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Stavskaya's Measure Is Weakly Gibbsian

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Abstract. Stavskaya's model is a one-dimensional Boolean probabilistic cellular automaton very similar to the contact and directed percolation processes. There is always an absorbing measure but close to the deterministic limit the model also shows a non-trivial invariant measure. We show that the latter "Stavskaya's" measure is weakly Gibbsian with an exponentially decaying interaction potential.

KEYWORDS: probabilistic cellular automata, Gibbs measure

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1. Introduction

Not much is known about global characterizations of stationary measures of interacting particle systems. There are of course the stochastic Ising model with reversible Gibbs measures as stationary distributions, see, e.g., [14], and weakly coupled systems with unique and Gibbsian stationary distributions, see, e.g., [6, 11]; for Gibbsian characterizations of transient measures, see [1]. In other cases, outside equilibrium, very little information is available about the Gibbsian nature of transient or of stationary measures. There are some negative results, e.g. in [13] about the extremal invariant measures of the voter model and there are some elementary surprises, like in the exceptional case of the invariance of the standard Ising model in dimensions $d \leq 2$ for a nonreversible dynamics, see [4].

In the present paper, we investigate the Gibbsianness of one of the simplest examples of an interacting particle system. We deal with Stavskaya's model, see also [15, 16, 18], which is a one-dimensional probabilistic cellular automaton. That model has a phase transition; for small noise there appears a non-trivial stationary measure. We show that it is weakly Gibbsian in the sense that there exists an exponentially decaying interaction potential for a full measure set of configurations, [5, 8]. To the best of our knowledge, our work establishes the only example so far away from the weak coupling regime, where a Gibbsian property can be established in a genuine nonequilibrium context.

The construction of the potential follows earlier work and methods that have been applied in [7, 9, 10] to control the so called Kozlov potential, [2]. A new mathematical difficulty in the present work is the presence of a hard core effect in the dynamics; the transition probabilities are not bounded away from zero. In fact, that effect is also responsible for why we think, see Remark 6.1 in Section 6, the measure is not Gibbsian. Nevertheless a weakly Gibbsian property can be established.

The reason to be interested in Gibbsian properties has a long history and motivations continue to come from various sides. Obviously Stavskaya's model is not so very realistic to be wholly useful in discussions on the physics and the construction of nonequilibrium statistical mechanics. Nevertheless, its simplicity combined with the presence of a phase transition make it into an interesting test case for attempts to characterize transient and stationary measures for spatially extended stochastic dynamics. One of the important avenues that gets opened and is related to Gibbsianness is the validity of a variational principle. We hope to come back to that last point in a future publication. The main point however was already started in [9, 10]: to understand stationary measures as projections of Gibbs measures, see [3, 12].

2. Stavskaya's model

2.1. Definition

Stavskaya's model is a discrete time Markov process on $\{+,-\}^{\mathbf{Z}}$. We think of variables $\eta_t(i)$ at time t at sites $i \in \mathbf{Z}$ which are simultaneously updated according to the rule: $\eta_{t+1}(i) = +$ if $\eta_t(i-1) = \eta_t(i) = +$, and if otherwise, then $\eta_{t+1}(i) = +$ with probability $\varepsilon \in (0,1)$. That basic rule is abbreviated as, for $a, b = \pm$,

$$\begin{aligned} p(+ \mid a,b) &\equiv 1 \quad \text{if } a = b = +, \\ &\equiv \varepsilon \quad \text{otherwise}, \\ p(- \mid a,b) &\equiv 1 - p(+ \mid a,b), \end{aligned} \tag{2.1}$$

through which we define the transition probabilities

$$\mathrm{Prob}[\eta_{t+1}(i) = a_i, i \in A \mid \eta_t] \equiv \prod_{i \in A} p(a_i \mid \eta_t(i-1), \eta_t(i)) \tag{2.2}$$

for all finite $A \subset \mathbf{Z}$, $a_i = \pm$.

In that way the model defines a local and translation invariant probabilistic cellular automata with one parameter ε for which however the transition probabilities are not bounded away from zero. As a result, the state with all $\eta(i) = +$ is invariant for all ε.

Note also that Stavskaya's model is monotone in the sense that $p(+|a,b)$ is a non-decreasing function of a and b. That has some pleasant consequences. For example, if we start the dynamics from all minuses and we denote the evolved measure at time $t \geq 0$ by μ_t, then the weak limit

$$\lim_t \mu_t = \mu$$

exists and is stationary.

2.2. Directed percolation

It is useful to think about the previous rules in terms of a noisy perturbation of a deterministic (parallel) updating. We define therefore the map

$$\begin{aligned} M(a,b) &\equiv + \quad \text{if } \ a = b = +, \\ &\equiv - \quad \text{otherwise} \end{aligned} \tag{2.3}$$

and a Bernoulli random field $(e_t(i), i \in \mathbf{Z}, t = 1, 2, \ldots); e_t(i) = 0, 1$ with density $\text{Prob}[e_t(i) = 1] = \varepsilon$.

