolution in time of the three types of regions. Note particularly the role played by the relationships given in (1-2) and (1-3). In fact, these relationships were carefully created with Proposition 3 in mind. More simplistic approaches which might also lead to a result like Proposition 3, such as setting $\beta = -\infty$

so that (1-6) is satisfied already at time $t = s$, run into trouble later on when we want to prove survival under the generic eroder condition.

The collection of regions $D(\mathbf{y}_k, u_k; u)$ in the statement of Proposition 3 need not be the same as the maximal collection that was used in the definition of overlap and collision regions. However, if $D(\mathbf{y}, u; u)$ is an overlap or collision region, and if the collection of regions $D(\mathbf{y}_k, u_k; u)$ is the maximal collection that was used in the definition of $D(\mathbf{y}, u; u)$, then (1-2) ensures that

$$H_i(\mathbf{x}, s; t) \subseteq \bigcup_{k=1}^{n} H_i(\mathbf{y}, u_k; t) \tag{1-7}$$

for all $i = 1, \ldots, d+1$ and $t \geq s$. Thus, for this special case, we may replace '$\subseteq$' by '$=$' in (1-6).

Existence. It may not be immediately obvious that there exists a process $\mathcal{A}_t$ corresponding to the prescription given above. The main problem is that it is conceivable that inconsistencies arise due to the fact that we are working in an unbounded space. We also have the minor problem that our notation does not distinguish between two different regions that arise at the same time s and have the same initial base $\mathbf{x}$. One way to overcome these problems is by defining the process in terms of a finite volume limit. More precisely, for each j, we construct a process in which we only allow deaths to occur at points $(\mathbf{x}, s) \in \mathcal{P}$ such that $\|\mathbf{x}\| \leq j$, according to the prescription given above. Since Poisson point processes are almost surely finite on bounded sets, we do not need to worry about infinitely many regions arising before any finite time. Thus, a straightforward inductive procedure can be used to define the process up to time t for any $t \in [0, \infty)$ and any $j \geq 0$. Our only concern is that more than one region might arise at a point $(\mathbf{x}, s)$ in one of these processes. This could happen if two deaths occur at the same point in space-time, or if a death occurs at the same point $(\mathbf{x}, s)$ in space-time where a collision also occurs, or if two different overlap regions arise at time s with the same base $\mathbf{x}$. The first possibility has probability zero because the different Poisson point processes $\mathcal{P}_k$ are independent. The second possibility has probability zero because there are only finitely many collision points up to any finite time. A little thought reveals that for each fixed s, the set of points $\mathbf{y}_1$ such that a death at $(\mathbf{y}_1, s)$ can produce two different regions with the same base $\mathbf{x}$ is a finite set. It follows that the third possibility also has probability zero. The proof that these possibilities all have probability zero relies on the fact that the probability that a Borel set B contains two or more points of the random set $\mathcal{P}$ is $O(\lambda(B)^2)$.

We now take a limit as $j \to \infty$. The set of points $(\mathbf{x}, s)$ at which deaths occur converges to $\mathcal{P}$. Unfortunately, the set of points $(\mathbf{x}, s)$ at which overlap and collision regions arise does not necessarily increase as $j \to \infty$. Due to the maximality conditions imposed on the collections of regions $D(\mathbf{y}_k, u_k; s)$ used in the definitions of overlap and collision regions, a given point $(\mathbf{x}, s)$ may correspond to a region for one value of j but not for some larger value of j. However, it can be shown by an elementary but tedious proof (using the fact about Poisson point processes mentioned in the last sentence of the preceding paragraph) that the probability is zero that there exists a point $(\mathbf{x}, s)$ and integers $j_1 < j_2 < j_3$ such that there is an overlap region corresponding to $(\mathbf{x}, s)$ for $j = j_1$ and $j = j_3$ but not for $j = j_2$. A similar fact can be proved for collision regions. Thus, the set of points $(\mathbf{x}, s)$ at which overlap regions arise has a limit as $j \to \infty$, as does the set of points at which collision regions arise. It is easy to see that for those points $(\mathbf{x}, s)$ that lie in the limit sets, the corresponding quantities $\gamma_{i,j}(\mathbf{x}, s; t)$ form a decreasing sequence for each fixed i and t, so the corresponding regions $D_j(\mathbf{x}, s, \zeta; t)$ form an increasing sequence for fixed t. Therefore, the limits $\gamma_i(\mathbf{x}, s; t)$ and $D(\mathbf{x}, s; t)$ exist for all $t \geq s$. We leave it to the reader to check that for these limits, conditions (1-2) and (1-3) are satisfied. Existence follows.

It is natural to also ask about uniqueness. We are content that at least one process exists that agrees with our prescription, since uniqueness is not needed in our applications. A graphical proof of uniqueness similar to the uniqueness proof in Gray [7] can be given under condition (1-1).

Note that in the construction, the randomness comes only from the random sets $\mathcal{P}_k$. Once these point processes are fixed, the remaining objects $(D(\mathbf{x}, s; t), \mathcal{A}_t, A_t$, etc.) are determined. This observation allows us to make certain comparisons. We state one such comparison here. It can be proved by first considering the finite volume processes and then taking limits.

Proposition 4. *Suppose that $\{\mathcal{P}_k\}$ and $\{\mathcal{P}'_k\}$ are two collections of Poisson point processes defined as above. Assume that they are defined jointly on the same probability space, and that for each k, $\mathcal{P}_k \subseteq \mathcal{P}'_k$. Also assume that we have two sets of occupation rates and interaction rates such that $\alpha_i \geq \alpha'_i$, and $\beta \geq \beta'$. Let $\mathcal{A}_t$ and $\mathcal{A}'_t$ be the corresponding generic processes, with vacant regions A_t and A'_t. Then $A_t \subseteq A'_t$.*

We conclude this section with two examples.

Example 1. We consider the general one-dimensional model ($d = 1$). In this case, polyhedral regions are intervals, so $\mathcal{A}_t$ is a collection of closed intervals, and A_t is their union. Note that in this one-dimensional setting, overlap and collision regions are always contained in the union of the corresponding death regions. Thus, as far as the set A_t is concerned, when a collision or overlap occurs between two regions, we treat them as being part of a single interval, ignoring the motions of the overlapping endpoints and also ignoring the collision or overlap regions that are produced. The remaining endpoints are the endpoints of the disjoint intervals whose union is A_t. The left endpoints of these intervals move inward at some rate α_ℓ and their right endpoints move inward at rate α_r. Note that except when a death causes an interval of A_t to jump in size, the intervals of A_t expand, shrink, or stay the same size, depending on whether $\alpha_\ell + \alpha_r$ is negative, positive, or zero. The generic eroder condition is that $\alpha_\ell + \alpha_r > 0$. The generic analogue to the asymmetric contact process has rates $\alpha_\ell = 0$ and $\alpha_r = 1$.

Example 2. We give the parameters for the generic analogue of Toom's model. The orientation vectors are $\mathbf{n}_1 = (1,0), \mathbf{n}_2 = (0,1)$, and $\mathbf{n}_3 = -(\sqrt{2},\sqrt{2})/2$, with corresponding occupation rates $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \sqrt{2}/2$. The interactive rate is $\beta = -1$. A typical death region is a right triangle whose vertical and horizontal edges do not move, and whose hypotenuse moves inward. The generic eroder condition is satisfied, since such triangles vanish after one time unit. If one were to simulate Toom's model and its generic analogue on a computer, using the distance between pixels as the unit of measurement, the two would look very similar. The main difference would be that vacant area increases somewhat faster in the generic model when the regions that arise from interactions are in their 'β-growth stage'.

2. The renormalization argument and the main theorem. Let $\mathcal{A}_t$ be a generic process, defined as in the preceding section in terms of occupation rates α_i, interaction rate β, orientation vectors $\mathbf{n}_i$, size distribution μ, and death rate $\varepsilon \geq 0$. We think of each of α_i, β, $\mathbf{n}_i$, and μ as being fixed, and ε as being a variable parameter. Our main result is

Theorem 1. *If the generic eroder condition does not hold, then for all* $\varepsilon > 0$

$$\lim_{t\to\infty} P(\mathbf{y} \in A_t) = 1$$

uniformly in $\mathbf{y} \in \mathbf{R}^d$. *If the generic eroder condition holds, and if in addition the size distribution* μ *satisfies (1-1), then*

$$\lim_{\varepsilon\searrow 0} \limsup_{t\to\infty} P(\mathbf{y} \in A_t) = 0$$

uniformly in $\mathbf{y}$.

Proof. Since $N_t \subseteq A_t$, the first half of the theorem follows from Proposition 2. In order to prove the second half of the theorem, we first simplify matters somewhat. Let α and $\mathbf{v}$ be as in the proof of Proposition 2. We change coordinates linearly according to the following transformation:

$$(\mathbf{x}, t) \to (\mathbf{x} - t\mathbf{v}, t).$$

Note that this transformation does not change the time variable, nor does it change distances in the spatial variables. Once this change of coordinates is made, the occupation rates α_i are all changed into α. Of course, the interaction rate β is also changed, resulting in $d+1$ possibly different interaction rates. By Proposition 4, we may replace these different interaction rates by their minimum, which will be some negative number. By changing our time scale if necessary, we may assume that this new interaction rate is less than or equal to -1. Since the eroder condition holds, $\alpha > 0$ (before and after the change in time scale). By applying Proposition 4 again, we may assume that $\alpha \leq 1$. For the remainder of the proof, we will assume that these modifications have been made, namely, that $\alpha_i = \alpha \in (0,1]$ and $\beta \leq -1$. We also fix $\theta \in (0,1]$ such that (1-1) holds, and let

$$C = \sup_k [\mu(k) \exp(\theta k)].$$

Here is an outline of the proof. We will first partition the points in $\mathcal{P}$ into clusters defined in terms of the proximity of these points to one another. We will prove that the clusters are finite almost surely. To each cluster of cardinality 2 or greater, we will associated a point in space-time, and the resulting set of points will be called $\mathcal{P}^{(1)}$. We will use the random set $\mathcal{P}^{(1)}$ to define a new generic process $\mathcal{A}_t^{(1)}$. This process is a renormalization of $\mathcal{A}_t$. We will need to take care of some technical details concerning the random set $\mathcal{P}^{(1)}$ that underlies the construction of $\mathcal{A}_t^{(1)}$ in order to know that $\mathcal{A}_t^{(1)}$ behaves in every essential way like a generic process. We then continue inductively to define successive renormalizations $\mathcal{A}_t^{(2)}, \mathcal{A}_t^{(3)}, \dots$. It is then shown that the vacant region A_t of our original generic process is contained in $N_t \cup A_t^{(1)}$, the union of the noninteractive region of the original generic process and the vacant region of the renormalized generic process. An inductive argument then allows us to obtain an upper bound for the probability that a point $\mathbf{y}$ lies in A_t in terms of upper bounds for the probabilities that $\mathbf{y}$ lies in the union of the noninteractive regions of the sequence of renormalized generic processes. This upper bound is shown to converge to 0 as $\varepsilon \searrow 0$, uniformly in $\mathbf{y}$ and t.

