Clairvoyant scheduling of random walks

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Abstract

Two infinite walks on the same finite graph are called *compatible* if it is possible to introduce delays into them in such a way that they never collide. Years ago, Peter Winkler asked the question: for which graphs are two independent random walks compatible with positive probability. Up to now, no such graphs were found. We show in this paper that large complete graphs have this property. The question is equivalent to a certain dependent percolation with a power-law behavior: the probability that the origin is blocked at distance n but not closer decreases only polynomially fast and not, as usual, exponentially.

1 Introduction

1.1 The model

Let us call any strictly increasing sequence $t = (t(0) = 0, t(1), \dots)$ of integers a *delay sequence*. For an infinite sequence $z = (z(0), z(1), \dots)$, the delay sequence t introduces a timing arrangement in which the value $z(n)$ occurs at time $t(n)$. Given infinite sequences z_d and delay sequences t_d, for $d = 0, 1$, let $a, b \in \{0, 1\}$. We say that there is a *collision* at (a, n, k) if $t_a(n) \leqslant t_b(k) < t_a(n+1)$ and $z_b(k) = z_a(n)$, for $a \neq b$. We call the two sequences z_0, z_1 *compatible* if there are delay sequences for them that avoid collisions.

For a finite undirected graph, a Markov chain $Z(1), Z(2), \dots$ with values that are vertices in this graph is called a *random walk* over this graph if it moves, going from $Z(n)$ to $Z(n+1)$, from any vertex with equal probability to any one of its neighbors.

Take two infinite random sequences Z_d for $d = 0, 1$ independent from each other, both of which are random walks on the same finite undirected graph. Here, the delay sequence t_d can be viewed as causing the sequence Z_d to stay in state $z_d(n)$ between times $t_d(n)$ and $t_d(n+1)$. (See the example on the graph K_5 in Figure 1.) A collision occurs when the two
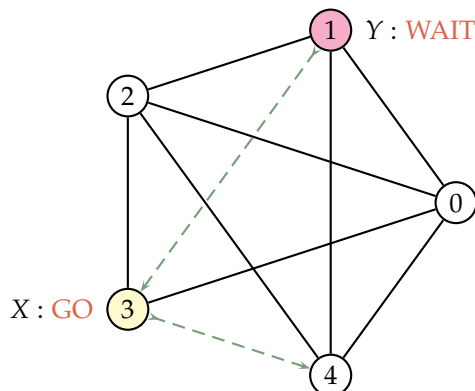
Figure 1: The clairvoyant demon problem. X, Y are "tokens" performing independent random walks on the same graph: here the complete graph K_5. A "demon" decides every time, whose turn it is. She is clairvoyant and wants to prevent collision.

delayed walks enter the same point of the graph. Our question is: are Z_0 and Z_1 compatible with positive probability? The question depends, of course, on the graph. Up to the present paper, no graph was known with an affirmative answer.

Consider the case when the graph is the complete graph K_m of size m. It is known that if $m \leqslant 3$ then the two sequences are compatible only with zero probability. Simulations suggest strongly that the walks do not collide if $m \geqslant 5$, and, somewhat less strongly, even for $m = 4$. The present paper proves the following theorem.

Theorem 1 (Main). *If m is sufficiently large then on the graph K_m, the independent random walks Z_0, Z_1 are compatible with positive probability.*

The upper bound computable for *m* from the proof is very bad. In what follows we will also use the simpler notation

$$X = Z_0, \quad Y = Z_1.$$

The problem, called the clairvoyant demon problem, arose first in distributed computing. The original problem was to find a leader among a finite number of processes that form the nodes of a communication graph. There is a proposed algorithm: at start, let each process have a "token". The processes pass the tokens around in such a way that each token performs a random walk. However, when two tokens collide they merge. Eventually, only one token will remain and whichever process has it becomes the leader. The paper [2] examined the algorithm in the traditional setting of distributed computing, when the timing of this procedure is controlled by an adversary. Under the (reasonable) assumption that the

adversary does not see the future sequence of moves to be made by the tokens, the work [2] gave a very good upper bound on the expected time by which a leader will be found. It considered then the question whether a clairvoyant adversary (a "demon" who sees far ahead into the future token moves) can, by controlling the timing alone, with positive probability, prevent two distinct tokens from ever colliding. The present paper solves Conjecture 3 of [2], which says that this is the case when the communication graph is a large complete graph.

   The proof is long and complex, but its construction is based some simple principles presented below in Section 2, after first transforming the problem into a percolation problem in Section 1.3. The rest of the paper is devoted mainly to proving Lemma 2.6, stated at the end of Section 2. The main ideas can be summarized as follows.

1. Reformulate into a percolation problem in the upper quarter plane, where closed sites (i, j) are those with $X(i) = Y(j)$ (Section 1.3).

2. Let us call these closed sites *obstacles*. These have long-range dependencies, making it hard to handle them directly: therefore for "bad events" (like when too many obstacles crowd together), we introduce a hierarchy of new kinds of obstacles in Section 3. These fall into two categories. Rectangular *traps*, like the original example, and infinite horizontal or vertical *walls* of various widths. A wall can only be penetrated at certain places.

3. Traps at the higher levels of the hierarchy are larger, walls are wider and denser. But these higher-level objects have smaller probability of occurrence, therefore on level k it will be possible to assume the local absence of level $k + 1$ objects with high probability. That percolation is possible under this assumption for each k, is proved in section 8.

4. The structure of random traps, walls, and so on, whose versions appear on all levels, is defined in Section 3. The higher-level objects are defined by a recursive procedure in Section 4.1, though some parameters will only be fixed in Section 6. Their combinatorial properties are proved in Section 4.2.

5. Those probabilistic estimates for the higher-level objects that can be proved without fixing all parameters are given in Sections 5. The rest is proved in Section 7.

1.2   Related synchronization problems

Let us define a notion of collision somewhat different from the previous section. For two infinite 0-1-sequences z_d ($d = 0, 1$) and corresponding delay sequences t_d we say that there is a *collision* at (a, n) if $z_a(n) = 1$, and there is no k such that $z_b(k) = 0$ and $t_a(n) = t_b(k)$, for $a \neq b$. We say that the sequences z_d are *compatible* if there is a pair of delay sequences t_d without collisions. It is easy to see that this is equivalent to saying that 0's can be deleted from both sequences in such a way that the resulting sequences have no collisions in the sense that they never have a 1 in the same position.

Suppose that for $d = 0, 1$, $Z_d = (Z_d(0), Z_d(1), \dots)$ are two independent infinite sequences of independent random variables where $Z_d(j) = 1$ with probability p and 0 with probability $1 - p$. Our question is: are Z_0 and Z_1 compatible with positive probability? The question depends, of course, on the value of p: intuitively, it seems that they are compatible if p is small.

Peter Winkler and Harry Kesten [5], independently of each other, found an upper bound smaller than 1/2 on the values p for which Z_0, Z_1 are compatible. Computer simulations by John Tromp suggest that when $p < 0.3$, with positive probability the sequences are compatible. The paper [4] proves that if p is sufficiently small then with positive probability, Z_0 and Z_1 are compatible. The threshold for p obtained from the proof is only 10^{-400}, so there is lots of room for improvement between this number and the experimental 0.3.

The author recommends the reader to refer to paper [4] while reading the present one. Its technique prefigures the one used here: the main architecture is similar, but many details are simpler.