Stavskaya's dynamics can thus be represented as

$$\eta_t(i) = (1 - e_t(i))\, M(\eta_{t-1}(i-1), \eta_{t-1}(i)) + e_t(i), \quad i \in \mathbf{Z},\ t = 1, 2, \ldots \tag{2.4}$$

(where we identify + with +1.) Obviously, $\eta_t(i) = -$ implies that $e_t(i) = 0$ and that $\eta_{t-1}(i-1)$ and $\eta_{t-1}(i)$ cannot both be +. That suggests a percolation process.

Consider the oriented graph with vertices $v = (i,t), i \in \mathbf{Z}, t = 0, 1, \ldots$ and directed edges going from (i,t) to $(i-1, t-1)$ and to $(i, t-1)$. We write $v \to v'$ if there is an edge from vertex v to vertex v'. A path is a sequence of vertices $v_1 \to v_2 \to \ldots \to v_n$.

Looking at (2.4), we can trace backwards in time where a particular $\eta_t(i) = -$ comes from. Whenever $\eta_t(i) = -$ there is a path $v_1 \to \ldots \to (k,s) \to \ldots \to v_t$, starting at $v_1 = (i,t)$ and ending in $v_t = (j,1)$ for some site $j \in \{i-t, \ldots, i\}$, on which $e_s(k) = 0$ for all vertices (k,s) on the path. The opposite holds as well: if there is such an oriented path on which the $e_s(k) = 0$ and if initially all $\eta_0(j) = -$, then $\eta_t(i) = -$.

2.3. Phase transition

The previous representation in terms of directed Bernoulli site percolation makes it clear that there is a phase transition. If ε is large, there will be a high density of "errors" $e_s(k) = 1$ and no percolation of zeroes will occur. Then, the

$\eta_t(i) = -$ will die out and the system has the trivial constant state $\eta(i) = +$ as unique invariant measure. If, on the other hand, ε is sufficiently small, the $e_k(s) = 0$ will percolate and a density of minuses will be maintained in the Stavskaya updating.

From now on we always start the dynamics from the constant minus state, $\eta_0(i) = -$ for all $i \in \mathbf{Z}$. We write Prob$[E]$ for the corresponding probability of an event E, measurable with respect to the $\eta_t(i), i \in \mathbf{Z}, t = 0, 1 \ldots$

There is a value $1/C > 0$ so that for $\varepsilon < 1/C$, there appears a second stationary measure μ having a finite density of minuses:

$$\mu_t[\eta(0) = -] \geq \mu[\eta(0) = -] = \lim_t \text{Prob}[\eta_t(0) = -] \geq 1 - C\varepsilon.$$

3. Potentials

A potential $U = (U_A)_A$ is a collection of real-valued functions U_A on $\{+, -\}^A$ for finite subsets $A \subset \mathbf{Z}$; $U_\emptyset = 0$. It is translation invariant when $U_A(\alpha) = U_{A+i}(\alpha')$ whenever $\alpha(j) = \alpha'(j + i)$ for all $i, j \in \mathbf{Z}$. To be useful, a potential has to obey certain summability properties.

A potential U is absolutely convergent at $\alpha \in \{+, -\}^{\mathbf{Z}}$ if for all finite $B \subset \mathbf{Z}$

$$\sum_{A \cap B \neq \emptyset} |U_A(\alpha)| < \infty. \tag{3.1}$$

Suppose that U is a potential and that there exists a tail field set $\Omega \subset \{+, -\}^{\mathbf{Z}}$ of points of absolute convergence of U (i.e., for all finite regions B the sums $\sum_{A \cap B \neq \emptyset} |U_A(\xi)|$ are well-defined whenever ξ coincides with some $\alpha \in \Omega$ outside a finite set). Then, for every finite $V \subset \mathbf{Z}$ and every $\alpha \in \Omega$ we can introduce the finite volume Gibbs measure

$$\mu_V^\alpha(\xi) \equiv \begin{cases} \exp\bigl\{-\sum_{A \cap V \neq \emptyset} U_A(\xi)\bigr\}/Z_V(\alpha) & \text{if } \xi = \alpha \text{ on } V^c, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where the normalization

$$Z_V(\alpha) \equiv \sum_{\xi(j) = \pm, j \in V} \exp\Bigl\{-\sum_{A \cap V \neq \emptyset} U_A(\xi_V \alpha_{V^c})\Bigr\} \tag{3.3}$$

is well-defined. Factors of temperature or *a priori* weights (reference measure) are supposed to be contained in the potential. The Dobrushin operator is then defined by taking expectations with respect to (3.2):

$$R_V^U(f)(\alpha) \equiv \int f(\xi)\mu_V^\alpha(d\xi) \tag{3.4}$$

mapping bounded measurable functions f on $\{+, -\}^{\mathbf{Z}}$ to functions $R_V^U(f)$ on Ω.