Definition of clusters. First we need some notation. Recall that $B_\delta(\mathbf{x})$ is the closed ball in $\mathbf{R}^d$ of radius δ centered at $\mathbf{x}$. Choose an integer $M \geq 1$ such that $R(1) \subseteq B_M(\mathbf{0})$. (Note that $R(r) \subseteq B_{Mr}(\mathbf{0})$ for all $r \geq 0$.) When measuring distance between points in $\mathbf{R}^d \times [0,\infty)$, we will use the max-norm:

$$\|(\mathbf{x}, s) - (\mathbf{y}, u)\|_{\max} = \max\{|s-u|, |x_1 - y_1|, \dots, |x_d - y_d|\},$$

where $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$. For $(\mathbf{x}, s) \in \mathbf{R}^d \times [0,\infty)$, we write

$$Q(\delta, \mathbf{x}, s) = [x_1 - \delta, x_1 + \delta) \times \cdots \times [x_d - \delta, x_d + \delta) \times ([s-\delta, s+\delta) \cap [0,\infty)),$$

which is a cube with sides of length 2δ, centered at $(\mathbf{x}, s)$, and truncated at $t = 0$ if necessary. For $(\mathbf{x}, s) \in \mathcal{P}$, define

$$S(\mathbf{x}, s) = \text{ the interior of } Q\left(\frac{5M^2|\beta|\Delta(\mathbf{x}, s)}{\alpha}, \mathbf{x}, s\right).$$

We think of $S(\mathbf{x}, s)$ as the cube of influence of the point $(\mathbf{x}, s)$. The quantity $5M^2|\beta|\Delta(\mathbf{x}, s)/\alpha$ is chosen for the proof of the relationship $A_t \subseteq N_t \cup \mathcal{A}_t^{(1)}$. For now, the reader should note that since the size of a death region $D(\mathbf{x}, s; t)$ shrinks at rate α and hence reaches 0 at time $s + \Delta(\mathbf{x}, s)/\alpha$, the quantity has been chosen large enough so that $D(\mathbf{x}, s; t) \subseteq S(\mathbf{x}, s)$ for all $t \geq s$. Let

$$\mathcal{S}(\mathbf{x}, s) = \text{ the component of } \bigcup_{(\mathbf{y},u)\in\mathcal{P}} S(\mathbf{y}, u) \quad \text{containing } S(\mathbf{x}, s)$$

and

$$\mathcal{C}(\mathbf{x}, s) = \mathcal{S}(\mathbf{x}, s) \cap \mathcal{P}.$$

We call $\mathcal{C}(\mathbf{x}, s)$ the **cluster** containing $(\mathbf{x}, s)$.

Finiteness of clusters. In order to show that the clusters are finite in cardinality almost surely, we will mainly rely on two basic properties of Poisson point processes which we now describe. The first is quite simple:

$$P(\sharp(B \cap \mathcal{P}) \geq 2) = O(\lambda(B)^2) \tag{2-1}$$

for Borel sets B. To state the second property, let $Q_1, \dots, Q_m$ be disjoint cubes in $\mathbf{R}^d \times [0, \infty)$. Then for all positive integers $k_1, \dots, k_m$,

$$P\left(\bigcap_{j=1}^{m}\{Q_j \cap \mathcal{P}_{k_j} \neq \emptyset\}\right) \leq \prod_{j=1}^{m}[\varepsilon\mu(k_j)\lambda(Q_j)]. \tag{2-2}$$

The reason we mention these properties now is that later in the proof, we will define clusters in terms of point processes that are not Poisson point processes, but which do satisfy versions of (2-1) and (2-2). The reader should take care to observe that these are the only special properties of Poisson point processes used in the following argument.

Fix a cube $Q \subseteq \mathbf{R}^d \times [0, \infty)$. We wish to obtain an upper bound for the probability that Q has nonempty intersection with some cluster with infinitely many points. For each positive integer k_0, define

$$G_0(k_0) = Q \cap \mathcal{P}_{k_0} \text{ and } G_0 = F_0 = \bigcup_{k_0=1}^{\infty} G_0(k_0).$$

Now proceed inductively. For positive integers $k_0, \dots, k_n$, let

$$G_n(k_0, \dots, k_n) = \{(\mathbf{x}, s) \in \mathcal{P}_{k_n} \setminus F_{n-1} : S(\mathbf{x}, s) \cap S(\mathbf{y}, u) \neq \emptyset \text{ for some } (\mathbf{y}, u) \in G_{n-1}(k_0, \dots, k_{n-1})\}$$

$$G_n = \bigcup_{k_0,\dots,k_n\geq 1} G_n(k_0, \dots, k_n)$$

$$F_n = F_{n-1} \cup G_n.$$

We may think of G_n as the n^{th} generation of a family of particles with multiple types. The particles are points $(\mathbf{x}, s)$ in $\mathcal{P}$. The type of a particle $(\mathbf{x}, s)$ is $\Delta(\mathbf{x}, s)$. A sequence of points $(\mathbf{x}_j, s_j), j = 0, \ldots, n$, such that

$$(\mathbf{x}_0, s_0) \in G(k_0),\ (\mathbf{x}_1, s_1) \in G(k_0, k_1),\ \ldots,\ (\mathbf{x}_n, s_n) \in G(k_0, \ldots, k_n),$$

is called an **ancestry of type** $(k_0, \ldots, k_n)$ if for each $j = 0, \ldots, n-1$,

$$S(\mathbf{x}_j, s_j) \cap S(\mathbf{x}_{j+1}, s_{j+1}) \neq \emptyset.$$

Every point in $G_n(k_0, \ldots, k_n)$ is the last point of one or more ancestries of type $(k_0, \ldots, k_n)$.

We will use (2-1) and (2-2) to obtain an upper bound for the probability that $G_n(k_0, \ldots, k_n)$ is nonempty, which will in turn give us an upper bound for the probability that G_n is nonempty and thus help us to analyze the size of the clusters that intersect Q. To this end, fix $\delta > 0$ and let

$$Q(\delta, \mathbf{y}_i, u_i),\ i = 1, 2, \ldots,$$

be a partition of $\mathbf{R}^d \times [0, \infty)$ into disjoint cubes with sides of length 2δ. For positive integers $k_0, \ldots, k_n$, let $I(k_0, \ldots, k_n)$ be the collection of all sequences $(i_0, \ldots, i_n)$ of distinct positive integers such that

$$Q(\delta, \mathbf{y}_{i_0}, u_{i_0}) \cap Q \neq \emptyset$$

and

$$\|(\mathbf{y}_{i_j}, u_{i_j}) - (\mathbf{y}_{i_{j+1}}, u_{i_{j+1}})\|_{\max} \leq 2\delta + 5M^2|\beta|(k_j + k_{j+1})/\alpha$$

for $j = 0, \ldots, n-1$. To understand this last expression, note that for any two consecutive points $(\mathbf{x}_j, s_j)$ and $(\mathbf{x}_{j+1}, s_{j+1})$ of an ancestry of type $(k_0, \ldots, k_n)$, the max-norm distance $\|(\mathbf{x}_j, s_j) - (\mathbf{x}_{j+1}, s_{j+1})\|_{\max}$ is at most $5M^2|\beta|(k_j + k_{j+1})/\alpha$. Thus, for any ancestry $(\mathbf{x}_0, s_0), \ldots, (\mathbf{x}_n, s_n)$ of type $(k_0, \ldots, k_n)$, as long as the points in the ancestry lie in different cubes of the partition, there exists a sequence $(i_0, \ldots, i_n) \in I(k_0, \ldots, k_n)$ such that

$$Q(\delta, \mathbf{y}_{i_j}, u_{i_j}) \cap \mathcal{P}_{k_j} \neq \emptyset,\ j = 0, \ldots, n. \tag{2-3}$$

The points in an ancestry of type $(k_0, \ldots, k_n)$ necessarily lie inside the cube

$$Q((5M^2|\beta|(k_0 + \cdots + k_n)/\alpha) + r, \mathbf{p}, t)$$

if $Q = Q(r, \mathbf{p}, t)$, so the probability that any two of the points in the ancestry lie in the same partitioning cube goes to 0 as $\delta \to 0$ by a straightforward application of (2-1). By (2-2), the probability of the event in (2-3) is bounded above by

$$\prod_{j=0}^{n} [\varepsilon(2\delta)^{d+1}\mu(k_j)]. \tag{2-4}$$

It follows that

$$\begin{aligned} &P(G_n(k_0,\dots,k_n)\neq\emptyset)\\ &\le \lim_{\delta\to 0}\sum_{(i_0,\dots,i_n)\in I(k_0,\dots,k_n)} P\left(\bigcap_{j=1}^{n}\{Q(\delta,\mathbf{y}_{i_j},u_{i_j})\cap\mathcal{P}_{k_j}\neq\emptyset\}\right)\\ &\le \varepsilon\mu(k_0)\lambda(Q)\prod_{i=1}^{n}[\varepsilon(5M^2|\beta|(k_{i-1}+k_i)/\alpha)^{d+1}\mu(k_i)]. \end{aligned}$$

This last expression is (crudely) bounded above by

$$\lambda(Q)(\varepsilon(10M^2|\beta|/\alpha)^{d+1})^{n+1}\prod_{i=0}^{n}[k_i^{2d+2}\mu(k_i)].$$

Since by elementary calculus

$$k^{2d+2}\exp(-k\theta/2)\le\left(\frac{2d+2}{\theta}\right)^{2d+2},$$

we have by (1-1)

$$\text{(2-5)}\qquad \frac{P(G_n(k_0,\dots,k_n)\neq\emptyset)}{\lambda(Q)}\le \left(\frac{K_1C\varepsilon|\beta|^{d+1}}{\alpha^{d+1}\theta^{2d+2}}\right)^{n+1}\exp(-\frac{\theta}{2}(k_0+\cdots+k_n)),$$

where

$$K_1=2(10M^2)^{d+1}(2d+2)^{2d+2}.$$

Summing (2-5) on $k_0,\dots,k_n$, we obtain

$$\text{(2-6)}\qquad \frac{P(G_n\neq\emptyset)}{\lambda(Q)}\le\left(\frac{K_1C\varepsilon|\beta|^{d+1}}{\alpha^{d+1}\theta^{2d+3}}\right)^{n+1}.$$

In order for a cluster to be infinite and also contain at least one point in Q, either $G_n\neq\emptyset$ for all n, or $G_n(k_0,\dots,k_n)$ has infinite cardinality for some n and some $k_0,\dots,k_n$. Since the points in $G_n(k_0,\dots,k_n)$ are confined by the construction to the intersection of $\mathcal{P}$ and a certain ball centered at the middle of Q, the sets $G_n(k_0,\dots,k_n)$ are finite almost surely (this follows from (2-1)). There are only countably many such sets. The probability in (2-6) converges to 0 as $n\to\infty$ for small $\varepsilon>0$, certainly for

$$\text{(2-7)}\qquad \varepsilon\le\frac{\alpha^{d+1}\theta^{2d+3}}{8K_1C|\beta|^{d+1}}.$$

(We have left some extra room in this condition because of what comes later.) Assuming (2-7), it follows that the probability that $G_n \neq \emptyset$ for all n is 0, so any cluster that intersects Q is finite almost surely. It is elementary to conclude that all clusters are finite almost surely for ε satisfying (2-7). We assume from now on in the proof that (2-7) is satisfied.

The first renormalization. In this part of the proof and the next, we will carry out a procedure that will be the first step in an induction. Our eventual goal will be to define point processes $\mathcal{P}_k^{(\ell)}$ and corresponding generic processes $\mathcal{A}_t^{(\ell)}$, such that analogues of (2-1) and (2-2) are satisfied.