1.3   A percolation

The clairvoyant demon problem has a natural translation into a percolation problem. Consider the lattice $\mathbb{Z}_+^2$, and a directed graph obtained from it in which each point is connected to its right and upper neighbor. For each i, j, let us "color" the ith vertical line by the state $X(i)$, and the jth horizontal line by the state $Y(j)$. The ingoing edges of a point (i, j) will be deleted from the graph if $X(i) = Y(j)$, if its horizontal and vertical colors coincide. We will also say that point (i, j) is *closed*; otherwise, it will be called *open*. (It is convenient to still keep the closed point (i, j) in the graph, even though it became unreachable from the origin.) The question is whether, with positive probability, an infinite oriented path starting from $(0, 0)$ exists in the remaining random graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}).$$

In [4], we proposed to call this sort of percolation, where two infinite random sequences X, Y are given on the two coordinate axes and the openness of a point or edge at position (i, j) depends on the pair $(X(i), Y(j))$, a *Winkler percolation*. This problem permits an interesting variation: undirected percolation, where the the whole lattice $\mathbb{Z}^2$ is present, and the edges are undirected. This variation has been solved, independently, in [6] and [1]. On the other hand, the paper [3] shows that the directed problem has a different nature, since it has power-law convergence (the undirected percolations have the usual exponential convergence).
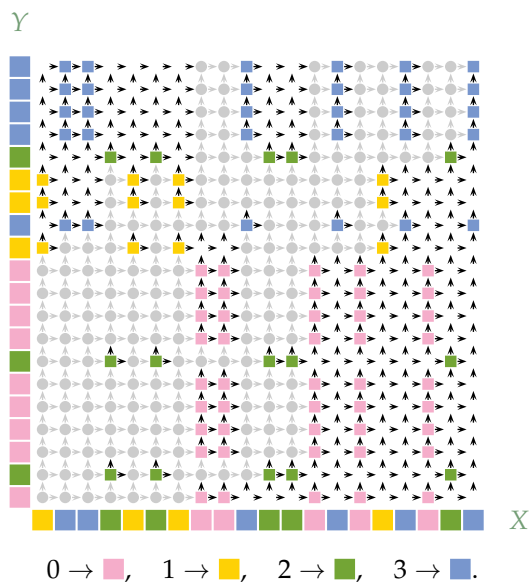
Figure 2: Percolation for the clairvoyant demon problem, for random walks on the complete graph K_4. Round light-grey dots mark the reachable points.

2 Outline of the proof

The proof introduces a whole army of auxiliary concepts, which are cumbersome to keep track of. The reader will find it helpful occasionally to refer to the glossary and notation index provided at the end of the paper.

2.1 Renormalization

The proof method used is *renormalization* (scale-up). An example of the ordinary renormalization method would be when, say, in an Ising model, the space is partitioned into blocks, spins in each block are summed into a sort of "superspin", and it is shown that the system of super-spins exhibits a behavior that is in some sense similar to the original system. We will also map our model repeatedly into a series of higher-order models similar to each other. However the definition of the new models is more complex than just taking the sums of some quantity over blocks. The model which will scale up properly may contain a number of new objects, and restrictions more combinatorial than computational in character.

The method is messy, laborious, and rather crude (rarely leading to the computation of exact constants). However, it is robust and well-suited to "error-correction" situations.

Here is a rough first outline.

1. Fix an appropriate sequence $\Delta_1 < \Delta_2 < \cdots$, of scale parameters for which $\Delta_{k+1} > 3\Delta_k$ holds[1]. (The actual values of our parameters will only be fixed in Section 6.) Let $\mathscr{F}_k$ be the event that point $(0,0)$ is blocked in the square $[0, \Delta_k]^2$. (In other applications, it could be some other *ultimate bad event*.) Throughout the proof, we will denote the probability of an event E by $\mathbf{P}(E)$. We want to prove

$$\mathbf{P}\big(\bigcup_k \mathscr{F}_k\big) < 1.$$

   This will be sufficient: if $(0,0)$ is not blocked in any finite square then by compactness (or by what is sometimes called König's Lemma), there is an infinite oriented path starting at $(0,0)$.

2. Identify some events that we call *bad events* and some others called *very bad events*, where the latter are much less probable.

   Define a series $\mathscr{M}^1, \mathscr{M}^2, \ldots$ of models similar to each other (in our case each based on the same directed lattice graph on $\mathbb{Z}_+^2$) where the very bad events of $\mathscr{M}^k$ become the bad events of $\mathscr{M}^{k+1}$. (The structure of these models will be introduced in Section 3, the actual recursive definition is carried out in Section 4.1, with the fixing of parameters left to Section 6.)

   Let $\mathscr{F}_k'$ hold iff some bad event of $\mathscr{M}^k$ happens in the square $[0, \Delta_{k+1}]^2$.

3. Prove

$$\mathscr{F}_k \subseteq \bigcup_{i \leqslant k} \mathscr{F}_i'. \tag{2.1}$$

   (This will follow from the structure of definitions in Section 4.1.)

4. Prove $\sum_k \mathbf{P}(\mathscr{F}_k') < 1$. (This results from the estimates in Section 6.)

In later discussions, we will frequently delete the index k from $\mathscr{M}^k$ as well as from other quantities defined for $\mathscr{M}^k$. In this context, we will refer to $\mathscr{M}^{k+1}$ as $\mathscr{M}^*$.

2.2 Application to our case

The role of the "bad events" of Subsection 2.1 will be played by *traps* and *walls*. The simplest kind of trap is a point (i, j) in the plane such that $X(i) = Y(j)$; in other words, a closed point. More generally, traps will be certain rectangles in the plane. We want to view the occurrence of two traps close to each other as a very bad event; however, this is justified only if this is indeed very improbable. Consider the events

$$\mathscr{A}_5 = \{X(1) = X(2) = X(3) = Y(5)\}, \quad \mathscr{A}_{13} = \{X(1) = X(2) = X(3) = Y(13)\}.$$

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[1] In the present paper, the actual quotient between these parameters will be more than 10^7.
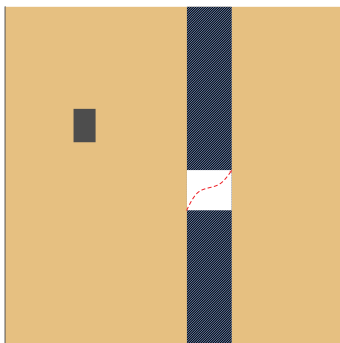
Figure 3: A trap and a wall with a hole.

(For simplicity, this example assumes that the random walk has the option of staying at the same point, that is loops have been added to the graph K_m.) The event $\mathscr{A}_5$ makes the rectangle $[1,3] \times \{5\}$ a trap of size 3, and has probability m^{-3}. Similarly for the event $\mathscr{A}_{13}$ and the rectangle $[1,3] \times \{13\}$. However, these two events are not independent: the probability of $\mathscr{A}_5 \cap \mathscr{A}_{13}$ is only m^{-4}, not m^{-6}. The reason is that the event $\mathscr{E} = \{X(1) = X(2) = X(3)\}$ significantly increases the conditional probability that, say, the rectangle $[1,3] \times \{5\}$ becomes a trap. In such a case, we will want to say that event $\mathscr{E}$ creates a *vertical wall* on the segment $(0,3]$.