Definition 3.1. A probability measure ν on $\{+,-\}^{\mathbf{Z}}$ is weakly Gibbsian if there exists a potential U and a tail field set Ω of points of absolute convergence of U such that

1. $\nu(\Omega) = 1$.
2. For all finite $V \subset \mathbf{Z}$, for all events $\mathcal{B}$ measurable in V^c and for every bounded measurable function f

$$\int_{\mathcal{B}} f \, d\nu = \int_{\mathcal{B}} R_V^U(f) \, d\nu. \tag{3.5}$$

4. Main results

Our main result is that μ is a weakly Gibbsian measure for an exponentially decaying interaction. To explain and to be more specific, we need some further notation.

Look at a large time τ to the sites in $\Lambda \equiv \{0, 1, \ldots, N\}$. Ask for the probability that the $\eta_\tau(i), i \in \Lambda$, take specific values $\alpha(i)$. Then,

$$\text{Prob}[\eta_\tau(i) = \alpha(i), i \in \Lambda] \equiv \text{Prob}[\eta_\tau(i) = -, i \in \Lambda] \ \exp[-H_N^\tau(\alpha)] \tag{4.1}$$

defines a "Hamiltonian" H_N^τ, $\alpha \in \{+,-\}^{N+1}$.

The problem is to understand that Hamiltonian is a sum of potentials $U_A^{N,\tau}$, indexed by the finite sets $A \subset \mathbf{Z}$. We must then show that $U_A^{N,\tau}(\alpha)$ decays sufficiently fast, uniformly in N and τ, as the set A gets a large diameter, at least when α is "typical" for μ.

To be specific about the typicality, we introduce the set

$$\Omega \equiv \bigcap_{k \in \mathbf{Z}} \Omega_k, \qquad \Omega_k \equiv \Omega_k^L \cap \Omega_k^R \tag{4.2}$$

with

$$\Omega_k^R \equiv \left\{ \alpha \in \{+,-\}^{\mathbf{Z}} : \exists\, \ell(\alpha) < +\infty, \ \sum_{j=k}^{\ell} \frac{\alpha(j)}{\ell - k} \leq -1/2 \ \text{ for all } \ell \geq \ell(\alpha) + k \right\}$$

and analogously for Ω_k^L (summing to the left of k).

Clearly Ω is translation invariant and is tail. We consider μ_τ, the measure at time τ when starting Stavskaya's PCA from all minuses.

Proposition 4.1. *For ε sufficiently small and for all $\tau \geq 0$*

$$\mu_\tau(\Omega) = \mu(\Omega) = 1. \tag{4.3}$$

Proof. Consider the probability

$$\text{Prob}\Big[\frac{1}{\ell}\sum_{i=0}^{\ell}\eta_\tau(i) > -1/2\Big]$$

for ℓ large. By monotonicity, that probability is smaller than the one for $\tau = +\infty$, i.e., in the stationary measure $\mu_{+\infty} = \mu$, and converges to

$$\mu\Big[\frac{1}{\ell}\sum_{i=0}^{\ell}\eta(i) > -1/2\Big]. \tag{4.4}$$

These probabilities can be rephrased in the percolation language of Section 2.2. The fraction of minus values in $\{\eta_\tau(0), \eta_\tau(1), \ldots, \eta_\tau(\ell)\}$ equals the fraction of sites in the set $\{0, 1, \ldots, \ell\}$ from which there is a path of $e_s(k) = 0$ to some vertex at time zero. Uniformly in τ, with probability going to one exponentially fast as ℓ goes to infinity, that fraction is larger than $1 - g(\varepsilon)$ with $g(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. As a consequence, (4.4) goes to zero exponentially fast in ℓ whenever $g(\varepsilon) < 1/4$. We can choose $\varepsilon > 0$ small enough so that is the case.

By translation invariance and subadditivity it suffices to show $\mu(\Omega_0^R) = 1$.

$$\mu\Big[\exists \ell_0, \forall \ell \geq \ell_0 : \frac{1}{\ell}\sum_{i=0}^{\ell}\eta(i) \leq -1/2\Big] \geq 1 - \sum_{\ell=L}^{\infty}\mu\Big[\frac{1}{\ell}\sum_{i=0}^{\ell}\eta(i) > -1/2\Big], \quad \forall L. \tag{4.5}$$

The last sum is converging and can be made arbitrarily small by taking L large. □

Proposition 4.2. *There are $U_A^{N,\tau}(\alpha)$ non zero only if $A = \{k, k+1, \ldots, \ell\}$ and $\alpha(k) = + = \alpha(\ell)$ for some $0 \leq k \leq \ell \leq N$ so that in* (4.1)

$$H_N^\tau(\alpha) = \sum_{A \subset \Lambda} U_A^{N,\tau}(\alpha)$$

and for small enough $\varepsilon > 0$,

$$|U_A^{N,\tau}(\alpha)| \leq -3\,\ln\varepsilon \tag{4.6}$$

while there is $F < +\infty$,

$$|U_A^{N,\tau}(\alpha)| \leq F\exp[-V(\varepsilon)\,a] \tag{4.7}$$

for large diameter $|A| = a$, with $V(\varepsilon) \uparrow +\infty$ as $\varepsilon \downarrow 0$ whenever $\alpha \in \Omega$.