To begin the first stage of the induction, we define

$$\mathcal{P}^{(1)} = \{(\mathbf{x}, s) \in \mathcal{P} : \sharp(\mathcal{C}(\mathbf{x}, s)) \geq 2 \text{ and } u \geq s \text{ for all } (\mathbf{y}, u) \in \mathcal{C}(\mathbf{x}, s)\}.$$

A point is in $\mathcal{P}^{(1)}$ if it is the 'oldest' member of a cluster containing at least 2 points. The fact that we only take points from clusters of size two or more will imply that for small $\varepsilon > 0$, the points in $\mathcal{P}^{(1)}$ are considerably sparser than the ones in $\mathcal{P}$. Since the clusters are finite almost surely, every cluster with at least 2 points contains a unique oldest member almost surely; there is therefore a one-to-one correspondence between clusters of cardinality greater than or equal to 2 and points in $\mathcal{P}^{(1)}$. For $(\mathbf{x}, s) \in \mathcal{P}^{(1)}$, let

$$\Delta^{(1)}(\mathbf{x}, s) = \lfloor 2 \inf\{r \geq 0 : \mathcal{S}(\mathbf{x}, s) \subseteq B_r(\mathbf{x}) \times [s - r, s + r]\} \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes 'greatest integer less than or equal to', and let

$$\mathcal{P}_k^{(1)} = \{(\mathbf{x}, s) \in \mathcal{P}^{(1)} : \Delta^{(1)}(\mathbf{x}, s) = k\}.$$

Also let

$$\alpha^{(1)} = \alpha/2 \text{ and } \beta^{(1)} = 2\beta.$$

We now have all the ingredients for defining a new generic process $\mathcal{A}_t^{(1)}$, using the sets $\mathcal{P}_k^{(1)}$ in place of the Poisson point processes $\mathcal{P}_k$, and the quantities $\Delta^{(1)}(\mathbf{x}, s)$ in the place of the quantities $\Delta(\mathbf{x}, s)$. We set all the occupation rates equal to $\alpha^{(1)}$, and the interactive rates equal to $\beta^{(1)}$. We use the same orientation vectors $\mathbf{n}_i$ as before. Since $\mathcal{P}^{(1)} \subseteq \mathcal{P}$, the finite volume construction outlined in Section 1 can be carried out, even though the sets $\mathcal{P}_k^{(1)}$ are not Poisson point processes. Thus, we can construct a collection $\mathcal{A}_t^{(1)}$ of polyhedral regions $D^{(1)}(\mathbf{x}, s; t)$, determined by the half spaces $H_i^{(1)}(\mathbf{x}, s; t)$ that move according to functions $\gamma_i^{(1)}(\mathbf{x}, s; t)$. We also have the vacant regions $A_t^{(1)}$ and noninteractive vacant regions $N_t^{(1)}$, all in analogy with what was done in the construction of the original process $\mathcal{A}_t$. The process $\mathcal{A}_t^{(1)}$ is our renormalization of $\mathcal{A}_t$. We remark that $\Delta^{(1)}(\mathbf{x}, s)$ has been chosen large enough so that for any point $(\mathbf{y}, u) \in \mathcal{C}(\mathbf{x}, s)$, $D(\mathbf{y}, u; u) \subseteq D^{(1)}(\mathbf{x}, s; u)$. This follows from the fact that the size of $D^{(1)}(\mathbf{x}, s; t)$ shrinks at rate $\alpha^{(1)} \le 1/2$, so that the space-time set $\mathcal{S}(\mathbf{x}, s)$ is contained in the union of the space-time sets $D^{(1)}(\mathbf{x}, s; t) \times \{t\}$, $t \ge s$.

Showing that $\mathcal{P}^{(1)}$ satisfies analogues of (2-1) and (2-2). We wish to continue the renormalization procedure inductively. We cannot do this immediately because the point processes $\mathcal{P}_k^{(1)}$ are not Poisson point processes. We must first prove the appropriate versions of (2-1) and (2-2). Since $\mathcal{P}^{(1)} \subseteq \mathcal{P}$, the following analogue of (2-1) is immediate:

(2-1a) $$P(\sharp(B \cap \mathcal{P}^{(1)}) \ge 2) = O(\lambda(B)^2) \quad \text{for Borel sets } B.$$

We must now find $\varepsilon^{(1)} > 0$, $\theta^{(1)} > 0$, $C^{(1)} < \infty$, and quantities $\mu^{(1)}(k)$ such that

(1-1a) $$\sup_k [\exp(\theta^{(1)} k)\, \mu^{(1)}(k)] = C^{(1)},$$

and such that for all collections of disjoint cubes $Q_1, \dots, Q_m$ and positive integers $k_1, \dots, k_m$,

(2-2a) $$P\left(\bigcap_{j=1}^m \{Q_j \cap \mathcal{P}_{k_j}^{(1)} \ne \emptyset\}\right) \le \prod_{j=1}^m [\varepsilon^{(1)} \mu^{(1)}(k_j) \lambda(Q_j)].$$

We choose

$$\varepsilon^{(1)} = \varepsilon^2 \quad \theta^{(1)} = \frac{\theta\alpha}{80 M^2 |\beta|}$$

$$C^{(1)} = \frac{32 K_1^2 C^2 \beta^{2d+2}}{\alpha^{2d+2} \theta^{4d+6}} \quad \mu^{(1)}(k) = C^{(1)} \exp(-\theta^{(1)} k).$$

(Note that $\mu^{(1)}$ automatically satisfies (1-1a). It is not necessary for the numbers $\mu^{(1)}(k)$ to sum to 1.)

We first prove (2-2a) for the case $m = 1$. Let Q_1 be a cube in $\mathbf{R}^d \times [0, \infty)$. We partition Q_1 into small disjoint cubes with sides of length 2η. Let Q be one such cube, and let the sets $G_n(k_0, \dots, k_n)$ be defined in terms of Q as in the preceding part of this proof (where we showed that the clusters are finite). Now look back at our definition of $\mathcal{P}_k^{(1)}$. If $(\mathbf{x}, s)$ is a point in $\mathcal{P}_k^{(1)}$, then some other point $(\mathbf{y}, u)$ in $\mathcal{C}(\mathbf{x}, s)$ lies outside the cube $Q(K, \mathbf{x}, s)$, where

$$K = \frac{k}{2} - (\Delta(\mathbf{x}, s) + \Delta(\mathbf{y}, u)) 5 M^2 |\beta| / \alpha.$$

If $(\mathbf{x}, s) \in Q$ and Q contains no other points of $\mathcal{P}$, then there is an ancestry from $(\mathbf{x}, s)$ to $(\mathbf{y}, u)$ of type $(k_0, \dots, k_n)$ for some $n \ge 1$ and integers $k_0, \dots, k_n$ (the case $n = 0$ can only occur if Q contains at least 2 points of $\mathcal{P}$). By the definition of ancestries, the max-norm distance between the i^{th} point and the $(i+1)^{\text{st}}$ point in such an ancestry is no more than $(k_i + k_{i+1})5M^2|\beta|/\alpha$. Therefore, by the triangle inequality,

$$\|(\mathbf{x},s)-(\mathbf{y},u)\|_{\max} \le (k_0+2k_1+2k_2+\cdots+2k_{n-1}+k_n)5M^2|\beta|/\alpha.$$

It follows that if Q contains at most one point of $\mathcal{P}$ and $Q\cap\mathcal{P}_k^{(1)}\ne\emptyset$, then there exists an integer $n\ge 0$ and positive integers $k_0,\dots,k_n$ such that $G_n(k_0,\dots,k_n)$ is nonempty and

$$(k_0+2k_1+\cdots+2k_{n-1}+k_n)5M^2|\beta|/\alpha \ge K,$$

where K is defined above. Since in this case $\Delta(\mathbf{x},s)=k_0$ and $\Delta(\mathbf{y},u)=k_n$, we have

$$k_0+\cdots+k_n\ge Ak, \quad \text{where} \quad A=\frac{\alpha}{20M^2|\beta|}.$$

Since the probability that Q contains two or more points of $\mathcal{P}$ is $O(\lambda(Q)^2)$ by (2-1), we have by (2-5)

$$\begin{aligned}
&\frac{P(Q\cap\mathcal{P}_k^{(1)}\ne\emptyset)}{\lambda(Q)} - O(\lambda(Q))\\
&\le \sum_{n=1}^{\infty}\ \sum_{k_0+\cdots+k_n\ge Ak} \frac{P(G_n(k_0,\dots,k_n)\ne\emptyset)}{\lambda(Q)}\\
&\le \sum_{n=1}^{\infty}\ \sum_{k_0+\cdots+k_n\ge Ak} \left(\frac{K_1 C\varepsilon|\beta|^{d+1}}{\alpha^{d+1}\theta^{2d+2}}\right)^{n+1} \exp(-\frac{\theta}{2}(k_0+\cdots+k_n))\\
&\le \exp(-A\theta k/4)\sum_{n=1}^{\infty}\left(\frac{4K_1 C\varepsilon|\beta|^{d+1}}{\alpha^{d+1}\theta^{2d+3}}\right)^{n+1}.
\end{aligned}$$

Now sum the geometric series. By (2-7) and our choice of $\varepsilon^{(1)}$, $\theta^{(1)}$, $C^{(1)}$, and $\mu^{(1)}$, we obtain

$$P(Q\cap\mathcal{P}_k^{(1)}\ne\emptyset)\le\varepsilon^{(1)}\mu^{(1)}(k)\lambda(Q)+O(\lambda(Q)^2). \tag{2-8}$$

The inequality in (2-8) holds for each of the cubes Q that partition the big cube Q_1. If we sum over all of the small cubes Q contained in Q_1 and let $\eta\searrow 0$, we obtain (2-2a) for $m=1$.

Now consider general $m\ge 1$ in (2-2a). By subadditivity, we may assume that the cubes Q_j all have sides of length less than 1. Decompose the events $\{Q_j\cap\mathcal{P}_{k_j}^{(1)}\ne\emptyset\}$ into events

$$G_{n_j j}(k_{0j},\dots,k_{n_j j})\ne\emptyset$$

with

$$k_{0j}+\cdots+k_{n_j j}\ge Ak_j$$

as above. Recall from the definition of $\mathcal{P}^{(1)}$ that only one point from a cluster can be a member of $\mathcal{P}^{(1)}$. Also recall from that definition that two points in $\mathcal{P}$ that lie in the same cube with sides of length less than 1 necessarily lie in the same cluster (since according to our assumptions, the quantities $M, |\beta|$, and $1/\alpha$ are all greater than or equal to 1). Therefore, if each Q_j contains a point in $\mathcal{P}^{(1)}$, then the sets

$$\bigcup_{n_j} \bigcup_{k_{0j},\dots,k_{n_j j}} G_{n_j j}(k_{0j},\dots,k_{n_j j})$$

are disjoint.