Though our study only concerns the integer lattice, it is convenient to use the notations of the real line and Euclidean plane. In particular, walls will be right-closed intervals. (Even though, of course, $(a,b] \cap \mathbb{Z} = [a+1,b] \cap \mathbb{Z}$, but we will not consider the interval $(4,9]$ to be automatically contained in $[5,10]$.) We will say that a certain rectangle *contains* a wall if the corresponding projection contains it, and that the same rectangle *intersects* a wall if the corresponding projection intersects it.

Traps will have low probability. If there are not too many traps, it is possible to get around them. On the other hand, to get through walls, one also needs extra luck: such lucky events will be called *holes* (see Figure 3). Our proof systematizes the above ideas by introducing an abstract notion of traps, walls and holes. We will have walls of many different types. To each (say, vertical) wall of a given type, the probability that a (horizontal) hole goes through it at a given point will be much higher than the probability that a (horizontal) wall of this type occurred at that point. Thus, the "luck" needed to go through some wall type is still smaller than the "bad luck" needed to create a wall of this type.

This model will be called a *mazery* $\mathscr{M}$ (a system for creating mazes). In any mazery, whenever it happens that walls and traps are well separated from each other and holes are not missing, then paths can pass through. (Formally, this claim will be called Lemma 8.1 (Approximation)—as the main combinatorial tool in a sequence of successive approxima-
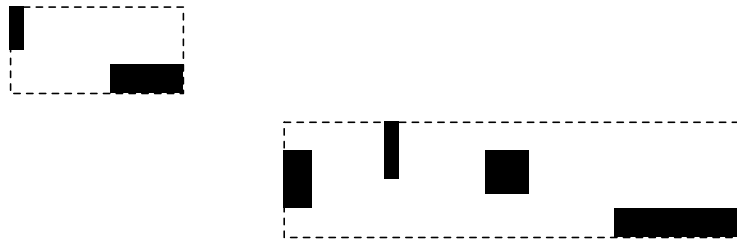
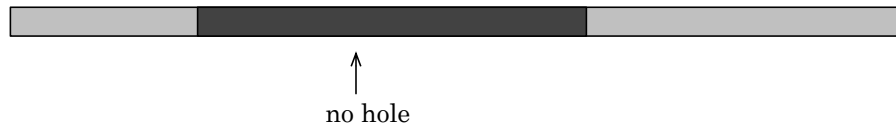Figure 4: An uncorrelated and a correlated compound trap

Figure 5: A missing-hole trap

tions.) Sometimes, however, unlucky events arise. These unlucky events can be classified in the types listed below. For any mazery $\mathcal{M}$, we will define a mazery $\mathcal{M}^*$ whose walls and traps correspond (essentially) to these typical unlucky events.

– A minimal rectangle enclosing two traps very close to each other, both of whose projections are disjoint, is an *uncorrelated compound trap* (see Figure 4).

– For both directions $d = 0, 1$, a (essentially) minimal rectangle enclosing 4 traps very close to each other, whose *d* projections are disjoint, is a *correlated compound trap* (see Figure 4).

– Whenever a certain horizontal wall *W* appears and at the same time there is a large interval without a vertical hole of $\mathcal{M}$ through *W*, this situation gives rise to a trap of $\mathcal{M}^*$ of the *missing-hole* kind (see Figure 5).

– A pair of very close walls of $\mathcal{M}$ gives rise to a wall of $\mathcal{M}^*$ called a *compound wall* (see Figure 6).

– A segment of the *X* sequence such that conditioning on it, a correlated trap or a trap of the missing-hole kind occurs with too high conditional probability, is a new kind of wall called an *emerging* wall. (These are the walls that, indirectly, give rise to all other walls.)

(The exact definition of these objects involves some extra technical conditions: here, we are just trying to give the general idea.)

At this point, it would be hard for the reader to appreciate that the set of kinds of objects (emerging traps and walls) is *complete*: that there is no need in any other ones. An
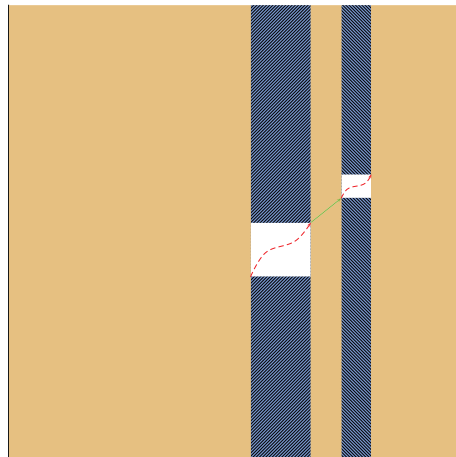
Figure 6: A compound wall, with a hole through it.

informal effort to try to prove Lemma 8.1 (Approximation) should give some insight: in other words, the reader should try to convince herself that as long as the kind of "very bad events" covered by the emerging traps and walls do not occur, percolation can occur.

There will be a constant

$$\chi = 0.015. \tag{2.2}$$

with the property that if a wall has probability p then a hole getting through it has probability lower bound p^χ. Thus, the "bad events" of the outline in Subsection 2.1 are the traps and walls of $\mathscr{M}$, the "very bad events" are (modulo some details that are not important now) the new traps and walls of $\mathscr{M}^*$. Let $\mathscr{F}, \mathscr{F}'$ be the events $\mathscr{F}_k, \mathscr{F}'_k$ formulated in Subsection 2.1. Thus, $\mathscr{F}'$ says that in $\mathscr{M}$ a wall or a trap is contained in the square $[0, \Delta^*]^2$.

Remark 2.1. The paper uses a number of constants:

$$c_1, c_2, c_3, H, \lambda, \Lambda, R_0, \gamma, \chi, \tau, \tau', \omega.$$

It would be more confusing to read (and a nightmare to debug) if we substituted numerical values for them everywhere. This is debatable in some cases, like $\lambda = \sqrt{2}$, $\omega = 4.5$, but even here, the symbol λ emphasizes that there is no compelling reason for using the exact value $\sqrt{2}$. There is a notation index allowing the reader to look up the definition of each constant, whenever needed. ⌟

We do not want to see all the details of $\mathscr{M}$ once we are on the level of $\mathscr{M}^*$: this was the reason for creating $\mathscr{M}^*$ in the first place. The walls and traps of $\mathscr{M}$ will indeed become transparent; however, some restrictions will be inherited from them: these are distilled in

the concepts of a *clean point* and of a *slope constraint*. Actually, we distinguish the concept of *lower left clean* and *upper right clean*. Let

$$\mathcal{Q}$$

be the event that point $(0,0)$ is not upper right clean in $\mathcal{M}$.

We would like to say that in a mazery, if points (u_0, u_1), (v_0, v_1) are such that for $d = 0,1$ we have $u_d < v_d$ and there are no walls and traps in the rectangle $[u_0, v_0] \times [u_1, v_1]$, then (v_0, v_1) is reachable from (u_0, u_1). However, this will only hold with some restrictions. What we will have is the following, with an appropriate parameter

$$0 \leqslant \sigma < 0.5.$$

Condition 2.2. Suppose that points $u = (u_0, u_1)$, $v = (v_0, v_1)$ are such that for $d = 0,1$ we have $u_d < v_d$ and there are no traps or walls contained in the rectangle between u and v. If u is upper right clean, v is lower left clean and these points also satisfy the slope-constraint

$$\sigma \leqslant \frac{v_1 - u_1}{v_0 - u_0} \leqslant 1/\sigma$$

then v is reachable from u. ⌟

We will also need sufficiently many clean points:

Condition 2.3. For every square $(a,b) + (0, 3\Delta]^2$ that does not contain walls or traps, there is a lower left clean point in its middle third $(a,b) + (\Delta, 2\Delta]^2$. ⌟

Lemma 2.4. *We have $\mathcal{F} \subseteq \mathcal{F}' \cup \mathcal{Q}$.*

Proof. (Please, refer to Figure 7.) Suppose that $\mathcal{Q}$ does not hold, then $(0,0)$ is upper right clean.