Theorem 4.1. *For all times $t > 0$ the evolved measure μ_t and the stationary measure μ are weakly Gibbsian for a translation invariant potential obeying the bounds* (4.6) *and* (4.7).

5. Proofs of Proposition 4.2 and Theorem 4.1

5.1. Telescoping potential

The potential will arise as terms in a telescoping expression for the Hamiltonian. The method goes back to [2] and was already applied for a similar problem in [7, 9, 10].

We start from the definition (4.1) for which we write

$$H_N^\tau(\alpha) = \sum_{k=0}^{N} \mathcal{F}_k(\alpha)$$

where

$$\mathcal{F}_k(\alpha) \equiv H_N^\tau(-,\dots,-,\alpha(k),\dots,\alpha(N)) - H_N^\tau(-,\dots,-,\alpha(k+1),\dots,\alpha(N)). \tag{5.1}$$

If therefore we put

$$\mathcal{U}_{k,\ell}^{N,\tau}(\alpha) \equiv \mathcal{F}_k(\alpha(0),\dots,\alpha(\ell),-,\dots,-) - \mathcal{F}_k(\alpha(0),\dots,\alpha(\ell-1),-,\dots,-), \tag{5.2}$$

then

$$H_N^\tau(\alpha) = \sum_{k=0}^{N}\sum_{\ell=k}^{N} \mathcal{U}_{k,\ell}^{N,\tau}(\alpha) \tag{5.3}$$

and $\mathcal{U}_{k,\ell}^{N,\tau}(\alpha)$ depends only on the values of $\alpha(k),\dots,\alpha(\ell)$. In that way we have defined our candidate potential

$$U_A^{N,\tau}(\alpha) \equiv \mathcal{U}_{k,\ell}^{N,\tau}(\alpha)$$

when $A = [k,\ell]$, non-zero only if A is a lattice interval $[k,\ell]$, and parameterized by N and τ.

5.2. Potential as correlation function

By construction, when $(\alpha(k),\alpha(\ell)) \neq (+,+)$, we have $\mathcal{U}_{k,\ell}^{N,\tau}(\alpha) = 0$. Henceforth, we put $\alpha(k) = + = \alpha(\ell)$. We also need not worry about the cases $\ell = k$ or $\ell = k+1$ which give no extra difficulties and continue with $k \le \ell - 2$.

We abbreviate

$$\text{Prob}[\eta_\tau(i) = \alpha(i), i \in \Lambda] = \mathcal{P}(\alpha)$$

so that

$$\exp\mathcal{U}_{k,\ell}^{N,\tau}(\alpha) = \frac{\mathcal{P}(-,\dots,-,\alpha(k+1),\dots,\alpha(\ell-1),+,-,\dots,-)}{\mathcal{P}(-,\dots,-,+,\alpha(k+1),\dots,\alpha(\ell-1),+,-,\dots,-)} \times \frac{\mathcal{P}(-,\dots,-,+,\alpha(k+1),\dots,-,\alpha(\ell-1),-,\dots,-)}{\mathcal{P}(-,\dots,-,\alpha(k+1),\dots,\alpha(\ell-1),-,\dots,-)}. \tag{5.4}$$

Factors can be taken together to write conditional expectations, e.g.,

$$\begin{aligned}
&\frac{\mathcal{P}(-,\ldots,-,+,\alpha(k+1),\ldots,\alpha(\ell-1),-,\ldots,-)}{\mathcal{P}(-,\ldots,-,+,\alpha(k+1),\ldots,\alpha(\ell-1),+,-,\ldots,-)}\\
&\quad = \mathsf{E}[f_\ell(\eta_{\tau-1}) \mid \eta_0 \equiv -, (\eta_\tau(i), i\in\Lambda) = (-,\ldots,-,+,\alpha(k+1),\ldots,\\
&\qquad \alpha(\ell-1),+,-,\ldots,-)]\\
&\quad \equiv \mathsf{E}^\alpha[f_\ell(\eta_{\tau-1})]
\end{aligned}$$

where

$$f_\ell(\eta) \equiv \frac{p(- \mid \eta(\ell-1),\eta(\ell))}{p(+ \mid \eta(\ell-1),\eta(\ell))}$$

and the expectation E^α is with respect to Stavskaya's PCA started from all $\eta_0(j) = -$ and always conditioned on $(\eta_\tau(i), i \in \Lambda) = (-,\ldots,-,+,\alpha(k+1),\ldots,\alpha(\ell-1),+,-,\ldots,-)$ (and hence also depending on N, k and ℓ). We will also use $\mathcal{P}^\alpha$ for the corresponding probability law.