Now we proceed as in the case $m = 1$. We have

$$P\left(\bigcap_{j=1}^{m}\{Q_j \cap \mathcal{P}_{k_j}^{(1)} \neq \emptyset\}\right) \leq \left(\sum_{n_1=0}^{\infty} \sum_{k_{01}+\cdots+k_{n_1 1}\geq Ak_{n_1}} \cdots \sum_{n_m=0}^{\infty} \sum_{k_{0m}+\cdots+k_{n_m m}\geq Ak_{n_m}}\right)' P\left(\bigcap_{j=1}^{m}\{G_{n_j j}(k_{0j},\dots,k_{n_j j}) \neq \emptyset\}\right),$$

where the notation $(\Sigma \dots \Sigma)'$ means that we only sum over terms for which the corresponding sets $G_{n_j j}(\cdots)$ are disjoint. To complete the proof of (2-2a), it is enough to show that under this disjointness assumption,

$$\frac{P\left(\bigcap_{j=1}^{m}\{G_{n_j j}(k_{0j},\dots,k_{n_j j}) \neq \emptyset\}\right)}{\lambda(Q_1)\cdots\lambda(Q_m)} \leq \prod_{j=1}^{m}\left[\left(\frac{K_1 C\varepsilon|\beta|^{d+1}}{\alpha^{d+1}\theta^{2d+2}}\right)^{n_j+1} \exp\left(\frac{\theta}{2}(k_{0j}+\cdots+k_{n_j j})\right)\right], \tag{2-9}$$

since the rest of the proof is then the same as in the case $m = 1$. Decompose each of the events $\{G_{n_j j}(\cdots) \neq \emptyset\}$ in (2-9) into events involving the small partitioning cubes that were used in the derivation of (2-5). This allows us to write the intersection in (2-9) as a disjoint union of events, each of which is an intersection of events of the form $\{Q(\delta, \mathbf{y}_i, u_i) \cap \mathcal{P}_k \neq \emptyset\}$ for various values of i and k. Because of the disjointness mentioned just before (2-9), the integers i involved in any one of these intersections are distinct. This fact allows us to use (2-2) to bound the probabilities of these intersections by products analogous to the product that appears in (2-4). Now summing over the decomposition and letting $\delta \to 0$ leads to (2-9), just as in the derivation of (2-5).

Successive renormalizations. Let us summarize what we have just accomplished. Using only (2-1), (2-2), and (2-7), we constructed point processes $\mathcal{P}_k^{(1)}$ which satisfy (2-1a) and (2-2a) for certain parameters $\varepsilon^{(1)}$, $\theta^{(1)}$, $C^{(1)}$, and $\mu^{(1)}(k)$. The generic process $\mathcal{A}_t^{(1)}$ corresponding to the point processes $\mathcal{P}_k^{(1)}$ and with rates $\alpha_i^{(1)}$ and $\beta^{(1)}$ is our first renormalization of $\mathcal{A}_t$. As long as the appropriate analogue of (2-7) is satisfied at each stage, we may continue to renormalize, thus inductively obtaining a sequence of collections of point processes $\{\mathcal{P}_k^{(\ell)}; k \geq 1\}$ and a sequence of corresponding generic processes $\mathcal{A}_t^{(\ell)}$, $\ell \geq 1$. The corresponding parameters are

$$\alpha^{(\ell)} = \alpha^{(\ell-1)}/2 = \alpha/2^\ell \qquad \beta^{(\ell)} = 2\beta^{(\ell-1)} = 2^\ell\beta \qquad \varepsilon^{(\ell)} = (\varepsilon^{(\ell-1)})^2 = \varepsilon^{2^\ell}$$

(2-10)

$$\theta^{(\ell)} = \frac{\theta^{(\ell-1)}\alpha^{(\ell-1)}}{80M^2|\beta^{(\ell-1)}|} \geq \frac{\theta\alpha^\ell}{(80M^2|\beta|)^\ell\, 4^{1+2+\cdots+(\ell-1)}} \geq \frac{\theta\alpha^\ell}{(M^2|\beta|)^\ell\, 2^{\ell^2+7\ell}}$$

$$C^{(\ell)} = \frac{32(K_1C^{(\ell-1)})^2(\beta^{(\ell-1)})^{2d+2}}{(\alpha^{(\ell-1)})^{2d+2}(\theta^{(\ell-1)})^{4d+6}} \qquad \mu^{(\ell)}(k) = C^{(\ell)}\exp(-\theta^{(\ell)}k).$$

(We write $\mathcal{A}_t^{(0)} = \mathcal{A}_t$, $\varepsilon^{(0)} = \varepsilon$, etc.) We remark that as before, clusters $\mathcal{C}^{(\ell-1)}(\mathbf{x}, s)$ are used to construct the point process $\mathcal{P}^{(\ell)}$. In this construction, the appropriate cubes of influence are of the form

$$S^{(\ell-1)}(\mathbf{x}, s) = \text{ the interior of } Q\left(\frac{5M^2|\beta^{(\ell-1)}|\Delta^{(\ell-1)}(\mathbf{x}, s)}{\alpha^{(\ell-1)}}, \mathbf{x}, s\right).$$

The renormalizations satisfy

(2-1b) $$P(\sharp(B \cap \mathcal{P}^{(\ell)}) \geq 2) = O(\lambda(B)^2) \quad \text{for Borel sets } B$$

and

(2-2b) $$P\left(\bigcap_{j=1}^m \{Q_j \cap \mathcal{P}_{k_j}^{(\ell)} \neq \emptyset\}\right) \leq \prod_{j=1}^m [\varepsilon^{(\ell)}\mu^{(\ell)}(k_j)\lambda(Q_j)]$$

for disjoint cubes $Q_1, \ldots, Q_m$.

In order to be able to continue the induction at each stage, we only need the following analogue of (2-7):

(2-7b) $$\varepsilon^{(\ell)} \leq \frac{(\alpha^{(\ell)})^{d+1}(\theta^{(\ell)})^{2d+3}}{8K_1C^{(\ell)}|\beta^{(\ell)}|^{d+1}}.$$

We will be able to obtain (2-7b) because of the simple but important fact that $\varepsilon^{(\ell)}$ goes to 0 very fast as $\ell \to \infty$. The quantities $\alpha^{(\ell)}, \theta^{(\ell)}$ go to 0 much more slowly, just as the quantity $|\beta^{(\ell)}|$ goes to ∞ much more slowly. The quantity $C^{(\ell)}$ is the only parameter that grows at a rate that is comparable to the rate at which $\varepsilon^{(\ell)}$ decreases, but for ε small enough, $\varepsilon^{(\ell)}$ will dominate $C^{(\ell)}$.

We use (2-10) to substitute for $\alpha^{(\ell)}$ in terms of α, for $\beta^{(\ell)}$ in terms of β, and the given lower bound for $\theta^{(\ell)}$ in terms of α, β, and θ. After these substitutions it is apparent that (2-7b) is implied by

$$\varepsilon^{(\ell)} \leq \frac{(\alpha/|\beta|)^{(2d+3)\ell+d+1}(\theta/M^{2\ell})^{2d+3}}{K_1 C^{(\ell)} 2^{(2d+3)(\ell^2+7\ell)+3}}.$$

It is clear that this last inequality is in turn implied by

$$\varepsilon^{(\ell)} \leq \frac{1}{K_2^{(\ell+1)^2+4} C^{(\ell)}} \tag{2-11}$$

for some constant K_2 which depends on α, β, θ, M, and d. After we put further assumptions on the constant K_2, we will prove inductively that (2-11) is satisfied for all ℓ if it is satisfied for $\ell = 0$. (Our choice of the exponent in (2-11) is dictated by the way in which the induction argument works.)

By using (2-10) as in the preceding paragraph, it is elementary to check that there exists some constant K_3 depending only on α, β, θ, M, and d such that

$$C^{(\ell)} \leq K_3^{\ell^2+1}(C^{(\ell-1)})^2.$$

Let us assume that the constant K_2 from the preceding paragraph is chosen large enough so that $K_2 \geq (K_3)^2$. Assume inductively that (2-11) is satisfied for some $\ell \geq 0$. Then

$$\begin{aligned}
\varepsilon^{(\ell+1)} &= (\varepsilon^{(\ell)})^2 \\
&\leq \frac{1}{(C^{(\ell)})^2 K_2^{2(\ell+1)^2+8}} \\
&\leq \frac{1}{C^{(\ell+1)}(K_2^{2(\ell+1)^2+8}/K_3^{(\ell+1)^2+1})} \\
&\leq \frac{1}{C^{(\ell+1)}(K_2)^{(3(\ell+1)^2+15)/2}}.
\end{aligned}$$

Since the exponent on the constant K_2 in the last line is bounded below by $(\ell+2)^2+4$, the inductive step in the proof of (2-11) is completed. In other words, if (2-11) is satisfied for $\ell = 0$, it is satisfied for all $\ell \geq 0$. We conclude that for all $\ell \geq 0$, the point processes $\mathcal{P}_k^{(\ell)}$ and corresponding generic processes $\mathcal{A}_t^{(\ell)}$ can be constructed so that (2-1b), (2-2b), and (2-7b) are satisfied, provided the original death rate ε satisfies

$$\varepsilon < \frac{1}{(K_2)^5 C}. \tag{2-12}$$

For the rest of the proof, we will assume that (2-12) holds.

Comparing the different renormalizations. Let $N_t^{(\ell)}$ and $A_t^{(\ell)}$ be respectively the noninteractive and vacant regions corresponding to the generic process $\mathcal{A}_t^{(\ell)}$ constructed in the preceding part of the proof. We will prove that for all $\ell \geq 0$,

$$A_t^{(\ell)} \subseteq N_t^{(\ell)} \cup A_t^{(\ell+1)} \quad \text{a.s. for all } t \geq 0. \tag{2-13}$$

This relationship between the different renormalized processes is the key to the entire proof. Our definitions of clusters and of the sizes $\Delta^{(1)}(\mathbf{x}, s)$ were designed with (2-13) in mind. As we will see in the final part of the proof, (2-13) will enable us to analyze the vacant region A_t in terms of the collection of noninteractive regions $N_t^{(\ell)}$ for $\ell = 0, 1, 2, \ldots$. It is relatively easy to work with the noninteractive regions because, for each ℓ, their faces all move inward at rate $\alpha^{(\ell)}$. Because of the rapidity with which $\varepsilon^{(\ell)}$ goes to 0 as ℓ increases, the noninteractive regions are extremely sparse for large ℓ and small ε. These simple facts motivate the proof of the theorem.

Because of the inductive way in which the renormalization is carried out for each ℓ, it is sufficient to prove (2-13) for the case $\ell = 0$. We write N_t for $N_t^{(0)}$, A_t for $A_t^{(0)}$, etc. It will be seen that certain parts of the proof of (2-13) are most properly done in terms of the finite volume processes that were used in Section 1 to show the existence of generic processes. It is certainly sufficient to prove (2-13) with A_t replaced by the finite volume process $A_{t,j}$, since A_t is the limit of $A_{t,j}$ as $j \to \infty$. The main advantage of finite volume processes is that there are only finitely many regions present in such a process at any finite time t. However, almost all of the notation acquires an extra subscript when working with the finite volume processes, so we will be slightly informal here and continue to speak in terms of the limit A_t. We will indicate those places in the argument where complete rigor would require dealing first with the finite volume processes and then taking limits.