Suppose also that $\mathcal{F}'$ does not hold: then by Condition 2.3, there is a point $u = (u_0, u_1)$ in the square $[\Delta, 2\Delta]^2$ that is lower left clean in $\mathcal{M}$. This u also satisfies the slope condition $1/2 \leqslant u_1/u_0 \leqslant 2$ and is hence, by Condition 2.2, reachable from $(0,0)$. □

We will define a sequence of mazeries $\mathcal{M}^1, \mathcal{M}^2, \dots$ with $\mathcal{M}^{k+1} = (\mathcal{M}^k)^*$, with $\Delta_k \to \infty$. All these mazeries are on a common probability space, since $\mathcal{M}^{k+1}$ is a function of $\mathcal{M}^k$. All components of the mazeries will be indexed correspondingly: for example, the event $\mathcal{Q}_k$ that $(0,0)$ is not upper right clean in $\mathcal{M}^k$ plays the role of $\mathcal{Q}$ for the mazery $\mathcal{M}^k$. We will have the following property:

Condition 2.5. We have $\mathcal{Q}_k \subseteq \bigcup_{i < k} \mathcal{F}'_i$. ⌟

In other words, if there are no traps or walls of any order $i < k$ in the i-neighborhood of the origin, then the origin is upper-right clean on level k.
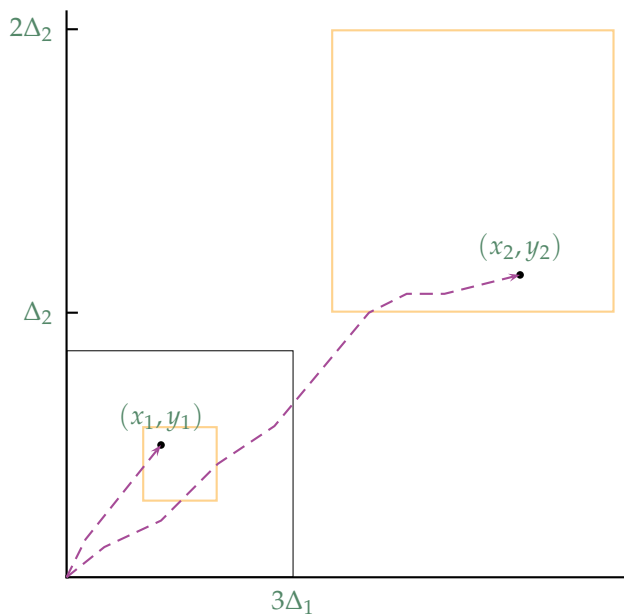
Figure 7: Illustration of Lemma 2.6 and Theorem 1

This, along with Lemma 2.4 implies $\mathscr{F}_k \subseteq \bigcup_{i \leqslant k} \mathscr{F}'_i$, which is inequality (2.1). Hence the theorem is implied by the following lemma, which will be proved after all the details are given:

For the following lemma, recall that m is the the number of "colors" in the percolation setting, and therefore $1/m$ upper-bounds the conditional probability of a trap $\{X(i) = Y(j)\}$ (under, for example, fixing $X(i)$).

Lemma 2.6 (Main). *If m is sufficiently large then the sequence $\mathscr{M}^k$ can be constructed, in such a way that it satisfies all the above conditions and also*

$$\sum_k \mathbf{P}(\mathscr{F}'_k) < 1. \tag{2.3}$$

2.3 The rest of the paper

The proof structure is quite similar to [4]. That paper is not simple, but it is still simpler than the present one, and we recommend very much looking at it in order to see some of the ideas going into the present paper in their simpler, original setting. Walls and holes, the general form of the definition of a mazery and the scale-up operation are similar. There are, of course, differences: traps are new.

Section 3 defines the random structures called mazeries. This is probably the hardest to absorb, since the structure has a large number of ingredients and required properties called Conditions. The proof in the sections that follow will clarify, however, the role of each concept and condition.

Section 4 defines the scale-up operation $\mathcal{M}^k \mapsto \mathcal{M}^{k+1}$. It also proves that scale-up preserves almost all *combinatorial* properties, that is those that do not involve probability bounds. The one exception is the reachability property, formulated by Lemma 8.1 (Approximation): its proof is more complex, and is postponed to Section 8.

Section 5 estimates how the probability bounds are transformed by the scale-up operation. Section 6 specifies the parameters in such a way that guarantees that the probability conditions are also preserved by scale-up. Section 7 carries out the computations leading to the proof of those conditions.

Section 9 ties up all threads into the proof of Lemma 2.6 and the proof of the theorem.

3 Mazeries

This section consists almost exclusively of definitions.

3.1 Notation

The notation (a,b) for real numbers a,b will generally mean for us the pair, and not the open interval. Occasional exceptions would be pointed out separately. We will use

$$a \wedge b = \min(a,b), \quad a \vee b = \max(a,b).$$

As mentioned earlier, we will use intervals on the real line and rectangles over the Euclidean plane, even though we are really only interested in the lattice $\mathbb{Z}_+^2$. To capture all of $\mathbb{Z}_+$ this way, for our right-closed intervals $(a,b]$, we allow the left end a to range over all the values $-1, 0, 1, 2, \ldots$. For an interval $I = (a,b]$, we will denote

$$X(I) = (X(a+1), \ldots, X(b)).$$

The *size* of an interval I with endpoints a,b (whether it is open, closed or half-closed), is denoted by $|I| = b - a$. By the *distance* of two points $a = (a_0, a_1)$, $b = (b_0, b_1)$ of the plane, we mean

$$|b_0 - a_0| \vee |b_1 - a_1|.$$

The *size* of a rectangle

$$\mathrm{Rect}(a,b) = [a_0, b_0] \times [a_1, b_1]$$

in the plane is defined to be equal to the distance between a and b. For two different points $u = (u_0, u_1)$, $v = (v_0, v_1)$ in the plane, when $u_0 \leqslant v_0$, $u_1 \leqslant v_1$:

$$\text{slope}(u, v) \quad = \frac{v_1 - u_1}{v_0 - u_0},$$
$$\text{minslope}(u, v) = \min\big(\text{slope}(u, v), 1/\text{slope}(u, v)\big).$$

We introduce the following partially open rectangles

$$\text{Rect}^{\rightarrow}(a, b) = (a_0, b_0] \times [a_1, b_1],$$
$$\text{Rect}^{\uparrow}(a, b) \ = [a_0, b_0] \times (a_1, b_1]. \tag{3.1}$$

The relation

$$u \rightsquigarrow v$$

says that point v is reachable from point u (the underlying graph will always be clear from the context). For two sets A, B in the plane or on the line,

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

3.2   The structure

The tuple

All our structures defined below refer to "percolations" over the same lattice graph, in $\mathbb{Z}_+^2$, defined by the pair of sequences of random variables

$$Z = (X, Y) = (Z_0, Z_1),$$

where $Z_d = (Z_d(0), Z_d(1), \dots)$ with $Z_d(t) \in \{1, \dots, m\}$ independent random walks on the set $\{1, \dots, m\}$ of nodes of the graph K_m for some fixed m.