As a consequence, proceeding similarly for the other fractions in the right-hand side of (5.4), we obtain

$$\mathcal{U}^{N,\tau}_{k,\ell}(\alpha) = \ln\Big[1 + \frac{\mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}))\,\mathsf{E}^\alpha(f_k(\eta_{\tau-1})) - \mathsf{E}^\alpha(f_\ell(\eta_{\tau-1})f_k(\eta_{\tau-1}))}{\mathsf{E}^\alpha(f_\ell(\eta_{\tau-1})f_k(\eta_{\tau-1}))}\Big]. \tag{5.5}$$

We start with the denominator $\mathsf{E}^\alpha(f_k(\eta_{\tau-1})f_\ell(\eta_{\tau-1}))$ in (5.5). Clearly

$$\big(\eta_{\tau-1}(k-1),\eta_{\tau-1}(k)\big) = (+,+)$$

or

$$\big(\eta_{\tau-1}(\ell-1),\eta_{\tau-1}(\ell)\big) = (+,+)$$

implies that either $f_k(\eta_{\tau-1})$ or $f_\ell(\eta_{\tau-1})$ will equal zero. On the other hand, when $\big(\eta_{\tau-1}(k-1),\eta_{\tau-1}(k)\big) \neq (+,+)$ and $\big(\eta_{\tau-1}(\ell-1),\eta_{\tau-1}(\ell)\big) \neq (+,+)$, then $f_k(\eta_{\tau-1})f_\ell(\eta_{\tau-1}) = (1-\varepsilon)^2/\varepsilon^2$. Therefore,

$$\mathsf{E}^\alpha(f_k(\eta_{\tau-1})f_\ell(\eta_{\tau-1})) = \Big(\frac{1-\varepsilon}{\varepsilon}\Big)^2\,\mathcal{P}^\alpha[F] \tag{5.6}$$

where F denotes the event that $\big(\eta_{\tau-1}(k-1),\eta_{\tau-1}(k)\big) \neq (+,+)$ and $\big(\eta_{\tau-1}(\ell-1),\eta_{\tau-1}(\ell)\big) \neq (+,+)$.

Remember now that in $\mathcal{P}^\alpha$ we condition, among other things, on $\eta_\tau(k) = \eta_\tau(\ell) = +$ so that F cannot occur when $e_\tau(k) = 0$ or $e_\tau(\ell) = 0$. If on the other hand, $e_\tau(k) = 1$ and $e_\tau(\ell) = 1$, then $\eta_{\tau-1}(k-1)$ and $\eta_{\tau-1}(\ell)$ are free and the conditioning on α has no influence on them. More precisely,

$$\begin{aligned}
\mathcal{P}^\alpha[F] &= \mathcal{P}^\alpha[F \mid e_\tau(k)=1, e_\tau(\ell)=1]\,\mathcal{P}^\alpha[e_\tau(k)=1, e_\tau(\ell)=1]\\
&\geq \mathcal{P}^\alpha[\eta_{\tau-1}(k-1) = -, \eta_{\tau-1}(\ell) = - \mid e_\tau(k)=1, e_\tau(\ell)=1]\\
&\quad \times \mathcal{P}^\alpha[e_\tau(k)=1, e_\tau(\ell)=1]\\
&\geq (1-2C\varepsilon)\,\mathcal{P}^\alpha[e_\tau(k)=1, e_\tau(\ell)=1].
\end{aligned}$$

The last factor is the probability that the noise acts at time τ to decide the value at sites k and ℓ. That is independent of the values of $\alpha(k+1), \ldots, \alpha(\ell-1)$, hence larger than ε^2; conclusion:

$$\mathsf{E}^\alpha(f_k(\eta_{\tau-1}) f_\ell(\eta_{\tau-1})) \geq \Big(\frac{1-\varepsilon}{\varepsilon}\Big)^2 (1-2C\varepsilon)\,\varepsilon^2. \tag{5.7}$$

The functions f_k also satisfy the upper bound $(1-\varepsilon)/\varepsilon$. That takes care of an upper bound to (5.5). A lower bound is obtained in exactly the same way. We thus see from (5.5) that the potential is uniformly bounded:

$$|\mathcal{U}^{N,\tau}_{k,\ell}(\alpha)| \leq \ln\Big[1 + \frac{2}{(1-2C\varepsilon)\varepsilon^2}\Big]$$

thereby proving the boundedness of the potential, uniformly in N and τ, see (4.6).

5.3. Percolation event

We must prove that $\mathcal{U}^{N,\tau}_{k,\ell}(\alpha)$ decays sufficiently fast in $|\ell - k|$, uniformly in $N \uparrow +\infty$ and $\tau \uparrow +\infty$, at least for a large class of α, i.e., for $\alpha \in \Omega$.