For each point $(\mathbf{x}, s)$ corresponding to a region $D(\mathbf{x}, s; t)$ in the generic process $\mathcal{A}_t$, we wish to define a point $(\xi(\mathbf{x}, s), \sigma(\mathbf{x}, s)) \in \mathcal{P}$ which marks 'the most recent death involved' in the appearance of the region $D(\mathbf{x}, s; t)$. For death regions $D(\mathbf{x}, s; t)$ we simply let $(\xi(\mathbf{x}, s), \sigma(\mathbf{x}, s)) = (\mathbf{x}, s)$. For overlap and collision regions, we recall the maximal collection of regions $D(\mathbf{y}_k, u_k; s)$ used in the definitions of overlap and collision regions in Section 1. Assuming that $\xi(\mathbf{y}_k, u_k)$ and $\sigma(\mathbf{y}_k, u_k)$ are defined for each k, choose i such that $\sigma(\mathbf{y}_i, u_i) = \max_k \sigma(\mathbf{y}_k, u_k)$, and then define

$$(\xi(\mathbf{x}, s), \sigma(\mathbf{x}, s)) = (\xi(\mathbf{y}_i, u_i), \sigma(\mathbf{y}_i, u_i)).$$

In the case of an overlap region $D(\mathbf{x}, s; s)$, we have $(\xi(\mathbf{x}, s), \sigma(\mathbf{x}, s)) = (\mathbf{y}_1, s)$, since in the definition of an overlap region, $D(\mathbf{y}_1, u_1; s)$ is a death region with $u_1 = s$. For collision regions $D(\mathbf{x}, s; s)$, complete rigor would require that we work with the finite volume process, both so that the maximum in the definition of $\sigma(\mathbf{x}, s)$ is actually achieved and so that the induction implied in the phrase "assuming that $\xi(\mathbf{y}_k, u_k)$ and $\sigma(\mathbf{y}_k, u_k)$ are defined" can be successfully carried out.

We claim the following:

$$\text{(2-14)} \quad \text{if} \quad \mathbf{p} \in D(\mathbf{x}, s; t), \quad \text{then} \quad \|\mathbf{p} - \xi(\mathbf{x}, s)\| \leq M[\Delta(\xi(\mathbf{x}, s), \sigma(\mathbf{x}, s)) + |\beta|(t - \sigma(\mathbf{x}, s))].$$

This statement is obvious for death regions $D(\mathbf{x}, s; t)$, since M has been chosen so that $D(\mathbf{x}, s; s) \subseteq B_{\Delta(\mathbf{x},s)M}(\mathbf{x})$, and since death regions shrink in size. For overlap and collision regions $D(\mathbf{x}, s; t)$, let $(\mathbf{y}_i, u_i)$ be as in the preceding paragraph. Then (2-14) is true, provided it is true with $(\mathbf{x}, s)$ replaced by $(\mathbf{y}_i, u_i)$, because $D(\mathbf{x}, s; s) \subseteq D(\mathbf{y}_i, u_i; s)$, and after time s, the radius of the smallest ball that contains $D(\mathbf{x}, s; t)$ increases no faster than rate $M|\beta|$. The induction argument implied by the phrase "provided it is true with $(\mathbf{x}, s)$ replaced by $(\mathbf{y}_i, u_i)$" is properly carried out in the finite volume setting.

We complete the proof of (2-13) (for $\ell = 0$) by showing that for any overlap or collision region $D(\mathbf{y}, u; t)$, there exists a region $D^{(1)}(\mathbf{x}, s; t) \in \mathcal{A}_t^{(1)}$ such that

$$\text{(2-15)} \quad H_i(\mathbf{y}, u; v) - (\alpha(v - \sigma(\mathbf{y}, u))/2)\mathbf{n}_i \subseteq H_i^{(1)}(\mathbf{x}, s; v) \quad \text{for all } v \in [u, t] \text{ and } i = 1, \ldots, d+1.$$

This last expression is just a way of saying that the i^{th} face of the region $D(\mathbf{y}, u; v)$ is at least $\alpha(v - \sigma(\mathbf{y}, u))/2$ units inside of the corresponding face of the region $D^{(1)}(\mathbf{x}, s; v)$. The significance of the factor $\alpha/2$ will be seen in the proof of (2-17) below. It is clear that (2-13) (for $\ell = 0$) holds if (2-15) is true for all overlap and collision regions $D(\mathbf{y}, u; t)$, since any death region $D(\mathbf{y}, u; t)$ is contained in N_t.

Let $D(\mathbf{y}, u; t)$ be a collision or overlap region, and let $D(\mathbf{y}_k, u_k; u)$, $k = 1, \ldots, n$, be the maximal collection of regions used in the definition of $D(\mathbf{y}, u; t)$ in Section 1 (we know that n is finite in the finite volume setting). We now make the assumption that (2-15) is satisfied for all overlap and collision regions that arise prior to time u. More precisely, we assume that for any overlap or collision region $D(\mathbf{y}', u'; t')$ such that $u' < u$ and $t' \leq t$, there exists a region $D^{(1)}(\mathbf{x}', s'; t')$ such that (2-15) is satisfied with $(\mathbf{y}, u)$ replaced by $(\mathbf{y}', u')$, $(\mathbf{x}, s)$ replaced by $(\mathbf{x}', s')$, and t replaced by t'. As above, we are setting up an induction that can be rigorously carried out in the finite volume setting. For this reason, we refer to the assumption just made as the 'inductive hypothesis'. Note that in particular, this inductive hypothesis applies with $D(\mathbf{y}', u'; t') = D(\mathbf{y}_k, u_k; t)$ for all k such that $D(\mathbf{y}_k, u_k; t)$ is not a death region, since for each such k, it follows from our construction of generic processes that $u_k < u$. For these values of k, let $(\mathbf{x}_k, s_k)$ be the point $(\mathbf{x}', s')$ provided by the inductive hypothesis.

We now break the argument into three cases: (i) none of the regions $D(\mathbf{y}_k, u_k; u)$ is a death region; (ii) only $D(\mathbf{y}_1, u_1; u)$ is a death region; (iii) $D(\mathbf{y}_1, u_1; u), \dots, D(\mathbf{y}_m, u_m; u)$ are death regions for some m with $2 \le m \le n$, but none of the remaining regions is a death region. It will be seen that cases (i) and (iii) are relatively easy. Case (ii) is the critical case. The constants used in the definition of clusters were chosen with this case in mind.

First consider case (i). By Proposition 3 and (1-7), there is a region $D^{(1)}(\mathbf{x}, s; t)$ such that

$$\bigcap_{k=1}^{n} D^{(1)}(\mathbf{x}_k, s_k; v) \subseteq D^{(1)}(\mathbf{x}, s; v) \quad \text{for} \quad v \in [u, t], \tag{1-4a}$$

and either

$$\frac{d\gamma_i^{(1)}}{dt}(\mathbf{x}, s; v) = \beta^{(1)} \quad \text{for} \quad v \in [u, t), \tag{1-5a}$$

or

$$\bigcup_{k=1}^{n} H_i^{(1)}(\mathbf{x}_k, s_k; t) = H_i^{(1)}(\mathbf{x}, s; t). \tag{1-6a}$$

We claim that this region $D^{(1)}(\mathbf{x}, s; t)$ satisfies (2-15). If (1-6a) holds, (2-15) follows from the inductive hypothesis (with $(\mathbf{y}', u') = (\mathbf{y}_k, u_k)$ and $(\mathbf{x}', s') = (\mathbf{x}_k, s_k)$) and the fact that $\sigma(\mathbf{y}, u) \ge \sigma(\mathbf{y}_k, u_k)$ for all k. Suppose that (1-5a) holds instead. By (1-4a),

$$\begin{aligned} H_i(\mathbf{y}, u; u) - &\frac{\alpha}{2}(u - \sigma(\mathbf{y}, u))\mathbf{n}_i \\ &= \bigcap_{k=1}^{n} H_i(\mathbf{y}_k, u_k; u) - \frac{\alpha}{2}(u - \sigma(\mathbf{y}, u))\mathbf{n}_i \\ &\subseteq \bigcap_{k=1}^{n} H_i^{(1)}(\mathbf{x}_k, s; u) \quad \subseteq \; H_i^{(1)}(\mathbf{x}, s; u). \end{aligned} \tag{2-16}$$

Since the derivative of $\gamma_i(\mathbf{y}, u; v)$ with respect to v is at least $\beta = \beta^{(1)}/2$, it follows from (1-5a) that the distance between the i^{th} face of $D(\mathbf{y}, u; v)$ and the corresponding face of $D^{(1)}(\mathbf{x}, s; v)$ increases at least at rate $|\beta^{(1)} - \beta| = |\beta| \ge 1 > \alpha/2$. This fact combined with (2-16) gives us (2-15).

Next consider case (ii). The argument for this case will be the same as the previous case, once we find a region $D^{(1)}(\mathbf{x}_1, s_1; t)$ such that for all $i = 1, \dots, d+1$ and $v \in [u, t]$,

$$H_i(\mathbf{y}_1, u_1; v) - (\alpha(v - \sigma(\mathbf{y}_1, u_1))/2)\mathbf{n}_i \subseteq H_i^{(1)}(\mathbf{x}_1, s_1; v). \tag{2-15a}$$

(Since $D(\mathbf{y}_1, u_1; u)$ is a death region, our inductive hypothesis does not help us directly in this case. It would do us no good to try to strengthen the inductive hypothesis, since (2-15a) is not true in general for death regions.) Recall that the faces of the death region $D(\mathbf{y}_1, u_1; t)$ all move inward at rate α, and that the faces of regions in the process $\mathcal{A}_t^{(1)}$ all either move outward, or they move inward at rate $\alpha^{(1)} = \alpha/2$. Thus, if a region $D^{(1)}(\mathbf{x}_1, s_1; v)$ contains a death region $D(\mathbf{y}_1, s_1; v)$ at some time v, then for each i the distance between the i^{th} faces of these two regions increases at least at rate $\alpha/2$ after time v. Hence, to prove (2-15a), it is sufficient to prove that there exists a region $D^{(1)}(\mathbf{x}_1, s_1; t)$ such that for some $v \in [u_1, u]$,

$$D(\mathbf{y}_1, u_1; v) \subseteq D^{(1)}(\mathbf{x}_1, s_1; v). \tag{2-17}$$

To prove (2-17), we first need to find some region $D(\mathbf{z}, w; v)$ with $w < u$ and $v \in [u_1, u]$, such that

$$D(\mathbf{z}, w; v) \cap D(\mathbf{y}_1, u_1; v) \neq \emptyset \quad \text{and} \quad (\xi(\mathbf{z}, w), \sigma(\mathbf{z}, w)) \neq (\mathbf{y}_1, u_1).$$

(It will become apparent later how such a region will help us.) If there is at least one value of $k \geq 2$ such that $(\xi(\mathbf{y}_k, u_k), \sigma(\mathbf{y}_k, u_k)) \neq (\mathbf{y}_1, u_1)$, then we can let $(\mathbf{z}, w) = (\mathbf{y}_k, u_k)$ and $v = u$, since in this case $u_k < u$ and $D(\mathbf{y}_1, u_1; u)$ has nonempty intersection with all of the regions $D(\mathbf{y}_k, u_k; u)$. If on the other hand, $(\xi(\mathbf{y}_k, u_k), \sigma(\mathbf{y}_k, u_k)) = (\mathbf{y}_1, u_1)$ for all $k \geq 2$, then it follows from the definitions that all of the regions $D(\mathbf{y}_k, u_k; u)$ for $k \geq 2$ necessarily arose due to an overlap interaction caused by the appearance of the death region $D(\mathbf{y}_1, u_1; u_1)$. In this case, the overlap was necessarily between $D(\mathbf{y}_1, u_1; u_1)$ and some region $D(\mathbf{z}, w; u_1)$ with $w < u_1 \leq u$, so we have our desired region $D(\mathbf{z}, w; v)$ with $v = u_1$. We have now chosen the value of v for which (2-17) will be satisfied. It remains to use $D(\mathbf{z}, w; v)$ to find the appropriate region $D^{(1)}(\mathbf{x}_1, s_1; v)$.