A *mazery*

$$\mathbb{M} = (\mathcal{M}, \Delta, \sigma, R, w, q) \tag{3.2}$$

consists of a random process $\mathcal{M}$, the parameters $\Delta > 0$, $\sigma \geqslant 0$, $R > 0$, and the probability bounds $w > 0$, q, all of which will be detailed below, along with conditions that they must satisfy. Let us describe the random process

$$\mathcal{M} = (Z, \mathcal{T}, \mathcal{W}, \mathcal{B}, \mathcal{C}, \mathcal{S}).$$

We have the random objects

$$\mathcal{T}, \quad \mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1), \quad \mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1), \quad \mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1), \quad \mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2).$$

all of which are functions of Z. The set $\mathcal{T}$ of random *traps* is a set of some closed rectangles of size $\leqslant \Delta$. For trap $\text{Rect}(a, b)$, we will say that it *starts* at its lower left corner a.

Definition 3.1 (Wall values). To describe the process $\mathscr{W}$, we introduce the concept of a *wall value* $E = (B, r)$. Here B is the *body* which is a right-closed interval,[2] and *rank*

$$r \geqslant R > 0.$$

We write $\text{Body}(E) = B$, $|E| = |B|$. We will sometimes denote the body also by E. Let Wvalues denote the set of all possible wall values. ⌟

Walls will arise in a variety of ways, but the properties we are interested in will only depend on the body and the rank. Walls of higher rank have a smaller probability upper bound of occurrence, and smaller probability lower bound of holes through them. They arise at higher levels in the hierarchical construction, but the rank depends on more details of how the wall arose than just the level of the mazery it is in.

Let

$$\mathbb{Z}_+^{(2)}$$

denote the set of pairs (u, v) with $u < v$, $u, v \in \mathbb{Z}_+$. The random objects

$$\mathscr{W}_d \subseteq \mathscr{B}_d \subseteq \text{Wvalues},$$
$$\mathscr{S}_d \subseteq \mathscr{C}_d \subseteq \mathbb{Z}_+^{(2)} \times \{-1, 1\} \text{ for } d = 0, 1,$$
$$\mathscr{S}_2 \subseteq \mathbb{Z}_+^{(2)} \times \mathbb{Z}_+^{(2)} \times \{-1, 1\} \times \{0, 1, 2\}$$

are also functions of Z. (Note that we do not have any $\mathscr{C}_2$.)

Definition 3.2 (Barriers and walls). The elements of $\mathscr{W}_d$ and $\mathscr{B}_d$ are called *walls* and *barriers* of Z_d respectively, where the sets $\mathscr{W}_d, \mathscr{B}_d$ are functions of Z_d. In particular, elements of $\mathscr{W}_0$ are called *vertical walls*, and elements of $\mathscr{W}_1$ are called *horizontal walls*. Similarly for barriers. When we say that a certain interval contains a wall or barrier we mean that it contains its body.

A right-closed interval is called *external* if it intersects no walls. A wall is called *dominant* if it is surrounded by external intervals each of which is either of size $\geqslant \Delta$ or is at the beginning of $\mathbb{Z}_+$. Note that if a wall is dominant then it contains every wall intersecting it. ⌟

The set of barriers is a random subset of the set of all possible wall values, and the set of walls is a random subset of the set of walls. In the scale-up operation, we first define barriers, and then we select the walls from among them. The form of the definition of barriers implies their simple dependency properties required in Condition 3.6.1, which then make simple probability upper bounds possible. These then hold for walls as well, since

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[2]This is different from the definition in the paper [4], where walls were open intervals.

walls are barriers. On the other hand, walls have the nicer combinatorial properties we need to prove eventually reachability (percolation).

The following definition uses the fact following from Condition 3.6.1b that whether an interval B is a barrier of the process X depends only $X(B)$.

Definition 3.3 (Potential wall). For a vertical wall value $E = (B, r)$ and a value of $X(B)$ making E a barrier of rank r we will say that E is a *potential vertical wall* of rank r if there is an extension of $X(B)$ to a complete sequence X that makes E a vertical wall of rank r. Similarly for horizontal walls. ⌟

Remarks 3.1. 1. We will see below that, for any rectangle with projections $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$. Also for any interval I, the event that it is a (say, vertical) barrier depends only on $X(I)$, but the same is not true of walls.

2. In the definition of the mazery $\mathcal{M}^{k+1}$ from mazery $\mathcal{M}^k$, we will drop *low rank* walls of $\mathcal{M}^k$, (those with $\leqslant R_{k+1}$). These walls will have high probability of holes through them, so reachability will be conserved.

   To control the proliferation of walls, a pair of close walls of $\mathcal{M}^k$ will give rise to a *compound wall* of $\mathcal{M}^{k+1}$ only if at least one of the components has low rank.

   ⌟

The following condition holds for the parts discussed above.

Condition 3.2. The parameter Δ is an upper bound on the size of every wall and trap. ⌟

Cleanness

Intuitively, a point x is clean in $\mathcal{M}^k$ when none of the mazeries $\mathcal{M}^i$ for $i < k$ has any bad events near x. This interpretation will become precise by the rescaling operation; at this point, we treat cleanness as a primitive, just like walls. Several kinds of cleanness are needed, depending on the direction in which the absence of lower-order bad events will be guaranteed.

The set $\mathcal{C}_d$ is a function of the process Z_d, and is used to formalize (encode) the notions of cleanness given descriptive names below.

Definition 3.4 (One-dimensional cleanness). For an interval $I = (a, b]$ or $I = [a, b]$, if $(a, b, -1) \in \mathcal{C}_d$ then we say that point b of $\mathbb{Z}_+$ is *clean* in I for the sequence Z_d. If $(a, b, 1) \in \mathcal{C}_d$ then we say that point a is clean in I. From now on, whenever we talk about cleanness of an element of $\mathbb{Z}_+$, it is always understood with respect to one of the sequences Z_d for $d = 0, 1$ (that is either for the sequence X or for Y).

Let us still fix a direction d and talk about cleanness, and so on, with respect to the sequence Z_d. A point $x \in \mathbb{Z}_+$ is called left-clean (right-clean) if it is clean in all intervals of
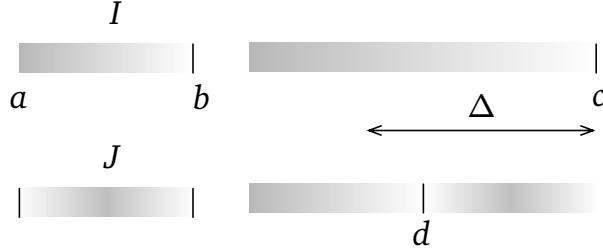
Figure 8: One-dimensional notions of cleanness. Point a is not clean in interval I, but point b is. Point c is left-clean. Interval J is inner clean. Point d is clean.
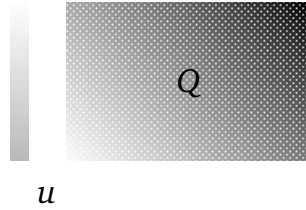

Figure 9: Cleanness in a rectangle. Point u is trap-clean in rectangle Q, but is not clean in it, since its projection is not clean in the corresponding projection of Q.

the form $(a, x]$, $[a, x]$ (all intervals of the form $(x, b]$, $[x, b]$). It is *clean* if it is both left- and right-clean. If both ends of an interval I are clean in I then we say I is *inner clean*.