From the above computations we conclude that in order to control the potential, we need to find an upper bound for the covariance appearing in (5.5). We use a standard trick to rewrite the covariance in a doubled space:

$$\begin{aligned}
&\mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}))\,\mathsf{E}^\alpha(f_k(\eta_{\tau-1})) - \mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}) f_k(\eta_{\tau-1}))\\
&\quad = \mathsf{E}^\alpha\left[f_\ell(\eta_{\tau-1})\,[\mathsf{E}^\alpha(f_k(\sigma_{\tau-1})) - \mathsf{E}^\alpha(f_k(\sigma_{\tau-1}) \mid \eta_{\tau-1}(\ell-1), \eta_{\tau-1}(\ell))]\right]\\
&\quad = \mathsf{E}^\alpha[f_\ell(\eta_{\tau-1})\,\mathsf{E}^{\alpha,\eta}(f_k(\eta_{\tau-1}) \times 1 - 1 \times f_k(\eta_{\tau-1}))]
\end{aligned} \tag{5.8}$$

where $\mathsf{E}^{\alpha,\eta}$ is the expectation with respect to the product coupling

$$\mathsf{P}^{\alpha,\eta} \equiv \mathcal{P}^\alpha \times \mathcal{P}^{\alpha,\eta} \tag{5.9}$$

where the second marginal is

$$\mathcal{P}^{\alpha,\eta} \equiv \mathcal{P}^\alpha[\,\cdot \mid \eta_{\tau-1}(\ell-1), \eta_{\tau-1}(\ell)]$$

The coupling defines a random field $((\sigma^1_s(j), \sigma^2_s(j)), (j,s) \in \mathcal{W}^\tau_N)$ on the space-time region

$$\mathcal{W}^\tau_N \equiv \{(i,t) \in \mathbb{Z}^2 \mid 0 \leq t \leq \tau,\, t-\tau \leq i \leq N\}.$$

As a first estimate, from (5.8)

$$\begin{aligned}
&|\,\mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}))\,\mathsf{E}^\alpha(f_k(\eta_{\tau-1})) - \mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}) f_k(\eta_{\tau-1}))|\\
&\quad \leq \frac{1-\varepsilon}{\varepsilon}\,|\,\mathsf{E}^{\alpha,\eta}((f_k(\eta_{\tau-1}) \times 1 - 1 \times f_k(\eta_{\tau-1}))|.
\end{aligned} \tag{5.10}$$

Suppose now that there is a fixed path γ which goes from a vertex $(j, \tau - 1)$ (for some $j \in \{k+1, \ell-2\}$) to a vertex $(j, 1)$ on which all $\sigma^1(v) = \sigma^2(v) = -, v \in \gamma$. Then, the function f_k is shielded away from what happens around the vertex $(\ell, \tau-1)$: conditioned on such a negative path, changing the configuration around $(\ell, \tau - 1)$ can have no influence on the distribution of f_k.

We formulate that more precisely using the percolation event

$$E^c \equiv \{(\sigma^1, \sigma^2) \mid \exists \gamma \in \Gamma_\tau^N, \forall (k,t) \in \gamma : (\sigma_t^1(k), \sigma_t^2(k)) = (-,-)\}$$

where Γ_τ^N is the collection of paths on $\mathcal{W}_N^\tau$ going from the lattice interval $[k + 1, \ell - 2]$ at time $\tau - 1$ to time 1. The complement of E^c is denoted by E. By construction

$$\mathsf{E}^{\alpha,\eta}[f_k(\eta_{\tau-1}) \times 1 - 1 \times f_k(\eta_{\tau-1})] = \mathsf{E}^{\alpha,\eta}[f_k(\eta_{\tau-1}) \times 1 - 1 \times f_k(\eta_{\tau-1} \mid E] \; \mathsf{P}^{\alpha,\eta}[E] \tag{5.11}$$

which is useful to continue our estimate (5.10). We get

$$\begin{aligned} &| \, \mathsf{E}^\alpha(f_\ell(\eta_{\tau-1})) \, \mathsf{E}^\alpha(f_k(\eta_{\tau-1})) - \mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}) f_k(\eta_{\tau-1}))| \\ &\quad \le 2\Big(\frac{1-\varepsilon}{\varepsilon}\Big)^2 \mathcal{P}^\alpha \times \mathcal{P}^\alpha[E]. \end{aligned} \tag{5.12}$$

We now have the coupling $\mathcal{P}^\alpha \times \mathcal{P}^\alpha$ with distribution obtained from Stavskaya's PCA and remembering that $\sigma_0(j) = -$ for all j, $\sigma_\tau(j) = -$ for all j except that $\sigma_\tau(k) = \sigma_\tau(\ell) = +$ and that $\sigma_\tau(i) = \alpha(i)$ for $i \in \{k+1, \ldots, \ell-1\}$.

It is left to prove that the probability of finding some sort of wall blocking directed $(-,-)$-percolation starting from some vertex $(i, \tau-1), i = k+1, \ldots, \ell-1$, is exceedingly small as $|k - \ell|$ grows, uniformly so in N and in τ, at least for $\alpha \in \Omega$ of (4.2).