Let us abbreviate by writing $(\xi, \sigma) = (\xi(\mathbf{z}, w), \sigma(\mathbf{z}, w))$. Note that both of the points (ξ, σ) and $(\mathbf{y}_1, u_1)$ are members of $\mathcal{P}$. If they lie in the same cluster, then there exists a point $(\mathbf{x}_1, s_1) \in \mathcal{P}^{(1)}$ such that $\mathcal{C}(\mathbf{y}_1, u_1) = \mathcal{C}(\xi, \sigma) = \mathcal{C}(\mathbf{x}_1, s_1)$. Now recall how the quantity $\Delta^{(1)}(\mathbf{x}_1, s_1)$ is defined. The remark at the end of the paragraph following the definition of $\Delta^{(1)}(\cdot)$ implies that $\Delta^{(1)}(\mathbf{x}_1, s_1)$ has been chosen large enough so that (2-17) is satisfied.

To finish case (ii), we must prove (2-17) under the assumption that $(\mathbf{y}_1, u_1)$ and (ξ, σ) do not lie in the same cluster. It follows from the definition of clusters that the death regions $D(\mathbf{y}_1, u_1; v)$ and $D(\xi, \sigma; v)$ are disjoint (see the comment made at the end of the section of the proof in which clusters were defined). But $D(\mathbf{y}_1, u_1; v)$ and $D(\mathbf{z}, w; v)$ are not disjoint, so $(\mathbf{z}, w) \neq (\xi, \sigma)$, implying that $D(\mathbf{z}, w; v)$ is not a death region. Since $w < u_1 \leq u$, the inductive hypothesis applies to $D(\mathbf{z}, w; v)$, giving us a region $D^{(1)}(\mathbf{x}_1, s_1; v)$ such that for all $i = 1, \ldots, d+1$,

$$H_i(\mathbf{z}, w; v) - (\alpha(v - \sigma))/2)\mathbf{n}_i \subseteq H_i^{(1)}(\mathbf{x}_1, s_1, v). \tag{2-15b}$$

We will show that the region $D^{(1)}(\mathbf{x}_1, s_1; v)$ found in the preceding paragraph satisfies (2-17). Since we have assumed that $(\mathbf{y}_1, u_1)$ and (ξ, σ) are not in the same cluster, it follows from the definition of clusters that either

$$\|\mathbf{y}_1 - \xi\| \geq 5M^2|\beta|(\Delta(\mathbf{y}_1, u_1) + \Delta(\xi, \sigma))/\alpha, \tag{2-18}$$

or

$$|u_1 - \sigma| \geq 5M^2|\beta|(\Delta(\mathbf{y}_1, u_1) + \Delta(\xi, \sigma))/\alpha. \tag{2-19}$$

We wish to show that both statements imply

$$v - \sigma \geq 4M(\Delta(\mathbf{y}_1, u_1) + \Delta(\xi, \sigma))/\alpha. \tag{2-20}$$

To see that (2-18) implies (2-20), we use (2-14), with $(\mathbf{x}, s)$ replaced by $(\mathbf{z}, w)$, t replaced by v, and with $\mathbf{p}$ equaling any point in $D(\mathbf{y}_1, u_1; v) \cap D(\mathbf{z}, w; v)$. Since the death region $D(\mathbf{y}_1, u_1; v)$ is contained in a ball of radius $M\Delta(\mathbf{y}_1, u_1)$, it follows from (2-14) that

$$\|\mathbf{y}_1 - \xi\| \leq M(\Delta(\mathbf{y}_1, u_1) + \Delta(\xi, \sigma) + |\beta|(v - \sigma)).$$

Combine this inequality with (2-18) to obtain (2-20). To see that (2-19) implies (2-20), it is sufficient to show that $u_1 > \sigma$ if (2-19) holds, since $v \geq u_1$ (and, of course, since $5M^2|\beta| > 4M$). We know that $u \geq \sigma$ and that the death region $D(\mathbf{y}_1, u_1; v)$ vanishes after time $v = u_1 + \Delta(\mathbf{y}_1, u_1)/\alpha$. If both $\sigma > u_1$ and (2-19) held, then the death region $D(\mathbf{y}_1, u_1; v)$ would be empty, contradicting the fact that it has nonempty intersection with $D(\mathbf{y}_k, u_k; v)$. This contradiction establishes the desired conclusion.

We have just proved that if $(\mathbf{y}_1, u_1)$ and (ξ, σ) do not lie in the same cluster, then (2-20) holds. Therefore, by (2-15b),

$$H_i(\mathbf{z}, w; v) - 2M(\Delta(\mathbf{y}_1, u_1) + \Delta(\xi, \sigma))\mathbf{n}_i \subseteq H_i^{(1)}(\mathbf{x}_1, s_1; v)$$

for $i = 1, \ldots, d+1$. Now use the fact that $D(\mathbf{y}_1, u_1; v)$ is contained in a ball of radius $M\Delta(\mathbf{y}_1, u_1)$ and has nonempty intersection with $D(\mathbf{z}, w; v)$ to conclude that (2-17) is satisfied.

Finally, we consider case (iii). As in case (ii), it is sufficient to prove the analogue of (2-17) for the regions $D(\mathbf{y}_k, u_k; u)$, $k = 1, \ldots, m$; namely, that for each of the death regions $D(\mathbf{y}_k, u_k; u)$, there exists a time $v \in [u_k, u]$ and a region $D^{(1)}(\mathbf{x}_k, s_k; v)$ such that $D(\mathbf{y}_k, u_k; v) \subseteq D^{(1)}(\mathbf{x}_k, s_k; v)$. Since the death regions $D(\mathbf{y}_k, u_k; t)$ have nonempty intersection for $k = 1, \ldots, m$, their corresponding cubes of influence $S(\mathbf{y}_k, u_k)$ have nonempty intersection (this follows from the comments made at the end of the section of the proof where clusters are defined). Therefore, the points $(\mathbf{y}_k, u_k)$, $k = 1, \ldots, m$, lie in the same cluster. Since $m \geq 2$, there is some point $(\mathbf{x}_1, s_1) \in \mathcal{P}^{(1)}$ that is also a member of that cluster. Let $(\mathbf{x}_k, s_k) = (\mathbf{x}_1, s_1)$ for $k = 1, \dots, m$. As explained in case (ii), $\Delta^{(1)}(\mathbf{x}_1, s_1)$ has been chosen large enough so that the analogue of (2-17) for the regions $D(\mathbf{y}_k, u_k; u)$ is satisfied for $k = 1, \dots, m$. This completes case (iii), so the proof of (2-15) and hence of (2-13) is finished.

The final upper bounds. It follows from (2-13) that

$$P(\mathbf{y} \in A_t) \;\le\; \limsup_{\ell\to\infty} P(\mathbf{y} \in A_t^{(\ell)}) \;+\; \sum_{\ell=0}^{\infty} P(\mathbf{y} \in N_t^{(\ell)}). \tag{2-21}$$

We will complete the proof of the theorem by obtaining appropriate bounds for the terms on the right side of (2-21). More precisely, we will show that the constant K_2 can be chosen sufficiently large so that for $\varepsilon > 0$ satisfying (2-12),

$$\limsup_{\ell\to\infty} P(\mathbf{y} \in A_t^{(\ell)}) = 0 \quad \text{for fixed } t, \text{ uniformly in } \mathbf{y}, \tag{2-22}$$

and that

$$\lim_{\varepsilon\searrow 0} \sum_{\ell=0}^{\infty} P(\mathbf{y} \in N_t^{(\ell)}) = 0 \quad \text{uniformly in } t \text{ and } \mathbf{y}. \tag{2-23}$$

We prove (2-23) first. Note that if $\mathbf{y} \in N_t^{(\ell)}$, then there exists a point $(\mathbf{x}, s) \in \mathcal{P}_k^{(\ell)}$ for some positive integer k such that

$$(\mathbf{x}, s) \in Q((Mk/\alpha^{(\ell)}), \mathbf{y}, t).$$

The reason for this is that the death region $D^{(\ell)}(\mathbf{x}, s; t)$ is always contained in the ball of radius Mk centered at $\mathbf{x}$, and furthermore, this death region vanishes after time $k/\alpha^{(\ell)}$. (We are also relying on the obvious inequality $Mk/\alpha^{(\ell)} \ge (k/\alpha^{(\ell)}) \vee (Mk)$.) Therefore, by (2-2b) with $m = 1$,

$$\begin{aligned} P(\mathbf{y} \in N_t^{(\ell)}) &\le \sum_{k=1}^{\infty} P(Q((Mk/\alpha^{(\ell)}), \mathbf{y}, t) \cap \mathcal{P}_k^{(\ell)} \ne \emptyset) \\ &\le \sum_{k=1}^{\infty} \varepsilon^{(\ell)} (2Mk/\alpha^{(\ell)})^{d+1} \mu^{(\ell)}(k) \\ &= \sum_{k=1}^{\infty} \varepsilon^{(\ell)} (2Mk/\alpha^{(\ell)})^{d+1} C^{(\ell)} \exp(-\theta^{(\ell)} k) \\ &\le \frac{\varepsilon^{(\ell)} 2^{2d+3} M^{d+1} C^{(\ell)} (d+1)!}{(\alpha^{(\ell)}\theta^{(\ell)})^{d+1}}. \end{aligned} \tag{2-24}$$

We have used the fact that

$$\sum_{k=1}^{\infty} k^{d+1} \exp(-ak) \leq (2/a)^{d+2}\,(d+1)!$$

for $0 < a < 1$.

Now use (2-10) to substitute for $\alpha^{(\ell)}, \beta^{(\ell)}$, and $\theta^{(\ell)}$ in terms of α, β, and θ. It follows, as in the argument preceding (2-12), that we can find a constant K_4 depending on α, β, θ, M, and d such that the expression on the right side of (2-24) is bounded above by

$$\varepsilon^{(\ell)} C^{(\ell)} K_4^{\ell^2+1}.$$

After substituting in (2-24), we obtain

$$P(\mathbf{y} \in N_t^{(\ell)}) \leq \frac{K_4^{\ell^2+1}}{K_2^{(\ell+1)^2+4}}. \tag{2-25}$$

Note that the right side of (2-25) does not depend on t or $\mathbf{y}$. As $\varepsilon \searrow 0$, we can let $K_2 \to \infty$ in accordance with (2-12). Thus (2-23) follows.