To every notion of one-dimensional cleanness there is a corresponding notion of *strong cleanness*, defined with the help of the process $\mathscr{S}$ in place of the process $\mathscr{C}$. ⌟

The relation of strong cleanness to cleanness is dual to the relation of walls to barriers: every strongly clean point is clean but not vice versa, and every wall is a barrier, but not vice versa. This duality is not accidental, since the scale-up operation will define strong cleanness recursively requiring the absence of nearby barriers, and cleanness requiring the absence of nearby walls.

Definition 3.5 (Trap-cleanness). For points $u = (u_0, u_1)$, $v = (v_0, v_1)$, $Q = \mathrm{Rect}^{\varepsilon}(u, v)$ where $\varepsilon = \rightarrow$ or $\uparrow$ or nothing, we say that point u is *trap-clean in Q* (with respect to the pair of sequences (X, Y)) if $(u, v, 1, \varepsilon') \in \mathscr{S}_2$, where $\varepsilon' = 0, 1, 2$ depending on whether $\varepsilon = \rightarrow$ or $\uparrow$ or nothing. Similarly, point v is *trap-clean in Q* if $(u, v, -1, \varepsilon') \in \mathscr{S}_2$. ⌟

It is not seen here, but the scale-up operation will define trap-cleanness recursively requiring the absence of nearby traps.

Definition 3.6 (Complex two-dimensional sorts of cleanness). We say that point u is *clean* in Q when it is trap-clean in Q and its projections are clean in the corresponding projections of Q.

If u is clean in all such left-open rectangles then it is called *upper right rightward-clean*. We delete the "rightward" qualifier here if we have closed rectangles in the definition here instead of left-open ones. (It is hopeless to illustrate visually the difference made by the "rightward" qualifier, but the distinction seems to matter in the proof.) Cleanness with qualifier "upward" is defined similarly. Lower left cleanness of v is defined similarly, using $(u, v, -1, \varepsilon')$, except that the qualifier is unnecessary: all our rectangles are upper right closed.

A point is called *clean* if it is upper right clean and lower left clean. If both the lower left and upper right points of a rectangle Q are clean in Q then Q is called *inner clean*. If the lower left endpoint is lower left clean and the upper right endpoint is upper right rightward-clean then Q is called *outer rightward-clean*. Similarly for *outer upward-clean* and *outer-clean*.

We will also use a *partial* version of cleanness. If point u is trap-clean in Q and its projection on the x axis is *strongly* clean in the same projection of Q then we will say that u is H-clean in Q. Clearly, if u is H-clean in Q and its projection on the y axis is clean in (the projection of) Q then it is clean in Q. We will call rectangle Q *inner* H-clean if both its lower left and upper right corners are H-clean in it.

The notion V-clean is defined similarly when we interchange horizontal and vertical. ⌟

Hops

Hops are intervals and rectangles for which we will be able to give some guarantees that they can be passed.

Definition 3.7 (Hops). A right-closed horizontal interval I is called a *hop* if it is inner clean and contains no vertical wall. A closed interval $[a, b]$ is a hop if $(a, b]$ is a hop. Vertical hops are defined similarly.

We call a rectangle $I \times J$ a *hop* if it is inner clean and contains no trap or wall. ⌟

Remark 3.3. An interval or rectangle that is a hop can be empty: this is the case if the interval is $(a, a]$, or the rectangle is, say, $\mathrm{Rect}^{\rightarrow}(u, u)$. ⌟

Definition 3.8 (Good sequences of walls). Two disjoint walls are called *neighbors* if the interval between them is a hop. A sequence $W_i \in \mathcal{W}$ of walls $i = 1, 2, \ldots$ along with the intervals $I_1, \ldots, I_{n-1}$ between them is called a *sequence of neighbor walls* if for all $i > 1$, W_i is a right neighbor of W_{i-1}. We say that an interval I is *spanned* by the sequence of neighbor walls $W_1, W_2, \ldots, W_n$ if $I = W_1 \cup I_1 \cup W_2 \cup \cdots \cup W_n$. We will also say that I is spanned by the sequence $(W_1, W_2, \ldots)$ if both I and the sequence are infinite and $I = W_1 \cup I_1 \cup W_2 \cup \ldots$.

If there is a hop I_0 adjacent on the left to W_1 and a hop I_n adjacent on the right to W_n (or the sequence W_i is infinite) then this system is called an *extended sequence of neighbor walls*. We say that an interval I is *spanned* by this extended sequence if $I = I_0 \cup W_1 \cup I_1 \cup \cdots \cup I_n$ (and correspondingly for the infinite case). ⌟

Holes

Definition 3.9 (Reachability)**.** To each mazery $\mathcal{M}$ belongs a random graph

$$\mathcal{V} = \mathbb{Z}_+^2, \quad \mathcal{G} = (\mathcal{V}, \mathcal{E})$$

where $\mathcal{E}$ is determined by the above random processes as in Subsection 1.3. We say that point v is *reachable* from point u in $\mathcal{M}$ (and write $u \rightsquigarrow v$) if it is reachable in $\mathcal{G}$. ⌟

Remark 3.4. According to our definitions in Subsection 1.3, point u itself may be closed even if v is reachable from u. ⌟

Intuitively, a hole is a place at which we can pass through a wall. We will also need some guarantees of being able to reach the hole and being able to leave it.

Definition 3.10 (Holes)**.** Let $a = (a_0, a_1)$, $b = (b_0, b_1)$, and let the interval $I = (a_1, b_1]$ be the body of a horizontal barrier B. For an interval $J = (a_0, b_0]$ with $|J| \leqslant |I|$ we say that J is a vertical *hole passing through* B, or *fitting* B, if $a \rightsquigarrow b$ within the rectangle $J \times [a_1, b_1]$.

The above hole is called *lower left clean*, upper right clean, and so on, if the rectangle is. Consider a point (u_0, u_1) with $u_i \leqslant a_i$, $i = 0, 1$. The hole J is called *good as seen from* point u if a is H-clean in $\text{Rect}^{\rightarrow}(u, a)$, and b is upper-right rightward H-clean (recall Definition 3.6). It is *good* if it is good as seen from any such point u. Note that this way the horizontal cleanness is required to be strong, but no vertical cleanness is required (since the barrier B was not required to be outer clean).

Horizontal holes are defined similarly.[3] ⌟

Remark 3.5. Note that the condition of passing through a wall depends on an interval slightly larger than the wall itself: it also depends on the left end of the left-open interval that is the body of the wall. ⌟

3.3 Conditions on the random process

Most of our conditions on the distribution of process $\mathcal{M}$ are fairly natural; however, the need for some of them will be seen only later. For example, for Condition 3.6.3d, only its special case (in Remark 3.7.2) is well motivated now: it says that through every wall

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[3] The notion of hole in the present paper is different from that in [4]. Holes are not primitives; rather, they are defined with the help of reachability.

there is a hole with sufficiently large probability. The general case will be used in the inductive proof showing that the hole lower bound also holds on compound walls after renormalization (going from $\mathcal{M}^k$ to $\mathcal{M}^{k+1}$).