5.4. Final estimates

Event E contains all (double) configurations (σ^1, σ^2) except those with a percolating path of $(-,-)$, starting somewhere in $[k+1, \ell-2]$ at time $t = \tau - 1$ up to $t = 1$. We must estimate its probability in the product coupling $\mathcal{P}^\alpha \times \mathcal{P}^\alpha$. The superscript α reminds us that we should also condition on having $\sigma_\tau(0) = \ldots = \sigma_\tau(k-1) = -, \sigma_\tau(k) = +, \sigma_\tau(k+1) = \alpha(k+1), \ldots, \sigma_\tau(\ell-1) = \alpha(\ell-1), \sigma_\tau(\ell) = +, \sigma_\tau(\ell+1) = \ldots = \sigma_\tau(N) = -$. Our first step is to get rid of this conditioning.

To that end we define the set $S(\alpha) = S \equiv \{i \in \Lambda, \alpha(i) = +\}$.

Lemma 5.1.

$$\mathcal{P}^\alpha \times \mathcal{P}^\alpha[E] \le \exp\Big[2|S(\alpha)| \, \ln\Big[1 + \frac{1-\varepsilon}{\varepsilon(1-C\varepsilon)}\Big]\Big] \, \mathcal{P} \times \mathcal{P}[E]. \tag{5.13}$$

Proof of (5.13). We start with a lower bound for $\mathcal{P}(\alpha)$. For every $i \in S$ we use the bound

$$p(+ \mid a,b) \geq p(- \mid a,b)\,\frac{\varepsilon}{1-\varepsilon}$$

to get

$$\mathcal{P}(\alpha) \geq (\frac{\varepsilon}{1-\varepsilon})^{|S|}\,\mathrm{Prob}[\sigma_\tau(i) = -, i \in \Lambda]. \tag{5.14}$$

Secondly, $p(+ \mid a,b)$ is either equal to 1 (when $(a,b) = (+,+)$) or equal to $\varepsilon p(-|a,b)/(1-\varepsilon)$ (when $(a,b) \neq (+,+)$). As a consequence,

$$\mathrm{Prob}[\sigma_\tau(i) = \alpha(i), i \in \Lambda \,|\, \sigma_{\tau-1}] = \Big(\frac{\varepsilon}{1-\varepsilon}\Big)^{|T|}\,\mathrm{Prob}[\sigma_\tau(i) = -, i \in \Lambda \backslash S \cup T \,|\, \sigma_{\tau-1}] \tag{5.15}$$

where $T = T(\sigma_{\tau-1}) \equiv \{i \in S, (\sigma_{\tau-1}(i-1), \sigma_{\tau-1}(i)) \neq (+,+)\}$. We combine (5.14) with (5.15) to get

$$\begin{aligned}&\mathrm{Prob}[\sigma_{\tau-1}(j) = \beta(j), j \in \Lambda^* \mid \sigma_\tau(i) = \alpha(i), i \in \Lambda]\\ &\quad = \Big(\frac{1-\varepsilon}{\varepsilon}\Big)^{|S\backslash T(\beta)|}\,\frac{\mathrm{Prob}[\sigma_\tau(i) = -, i \in \Lambda \setminus S \cup T(\beta) \mid \sigma_{\tau-1}(j) = \beta(j), j \in \Lambda^*]}{\mathrm{Prob}[\sigma_\tau(i) = -, i \in \Lambda]}\end{aligned} \tag{5.16}$$

for every $\beta(j) = \pm, j \in \Lambda^* \equiv \{-1, 0, 1, \ldots, N\}$. Observe that α has disappeared from the right-hand side in (5.16). We use that for (5.13):

$$\begin{aligned}\mathcal{P}^\alpha \times \mathcal{P}^\alpha[E] &= \sum_{T_1, T_2 \subset S} \mathcal{P}^\alpha \times \mathcal{P}^\alpha[E, T(\sigma^1_{\tau-1}) = T_1, T(\sigma^2_{\tau-1}) = T_2]\\ &\leq \sum_{T_1, T_2 \subset S} \Big(\frac{1-\varepsilon}{\varepsilon}\Big)^{|S\backslash T_1| + |S\backslash T_2|}\,\frac{\mathrm{Prob}[E, F_1, F_2]}{\mathrm{Prob}[\sigma_\tau(i) = -, i \in \Lambda]}\end{aligned} \tag{5.17}$$

where $F_k, k = 1,2$, is the event $\sigma^k_\tau(j) = -, j \in \Lambda \setminus S \cup T_k$. Each term in the last sum of (5.17) can be written as

$$\frac{\mathrm{Prob}[E, F_1, F_2]}{\mathrm{Prob}[\sigma_\tau(i) = -, i \in \Lambda]} = \frac{R_1}{R_2}$$

with $R_1 \equiv \dfrac{\mathrm{Prob}[E, F_1, F_2]}{\mathrm{Prob}[F_1, F_2]}$ and

$$R_2 \equiv \mathrm{Prob}[\sigma^1_\tau(j) = -, j \in S \setminus T_1 \mid F_1]\mathrm{Prob}[\sigma^2_\tau(j) = -, j \in S \setminus T_2 \mid F_2]. \tag{5.18}$$