We now prove (2-22). Define an auxiliary process $\hat{N}_t^{(\ell)}$ by

$$\hat{N}_t^{(\ell)} = \bigcup_{(\mathbf{x},s)\in\mathcal{P}^{(\ell)}:s\leq t} \; \bigcap_{i=1}^{d+1} H(\mathbf{n}_i, \mathbf{x} + (2^\ell \beta(t-s) - \Delta^{(\ell)}(\mathbf{x},s))\mathbf{n}_i).$$

This process is just like $N_t^{(\ell)}$, except that all of the occupation rates $\alpha_i^{(\ell)}$ have been replaced by the value $\beta^{(\ell)} = 2^\ell \beta$. The points at which deaths occur are the same for the processes $N_t^{(\ell)}$ and $\hat{N}_t^{(\ell)}$, so it is clear that $N_t^{(\ell)} \subseteq \hat{N}_t^{(\ell)}$. If we define the generic process $\hat{\mathcal{A}}_t^{(\ell)}$ corresponding to $\hat{N}_t^{(\ell)}$, Proposition 4 implies that $A_t^{(\ell)} \subseteq \hat{A}_t^{(\ell)}$. A little thought based on the details of the construction given in Section 1 reveals that $\hat{A}_t^{(\ell)} = \hat{N}_t^{(\ell)}$. Thus

$$P(\mathbf{y} \in A_t^{(\ell)}) \leq P(\mathbf{y} \in \hat{N}_t^{(\ell)}).$$

Now note that if $\mathbf{y} \in \hat{N}_t^{(\ell)}$, then $\mathbf{y}$ lies in some death region $\hat{D}^{(\ell)}(\mathbf{x}, s; t)$, where $(\mathbf{x}, s) \in \mathcal{P}_k^{(\ell)}$ for some $k > 0$. The faces of such a region move outward at speed $2^\ell |\beta|$, so $(\mathbf{y}, t) \in Q(M(2^\ell|\beta|(t-s)+k), \mathbf{x}, s)$, or equivalently,

$$(\mathbf{x}, s) \in Q(M(2^\ell|\beta|(t-s)+k), \mathbf{y}, t).$$

Noting that $M(2^\ell|\beta|(t-s)+k) \leq 2^{\ell+1} M|\beta|(t \vee 1)k$, it follows from (2-2b), as in (2-24), that

$$\begin{aligned} P(\mathbf{y} \in A_t^{(\ell)}) &\leq P(\mathbf{y} \in \hat{N}_t^{(\ell)}) \\ &\leq \sum_{k=1}^{\infty} \varepsilon^{(\ell)} (2^{\ell+2} M k |\beta| (t \vee 1))^{d+1} C^{(\ell)} \exp(-\theta^{(\ell)} k) \\ &\leq \varepsilon^{(\ell)} 2^{2d+3} (2^{\ell+2} (t \vee 1) M |\beta| / \theta^{(\ell)})^{d+1} C^{(\ell)} (d+1)!\,. \end{aligned}$$

As in the proof of (2-23), we may choose a constant K_5 depending only on α, β, θ, M, and d such that the term on the right side of this last expression is bounded above by

$$\frac{K_5^{\ell^2+1}(t \vee 1)^{d+1}}{K_2^{(\ell+1)^2+3}}. \tag{2-26}$$

By choosing $\varepsilon > 0$ sufficiently small, we can choose $K_2 > K_5$ in accordance with (2-12). For such a choice of ε and K_2, the expression in (2-26) converges to 0 as $\ell \to \infty$. Since $\mathbf{y}$ does not appear in this expression, (2-22) follows. ■

3. Toom's Theorem and other applications. Now that we know precisely which families of generic processes survive for small ε, we can use them as comparison processes for other models of interest. In this section, we will show how to carry out this comparison for the general class of discrete time processes studied by Toom [13]. As a result, we obtain a relatively easy proof of a generalization of the main theorem in [13].

Toom Processes. The models most directly related to our generic processes are certain probabilistic cellular automata that we call Toom processes. This is the class of processes to which Toom's Theorem applies. The two discrete time processes mentioned in the introduction, namely the basic discrete time contact process and Toom's model, are both in this class.

To fix notation, we will briefly describe the construction of Toom processes. A **(d-dimensional) Toom process with range** r is defined in terms of a **Toom rule**, which is a function

$$\varphi : \{\text{subsets of } \mathbf{Z}^d\} \to \{\text{subsets of } \mathbf{Z}^d\}$$

satisfying four conditions:

(1) (monotonicity) $S_1 \subseteq S_2 \Rightarrow \varphi(S_1) \subseteq \varphi(S_2)$;
(2) (translation invariance) $\varphi(S) + \mathbf{x} = \varphi(S + \mathbf{x})$;
(3) (range r) $\mathbf{x} \in \varphi(S) \Leftrightarrow \mathbf{x} \in \varphi(S \cap B_r(\mathbf{x}))$;
(4) (nontriviality) $\varphi(\emptyset) = \emptyset$ and $\varphi(\mathbf{Z}^d) = \mathbf{Z}^d$.

Fix a parameter value $\tilde{\varepsilon} \in [0,1]$, the **death rate**. For $(\mathbf{x}, t) \in \mathbf{Z}^d \times \{1,2,3,\dots\}$, let $M(\mathbf{x},t)$ be independent identically distributed Bernoulli random variables with $P(M(\mathbf{x},t) = 0) = \tilde{\varepsilon}$, and let

$$\tilde{\mathcal{P}} = \{(\mathbf{x}, t) : M(\mathbf{x}, t) = 0\}.$$

We now define the Toom process with rule φ, death rate $\tilde{\varepsilon}$ and initial state $\tilde{A}_0$, where $\tilde{A}_0$ is is an arbitrary subset of $\mathbf{Z}^d$. For $t = 0,1,2,\dots$, define inductively

$$\tilde{A}_{t+1} = \varphi(\tilde{A}_t) \cup \{\mathbf{x} : (\mathbf{x}, t+1) \in \tilde{\mathcal{P}}\}.$$

It is also natural to consider a somewhat more general class of processes as follows. For $(\mathbf{x}, t) \in \mathbf{Z}^d \times \{1, 2, 3, \dots\}$ let $\tilde{\Delta}(\mathbf{x}, t)$ be independent identically distributed positive integer-valued random variables, with probability density $\tilde{\mu}$. Let φ, $\tilde{\mathcal{P}}$, and $\tilde{A}_0$ be as above. The **generalized Toom process** with rule φ, death rate $\tilde{\varepsilon}$, initial state $\tilde{A}_0$ and size density $\tilde{\mu}$ is defined inductively by

$$\tilde{A}_{t+1} = \varphi(\tilde{A}_t) \cup \left(\bigcup_{(\mathbf{x}, t+1) \in \mathcal{P}} (B_{\tilde{\Delta}(\mathbf{x}, t+1)}(\mathbf{x}) \cap \mathbf{Z}^d) \right).$$

We may interpret a generalized Toom process as follows. The system follows the rule φ except near the random points $(\mathbf{x}, t) \in \tilde{\mathcal{P}}$. At such a point $(\mathbf{x}, t)$ in space-time, a 'catastrophe' occurs at time t which vacates all the sites within a certain random distance of $\mathbf{x}$.

We have used notation that is analogous to that used for generic processes. It is more standard to think of the state of one of these processes as a configuration of 0's and 1's located at the sites of the integer lattice $\mathbf{Z}^d$. In our notation, $\tilde{A}_t$ is the set of sites at which there is a 0 at time t, or in terms of population models, the vacant region.

Toom [13] gave an elegant criterion for survival at small death rates in families of Toom processes parametrized by the death rate. Fix a Toom rule φ with range r. For $t = 1, 2, 3, \dots$, let

$$\varphi^t = \varphi \circ \cdots \circ \varphi \qquad (t \text{ times}).$$

Toom's eroder condition. *For all finite sets $A \subseteq \mathbf{Z}^d$,*

$$\varphi^t(A) = \emptyset$$

for some value of $t \in \{1, 2, 3, \dots\}$ which may depend on A.

Toom's Theorem is as follows: Let $\tilde{A}_t$ be the Toom process with rule φ, death rate $\tilde{\varepsilon}$, and initial state $\tilde{A}_0 = \emptyset$. If Toom's eroder condition holds, then

$$\lim_{\tilde{\varepsilon} \searrow 0} \limsup_{t \to \infty} P(\mathbf{0} \in \tilde{A}_t) = 0,$$

whereas if Toom's eroder condition does not hold, the limit is 1. We will use generic processes and our Theorem 1 to prove the hard half of Toom's Theorem, namely the first part (the proof of the easy half is essentially the same as the proof of Proposition 2). In fact, as we will see, the same proof works for generalized Toom processes, provided the size distribution $\tilde{\mu}$ satisfies (1-1). Toom has communicated to one of us (L. Gray) that he was not able to prove the more general result, since contour methods do not seem to work well with generalized Toom processes.

Toom processes have a simple property that makes it easy for us to compare them to generic processes. For any point $\mathbf{p}$ and unit vector $\mathbf{n}$ in $\mathbf{R}^d$, let

$$\tilde{H}(\mathbf{n}, \mathbf{p}) = H(\mathbf{n}, \mathbf{p}) \cap \mathbf{Z}^d.$$

We say that a unit vector $\mathbf{n} \in \mathbf{R}^d$ is **rationalizable** if there exists a vector $\mathbf{x} \in \mathbf{Z}^d$ such that $\mathbf{n} = \mathbf{x}/\|\mathbf{x}\|$.

Proposition 5. *Let* $\mathbf{n}$ *be a rationalizable unit vector, and let* φ *be a Toom rule with range* r. *Let*

$$\alpha = \sup\{a : \varphi(\tilde{H}(\mathbf{n}, \mathbf{0})) \subseteq \tilde{H}(\mathbf{n}, a\mathbf{n})\}.$$

Then $\alpha \in [-r, r]$ *and*

$$\varphi^t(\tilde{H}(\mathbf{n}, \mathbf{0})) = \tilde{H}(\mathbf{n}, \alpha t \mathbf{n})$$

for all $t = 0, 1, 2, \ldots$.

This simple fact, which is also used by Toom [13], follows easily from the four properties in the definition of a Toom rule. We will not give the proof here.

For any rationalizable unit vector $\mathbf{n}$ and rule φ, we let $\alpha(\mathbf{n}, \varphi)$ be the quantity α given in Proposition 5. We now state an equivalent form of Toom's eroder condition:

Proposition 6. *Fix a* d*-dimensional Toom rule* φ *with range* r. *Then Toom's eroder condition is satisfied if and only if there exist* $d+1$ *rationalizable unit vectors* $\mathbf{n}_i \in \mathbf{R}^d$ *satisfying the conditions of Proposition 1, and a real number* $t \geq 0$, *such that*

$$\bigcap_{i=1}^{d+1} H(\mathbf{n}_i, (\alpha(\mathbf{n}_i, \varphi)t - 1)\mathbf{n}) = \emptyset. \tag{3-1}$$

This alternate form of the eroder condition is in fact the one that Toom used to prove his theorem. Note that if we set $\alpha_i = \alpha(\mathbf{n}_i, \varphi)$, (3-1) is the same as our generic eroder condition. The proof of Proposition 6 is easy in one direction, since (3-1) obviously implies Toom's eroder condition. To prove the opposite direction, one first shows that Toom's condition implies that there is a finite collection of rationalizable vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ such that (3-1) is satisfied with $d+1$ replaced by k. Then, assuming that $\alpha(\mathbf{n}_1, \varphi) = \max_i \alpha(\mathbf{n}_i, \varphi)$, one projects onto the $d-1$-dimensional hyperplane that forms the boundary of $H(\mathbf{n}_1, (\alpha(\mathbf{n}_1, \varphi)t - 1)\mathbf{n}_1)$. An inductive argument completes the proof. The details are left to the reader.