It is a fair question to ask at this point, why all these conditions are necessary, and whether they are sufficient. Unfortunately, at this point I can only answer that each condition will be used in the proof, suggesting their necessity (at least in this proof). On the other hand, the proof that the scale-up operation conserves the conditions, shows their sufficiency.

The combinatorial conditions derive ultimately from the necessity of proving Lemma 8.1 (Approximation). The dependency conditions and probability estimates derive ultimately from the necessity of proving inequality (2.3). But some conditions are introduced just in order to help the proof of the conservation in the scale-up.

Definition 3.11. The function

$$p(r,l) \tag{3.3}$$

is defined as the supremum of probabilities (over all points t) that any barrier with rank r and size l starts at t conditional over all possible conditions of the form $Z_d(t) = k$ for $k \in \{1,\dots,m\}$. ⌟

The constant χ has been introduced in (2.2). Its choice, as well as the choice of some of the other expressions we are about to introduce, will be motivated in Section 6. We will use three additional constants, c_1, c_2, c_3. Constant c_1 will be chosen at the end of the proof of Lemma 7.3, c_2 in the proof or Lemma 7.5, while c_3 will be chosen at the end of the proof of Lemma 7.10.

Definition 3.12. We will make use of constant

$$\lambda = 2^{1/2}. \tag{3.4}$$

⌟

Let us define a function that will serve as an an upper bound on $\sum_l p(r,l)$.

Definition 3.13 (Wall probability upper bound)**.** Let

$$p(r) = c_2 r^{-c_1} \lambda^{-r}. \tag{3.5}$$

⌟

The term $c_2 r^{-c_1}$ serves for absorbing some lower-order factors that arise in estimates like (5.19).

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Definition 3.14 (Hole probability lower bound). Let

$$h(r) = c_3 \lambda^{-\chi r}. \tag{3.6}$$

⌟

Here come the conditions.

Condition 3.6.

1. (Dependencies)

   a. For any rectangle $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$.
   b. For a vertical wall value E the event $\{E \in \mathscr{B}\}$ (that is the event that it is a vertical barrier) is a function of $X(\mathrm{Body}(E))$. Similarly for horizontal barriers.
   c. For integers $a < b$, and $d = 0,1$, the events defining strong cleanness, that is $\{(a,b,-1) \in \mathscr{S}_d\}$ and $\{(a,b,1) \in \mathscr{S}_d\}$, are functions of $Z_d((a,b])$.
      When Z is fixed, then for a fixed a, the (strong and not strong) cleanness of a in $(a,b]$ is decreasing as a function of $b - a$, and for a fixed b, the (strong and not strong) cleanness of b in $(a,b]$ is decreasing as a function of $b - a$. These functions reach their minimum at $b - a = \Delta$: thus, if x is (strongly or not strongly) left clean in $(x - \Delta, x]$ then it is (strongly or not strongly) left clean.
   d. For any rectangle $Q = I \times J$, the event that its lower left corner is trap-clean in Q, is a function of the pair $X(I), Y(J)$.
      Among rectangles with a fixed lower left corner, the event that this corner is trap-clean in Q is a decreasing function of Q (in the set of rectangles partially ordered by containment). In particular, the trap-cleanness of u in $\mathrm{Rect}(u,v)$ implies its trap-cleanness in $\mathrm{Rect}^{\rightarrow}(u,v)$ and in $\mathrm{Rect}^{\uparrow}(u,v)$. If u is upper right trap-clean in the left-open or bottom-open or closed square of size Δ, then it is upper right trap-clean in all rectangles Q of the same type. Similar statements hold if we replace upper right with lower left.

   Whether a certain wall value $E = (B,r)$ is a barrier depends only on $X(B)$. However, whether it is a wall may depend also on the values of X outside B. Similarly, whether a certain interval is inner clean may depend on the sequence outside it, but whether it is strongly inner clean does not.

2. (Combinatorial requirements)

   a. A maximal external interval (see Definition 3.2) of size $\geqslant \Delta$ or one starting at -1 is inner clean.
   b. An interval I that is surrounded by maximal external intervals is spanned by a sequence of neighbor walls (see Definition 3.8). This is true even in the case when I starts at 0 and even if it is infinite. To accomodate these cases, we require the following, which is somewhat harder to parse: Suppose that interval I is adjacent on the

left to a maximal external interval that either starts at -1 or has size $\geqslant \Delta$. Suppose also that it is either adjacent on the right to a similar interval or is infinite. Then it is spanned by a (finite or infinite) sequence of neighbor walls. In particular, the whole line is spanned by an extended sequence of neighbor walls.

c. If a (not necessarily integer aligned) right-closed interval of size $\geqslant 3\Delta$ contains no wall, then its middle third contains a clean point.

d. Suppose that a rectangle $I \times J$ with (not necessarily integer aligned) right-closed I, J with $|I|, |J| \geqslant 3\Delta$ contains no horizontal wall and no trap, and a is a right clean point in the middle third of I. There is an integer b in the middle third of J such that the point (a,b) is upper right clean. A similar statement holds if we replace upper right with lower left (and right with left). Also, if a is clean then we can find a point b in the middle third of J such that (a,b) is clean.

There is also a similar set of statements if we vary a instead of b.

3. (Probability bounds)

a. Given a string $x = (x(0), x(1), \ldots)$, a point (a,b), let $\mathscr{F}$ be the event that a trap starts at (a,b). Let $s \in \{1, \ldots, m\}$, then

$$\mathbf{P}[\mathscr{F} \mid X = x, \ Y(b-1) = s] \leqslant w.$$

The same is required if we exchange X and Y.

b. We have $p(r) \geqslant \sum_l p(r,l)$.

c. We require $q < 0.1$, and that for all $k \in \{1, \ldots, m\}$, for all $a < b$ and all $u = (u_0, u_1)$, $v = (v_0, v_1)$, for all sequences y, the following quantities are all $\leqslant q/2$:

$$\mathbf{P}[a \text{ (resp. } b) \text{ is not strongly clean in } (a,b] \mid X(a) = k], \qquad (3.7)$$
$$\mathbf{P}[u \text{ (resp. } v) \text{ is not trap-clean in } \mathrm{Rect}^{\rightarrow}(u,v) \mid X(u_0) = k, Y = y], \qquad (3.8)$$
$$\mathbf{P}[u \text{ (resp. } v) \text{ is not trap-clean in } \mathrm{Rect}(u,v) \mid X(u_0 - 1) = k, Y = y], \qquad (3.9)$$

and similarly with X and Y reversed.

d. Let $u \leqslant v < w$, and a be given with $v - u \leqslant 12\Delta$, and define

$$b = a + \lceil (v-u)/2 \rceil, \qquad c = a + (v-u) + 1.$$

Assume that $Y = y$ is fixed in such a way that B is a horizontal wall of rank r with body (v,w). For a $d \in [b, c-1]$ let $Q(d) = \mathrm{Rect}^{\rightarrow}((a,u), (d,v))$. Let $E = E(u,v,w; a)$ be the event (a function of X) that there is a d such that

   (i) A vertical hole fitting B starts at d.