It summarizes (5.17) to

$$\mathcal{P}^\alpha \times \mathcal{P}^\alpha[E] \leq \sum_{T_1, T_2 \subset S} \Big(\frac{1-\varepsilon}{\varepsilon}\Big)^{|S\backslash T_1| + |S\backslash T_2|}\,\frac{R_1}{R_2}. \tag{5.19}$$

R_1 is the probability that no directed percolation of $(-,-)$ occurs in the product coupling conditioned on F_1, F_2. By monotonicity $R_1 \leq \mathcal{P} \times \mathcal{P}[E]$. Similarly,

$$R_2 \geq (1 - C\varepsilon)^{|S \setminus T_1| + |S \setminus T_2|}$$

so that

$$\mathcal{P}^\alpha \times \mathcal{P}^\alpha[E] \leq \mathcal{P} \times \mathcal{P}[E] \sum_{T_1, T_2 \subset S} \left(\frac{1-\varepsilon}{\varepsilon(1-C\varepsilon)} \right)^{|S \setminus T_1| + |S \setminus T_2|} \tag{5.20}$$

which immediately yields the required (5.13) □

Lemma 5.2. *There is a constant $K < \infty$ so that*

$$\mathcal{P} \times \mathcal{P}[E] \leq (K\,\varepsilon)^{|k-\ell|}. \tag{5.21}$$

Proof of (5.21). We refer to Section 2.2: if there is $(0,0)$ percolation in the double Bernoulli field, then there is $(-,-)$ percolation of the Stavskaya fields. As a consequence, the event E implies that there is no path of $(0,0)$ for the field $(e_s(k)^1, e_s(k)^2)$ from the interval Λ at time τ to time one. For ε small, the density of $(0,0)$ is however arbitrarily close to one. Standard percolation results finish the proof. □

Proof of (4.7). We collect all estimates starting from (5.5) and (5.7). Finally we use (5.12) combined with (5.13) and (5.21). □

Proof of Theorem 4.1. To prove that the time-evolved measures μ_τ are weakly Gibbsian, it suffices to inspect (4.1) and to combine it with the results of Proposition 4.2. In particular, the measures μ_τ are translation invariant and the construction of the potential $(U_A^{N,\tau})$ is also translation invariant. The absolute convergence of the potential is uniform in N and each term

$$U_A^{N,\tau}(\alpha) \to U_A^\tau(\alpha)$$

converges to a well-defined limit as $N \uparrow +\infty$. That follows form the expression (5.5): the functions f_ℓ, f_k and the products $f_k f_\ell$ are all local non-negative decreasing functions. Therefore each expression like $\mathsf{E}^\alpha(f_\ell(\eta_{\tau-1}))$ which depends on N, converges monotonically as $N \uparrow +\infty$.

The proof that the stationary measure μ is also weakly Gibbsian proceeds similarly but now uses the uniformity in τ. Expressions like

$$\mathsf{E}[f_\ell(\eta_{\tau-1}) \mid \eta_\tau]$$

are monotone in τ, bounded and positive. The potential converges term by term and the bounds are preserved enabling the definition of the relative Hamiltonian for $\alpha \in \Omega$. □

6. Additional remarks

Remark 6.1. A natural question is to ask whether perhaps Stavskaya's measure is (fully) Gibbsian (and not just weakly Gibbsian). We believe (but do not prove) that the answer is negative. One may indeed wonder what happens in the case where $\alpha = +$. The following is crucial:

$$\text{Prob}[\eta_{\tau-1}(i) = +, i \in \Lambda^* \mid \eta_\tau(j) = +, j \in \Lambda] = \frac{\text{Prob}[\eta_{\tau-1}(i) = +, i \in \Lambda^*]}{\text{Prob}[\eta_\tau(i) = +, i \in \Lambda]}$$

does not decay with N. As a consequence, the probability to find a minus somewhere in Λ^* at time $\tau - 1$ does not go to one when Λ is full of pluses. That is why a good fraction of minuses in α over arbitrarily long intervals is needed to have good decay of the constructed potential.

Remark 6.2. The method of proof can be exported to various other PCA's. There is certainly no restriction to one dimension (see e.g. [7, 8]). Of course, the estimates do not come for free. So far, we have no answer to the question whether for example the extremal stationary measures in Toom's north-east-center model are (weakly) Gibbsian, [12, 17]. On the other hand, the extension to more general percolation processes, as e.g. defined in [18], does not seem to present extra difficulties. The application of our method to continuous time dynamics, like the contact process, has also not been tried yet but we do not expect important difficulties.

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