We now define, for each Toom rule that satisfies (3-1), a corresponding set of parameters for a generic process that will be used as a comparison process. Suppose that φ is a rule with range r that satisfies condition (3-1) for a given set of unit vectors $\mathbf{n}_i$. Let $\tilde{\mu}$ be a size density that satisfies (1-1), and define another size density μ by

$$\mu(\Delta + k) = \tilde{\mu}(k),$$

where Δ is the smallest integer greater than or equal to $2r + \sqrt{d}/2$. Choose $\tilde{\varepsilon} \geq 0$ and then let ε be the solution to the equation $\tilde{\varepsilon} = 1 - \exp(-\varepsilon)$. The reason for these choices of μ and ε will become clear later. Let $\tilde{A}_t$ be the generalized Toom process with rule φ, death rate $\tilde{\varepsilon}$, size density $\tilde{\mu}$ and initial state $\tilde{A}_0 = \emptyset$. Also let A_t be the vacant region of the generic process with orientations $\mathbf{n}_i$, occupation rates $\alpha_i = \alpha(\mathbf{n}_i, \varphi)$, interaction rate $\beta = -r$, death rate ε, and size density μ. We say that A_t is a **canonical comparison process** for $\tilde{A}_t$.

Theorem 2. *Let $\tilde{A}_t$ be the generalized Toom process with rule φ, death rate $\tilde{\varepsilon}$, size density $\tilde{\mu}$ satisfying (1-1), and initial state $\emptyset$. Suppose that φ satisfies (3-1) for a given set of unit vectors $\mathbf{n}_i$, and let A_t be the corresponding canonical comparison process. Then the processes A_t and $\tilde{A}_t$ may be defined jointly ('coupled') so that*

$$\tilde{A}_t \subseteq A_t$$

for all $t \geq 0$.

Proof. Let the process A_t be constructed as in Section 1. For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbf{Z}^d$ and $t = 1, 2, 3, \ldots$, define

$$M(\mathbf{x}, t) = \begin{cases} 0 & \text{if} \quad \mathcal{P} \cap Q(\frac{1}{2}, \mathbf{x}, t - \frac{1}{2}) \neq \emptyset \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathcal{P}$ is the Poisson point process with intensity parameter ε used in the construction of A_t. Note that $Q(\frac{1}{2}, \mathbf{x}, t - \frac{1}{2})$ is a unit cube. The random variables $M(\mathbf{x}, t)$ are independent identically distributed Bernoulli, with

$$P(M(\mathbf{x}, t) = 0) = \tilde{\varepsilon} = 1 - \exp(-\varepsilon).$$

For $(\mathbf{x}, t)$ such that $M(\mathbf{x}, t) = 0$, let

$$\hat{\Delta}(\mathbf{x}, t) = -\Delta + \sup\{k : \mathcal{P}_k \cap Q(\frac{1}{2},\ \mathbf{x},\ t - \frac{1}{2}) \neq \emptyset\}.$$

Note that these random variables are independent and identically distributed. We use them to construct a generalized Toom process $\hat{A}_t$ with rule φ and initial state $\emptyset$. Unfortunately, the common distribution of the random variables $\hat{\Delta}(\mathbf{x}, t)$ is not the same as $\tilde{\mu}$. However, it dominates $\tilde{\mu}$ in the appropriate sense, so that it is straightforward to construct a generalized Toom process $\tilde{A}_t$ with rule φ, death rate $\tilde{\varepsilon}$, size density $\tilde{\mu}$ and initial state $\emptyset$ so that $\tilde{A}_t \subseteq \hat{A}_t$. We leave the details to the reader. Note that Δ has been chosen large enough so that

$$\text{(3-2)} \quad M(\mathbf{x},t) = 0 \quad \text{and} \quad \hat{\Delta}(\mathbf{x},t) = k \quad \Rightarrow$$
$$\exists\,(\mathbf{y},u) \in \mathcal{P}_{k+\Delta} \cap Q(\frac{1}{2},\, \mathbf{x},\, t-\frac{1}{2}) \text{ such that } B_{k+r}(\mathbf{x}) \subseteq D(\mathbf{y},u;t).$$

To see that this is the case, note that for $(\mathbf{y},u)$ as in (3-2), (i) $\|\mathbf{x}-\mathbf{y}\| \le \sqrt{d}/2$, and (ii) the size of $D(\mathbf{y},u;t)$ is at least $k+r+\sqrt{d}/2$ since $t-u \le 1$.

The comparison. We will prove by induction on $t = 0, 1, 2, \ldots$, that

$$\text{(3-3)} \qquad \mathbf{p} \in \hat{A}_t \Rightarrow B_r(\mathbf{p}) \subseteq A_t,$$

from which it follows that $\hat{A}_t \subseteq A_t$, as desired. The case $t = 0$ is obvious since both processes have initial state $\emptyset$. Now let t be a positive integer. By (3-2), if $\mathbf{p}$ is a site that is vacated at time t due to a 'catastrophe' at some site $\mathbf{x}$ such that $\hat{\Delta}(\mathbf{x},t) = k$ and $\mathbf{p} \in B_k(\mathbf{x})$, then $B_r(\mathbf{p}) \subseteq A_t$. By the construction of Toom processes, all other points $\mathbf{p} \in \hat{A}_t$ are in $\varphi(\hat{A}_{t-1})$, so to complete the proof of (3-3), it is sufficient to prove

$$\text{(3-4)} \qquad \mathbf{p} \in \varphi(\hat{A}_{t-1}) \Rightarrow B_r(\mathbf{p}) \subseteq A_t.$$

Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be those points in $\hat{A}_{t-1}$ that are within distance r of $\mathbf{p}$:

$$\hat{A}_{t-1} \cap B_r(\mathbf{p}) = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}.$$

By the inductive hypothesis, there exist regions $D(\mathbf{y}_k, u_k; t-1)$ such that

$$\text{(3-5)} \qquad B_r(\mathbf{p}_k) \subseteq D(\mathbf{y}_k, u_k; t-1)$$

for $k = 1, \ldots, n$. Since the distance between each $\mathbf{p}_j$ and $\mathbf{p}$ is less than or equal to r, we also have

$$\text{(3-6)} \qquad \mathbf{p} \in \bigcap_{k=1}^{n} D(\mathbf{y}_k, u_k; t-1).$$

In order to prove (3-4), we first need the following statement:

$$\text{(3-7)} \qquad B_r(\mathbf{p}) \subseteq \bigcup_{k=1}^{n} H_i(\mathbf{y}_k, u_k; t) \quad \text{for all} \quad i = 1, \ldots, d+1.$$

(Refer back to (1-3) for the definition of the half-spaces $H_i(\mathbf{y}_k, u_k; t)$.) To prove (3-7), we choose real numbers q_{ik} such that

$$H_i(\mathbf{y}_k, u_k; t-1) = H(\mathbf{n}_i, q_{ik}\mathbf{n}_i)$$

for $i = 1, \ldots, d+1$ and $k = 1, \ldots, n$. Let

$$q_i = \min\{q_{ik} : k = 1, \ldots, n\}.$$

Since all of the functions γ_i in (1-3) have slopes which are bounded above by α_i,

$$H(\mathbf{n}_i, (q_i + \alpha_i)\mathbf{n}_i) \subseteq \bigcup_{k=1}^{n} H_i(\mathbf{y}_k, u_k; t) \quad \text{for } i = 1, \ldots, d+1. \tag{3-8}$$

Thus it is enough to show that $B_r(\mathbf{p})$ is contained in the set on the left of (3-8) for each i. Fix i. By (3-5), $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subseteq H(\mathbf{n}_i, (q_i + r)\mathbf{n}_i)$. Assuming that the hypothesis of (3-4) holds, it follows from the finite range and monotonicity properties that

$$\mathbf{p} \in \varphi(\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}) \subseteq \varphi(H(\mathbf{n}_i, (q_i + r)\mathbf{n}_i) \cap \mathbf{Z}^d).$$

By Proposition 5,

$$\mathbf{p} \in H(\mathbf{n}_i, (q_i + r + \alpha_i)\mathbf{n}_i),$$

from which it follows that

$$B_r(\mathbf{p}) \subseteq H(\mathbf{n}_i, (q_i + \alpha_i)\mathbf{n}_i),$$

completing the proof of (3-7).

To complete the proof of the theorem, we apply Proposition 3. Let $D(\mathbf{x}, s; u), u \geq s$, be a region satisfying (1-4) and either (1-5) or (1-6), with the regions $D(\mathbf{y}_k, u_k; u)$ being as above. The existence of such a region $D(\mathbf{x}, s; u)$ is guaranteed by Proposition 3. For all i such that (1-5) is satisfied, (3-6) and (1-4) imply that $B_r(\mathbf{p}) \subseteq H_i(\mathbf{x}, s; t)$ (since $\beta = -r$). For all i such that (1-6) is satisfied, (3-7) implies the same result. Proposition 3 says that every i must satisfy either (1-5) or (1-6), so

$$B_r(\mathbf{p}) \subseteq \bigcap_{i=1}^{d+1} H_i(\mathbf{x}, s; t) = D(\mathbf{x}, s; t) \subseteq A_t,$$

and the proof is complete. ■

Note that the size density μ of the canonical comparison process satisfies (1-1) if the size density $\tilde{\mu}$ of the corresponding generalized Toom process satisfies (1-1), and that the generic eroder condition is satisfied for the comparison process if Toom's eroder condition is satisfied for the rule of the generalized Toom process. Therefore, we have the following generalization of Toom's Theorem:

Corollary. *Let φ be a Toom rule which satisfies (3-1), and let $\tilde{\mu}$ be a discrete probability density on $\{1, 2, 3, \ldots\}$ satisfying (1-1). Let $\tilde{A}_t$ be the* *corresponding generalized Toom process with death rate $\tilde{\varepsilon}$ and initial state $\tilde{A}_0 = \emptyset$. Then*

$$\lim_{\tilde{\varepsilon} \searrow 0} \limsup_{t \to \infty} P(\mathbf{0} \in \tilde{A}_t) = 0.$$

APPLICATIONS TO OTHER MODELS. The class of Toom processes does not include all of the finite range translation invariant attractive (monotone) models that are of interest. In a Toom process, the conditional probability that there is a change at a site $\mathbf{x}$ at time t, given the state of the process at time $t-1$, is either 1, $1-\tilde{\varepsilon}$, $\tilde{\varepsilon}$, or 0. When $\tilde{\varepsilon} = 0$, the process is deterministic, so we may view Toom processes as $\tilde{\varepsilon}$ perturbations of deterministic cellular automata. The Corollary to Theorem 2 says that if a Toom rule φ satisfies Toom's eroder condition, then the deterministic process is stable under perturbation by $\tilde{\varepsilon}$ when the initial state is 'all occupied'.

In general Markovian discrete time models A_t, the conditional probability that there is a change at time t, given $A_{t-1} = A$, equals some quantity $c(\mathbf{x}, A)$ which may not be close to either 0 or 1. This is more in line with what happens in continuous time, where the rate at which there is a change at a site $\mathbf{x}$ at time t, conditioned on $A_{t-} = A$, is given by a flip rate $c(\mathbf{x}, A)$.

One can ask about the stability of these more general models under various types of perturbations of the rates $c(\mathbf{x}, A)$. We suggest that generic processes are quite useful for determining stability properties. In a future paper, we intend to prove stability for various classes of systems that don't fit into the context of the present paper. We have already found certain eroder conditions that lead to some new stability results. These eroder conditions also apply to the sexual contact process, and relatively simple proofs of both of the main stability results in Durrett and Gray [5] can be given. We are however a long way from finding necessary and sufficient conditions for general systems. In general, one needs some probabilistic analogue of Proposition 5.

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