   (ii) Rectangle $Q(d)$ contains no traps or vertical barriers, and is inner H-clean.
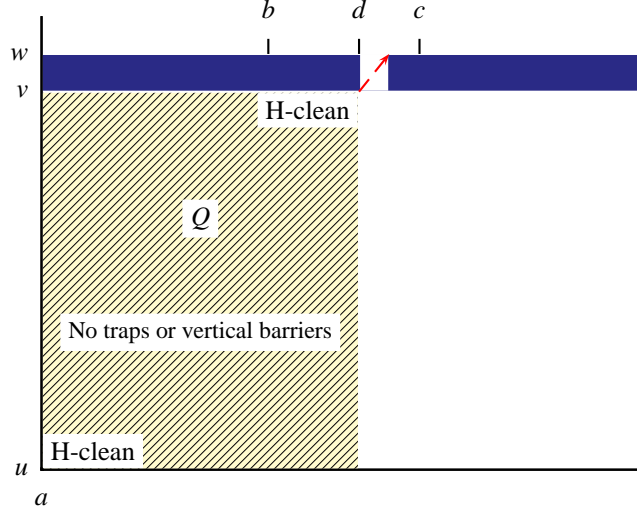
Figure 10: Condition 3.6.3d. The hatched rectangle is inner H-clean and contains no traps or vertical barriers.

Let $k \in \{1,\ldots,m\}$. Then

$$\mathbf{P}[E \mid X(a) = k, \ Y = y] \geqslant (c-b)^X h(r).$$

The same is required if we exchange horizontal and vertical.

⌟

Remarks 3.7.

1. Conditions 3.6.2c and 3.6.2d imply the following. Suppose that a right-upper closed square Q of size 3Δ contains no wall or trap. Then its middle third contains a clean point. In particular, this implies Condition 2.3.

2. The most important special case of Condition 3.6.3d is $v = u$, implying $b = a$, $c = b+1$: then it says that for any horizontal wall B of rank r, at any point a, the probability that there is a vertical hole passing through B at point a is at least $h(r)$.

⌟

The graph $\mathscr{G}$ defined in Definition 3.9 is required to satisfy the following conditions.

Condition 3.8 (Reachability). We require $0 \leqslant \sigma < 0.5$. Let u, v be points with $\mathrm{minslope}(u,v) \geqslant \sigma$. If they are the starting and endpoint of a rectangle that is a hop, then $u \rightsquigarrow v$. The rectangle in question is allowed to be bottom-open or left-open, but not both. ⌟

Example 3.9 (Base case). The clairvoyant demon problem can be seen as a special case of a mazery. Let $\Delta = 1$ for the scale parameter, and $\sigma = 0$, that is there is no lower bound on the minimum slope for the reachability condition. The parameters $R > 0$ and $0 < q < 0.1$ are chosen arbitrarily, and we choose

$$1 > w > \frac{1}{m-1},\tag{3.10}$$

where m is the size of the complete graph on which the random walks are performed.

Let $\mathscr{T} = \{ (i,j) \} : X(i) = Y(j)$, that is traps are points (i,j) with $X(i) = Y(j)$. We set $\mathscr{B}_d = \mathscr{W}_d = \emptyset$, that is there are no barriers (and then, of course, no walls).

Let $\mathscr{S}_d = \mathscr{C}_d = \mathbb{Z}_+^{(2)} \times \{-1,1\}$ for $d = 0,1$. In other words, every point is strongly clean in all one-dimensional senses. Also $\mathscr{S}_2 = \mathbb{Z}_+^{(2)} \times \mathbb{Z}_+^{(2)} \times \{-1,1\} \times \{0,1,2\}$, that is every point is trap-clean in all senses.

All combinatorial and dependency conditions are satisfied trivially. Of the probability bounds, Condition 3.6.3a is satisfied by our requirement (3.10). Since there are no walls and every point is clean in all possible ways, the other probability bounds are satisfied trivially.

Since now in a trap-free rectangle nothing blocks reachability, the reachability condition also holds trivially. Note that it is violated in the bottom-left open rectangle $(0,1] \times (0,1]$ if $X(0) = 1$, $X(1) = 2$, $Y(0) = 2$, $Y(1) = 1$. Indeed, the traps $(0,1)$, $(1,0)$ are not part of the rectangle where they are prohibited, but they block point $(0,0)$ from point $(1,1)$. ⌟

4   The scaled-up structure

We will use Example 3.9 as the mazery $\mathscr{M}^1$ in our sequence of mazeries $\mathscr{M}^1, \mathscr{M}^2, \ldots$ where $\mathscr{M}^{k+1} = (\mathscr{M}^k)^*$. In this section, we will define the scaling-up operation $\mathscr{M} \mapsto \mathscr{M}^*$: we still postpone to Section 6 the definition of several parameters and probability bounds for $\mathscr{M}^*$.

Let us recall the meaning of the scale-up operation from Section 2. Our final goal is to prove reachability of points far away from the origin, with large probability. In our model $\mathscr{M}$, reachability is guaranteed in a rectangle $Q = \text{Rect}(u,v)$ from u to v if u,v are inner clean in Q, and there are no traps or walls in Q. The absence of traps cannot be guaranteed in our base model when the rectangle is large. It should be sufficient for traps to be far from each other, but even this condition will fail occasionally.

The idea of the scale-up strategy is to define new kinds of "obstacles" on which we can blame such failures. If these obstacles have sufficiently small probability, then they can be regarded as the traps and walls of a model $\mathscr{M}^*$. The crucial combinatorial test of this procedure is the proof of the reachability condition in model $\mathscr{M}^*$, which is Lemma 8.1. This may not be so complicated in going up just one step from the base model $\mathscr{M}^1$ to $(\mathscr{M}^1)^* = \mathscr{M}^2$, since the base model has no walls. But other models $\mathscr{M}^k$ do have them:

in this case, reachability in $\mathcal{M}^{k+1}$ in a rectangle Q without walls means that Q may have walls and traps of $\mathcal{M}^k$, just no walls or traps of $\mathcal{M}^{k+1}$. So, the walls and traps of $\mathcal{M}^*$ impersonate the difficulties of passing through the walls and around the traps of $\mathcal{M}$. Walls of $\mathcal{M}^*$ will also be used as scapegoats for some excessive correlations among traps of $\mathcal{M}$.

4.1 The scale-up construction

Some of the following parameters will be given values only later, but they are introduced by name here.

Definition 4.1. Let Λ be a constant and let parameters f, g satisfy

$$\Lambda = 500,$$
$$\Delta/g \leqslant g/f < (0.5 - \sigma)/(2\Lambda). \tag{4.1}$$

⌟

The parameters $\Delta \ll g \ll f$ will be different for each level of the construction. The scale parameter Δ is part of the definition of a mazery. Here is the approximate meaning of f and g: We try not to permit walls closer than f to each other, and we try not to permit intervals larger than g without holes.

The following definition is already part of the scale-up.

Definition 4.2. Let $\sigma^* = \sigma + \Lambda g/f$. ⌟

The value Δ^* will be defined later, but we will guarantee the inequality

$$3f \leqslant \Delta^*. \tag{4.2}$$

After defining the mazery $\mathcal{M}^*$, eventually we will have to prove the required properties. To be able to prove Condition 3.8 for $\mathcal{M}^*$, we will introduce some new walls and traps in $\mathcal{M}^*$ whenever some larger-scale obstacles prevent reachability. There will be two kinds of new walls, so-called *emerging* walls, and *compound* walls. A pair of traps too close to each other will define, under certain conditions, a compound trap, which becomes part of $\mathcal{M}^*$. A new kind of trap, called a trap of the missing-hole kind will arise when some long stretch of a low-rank wall is without a hole.

For the new value of R we require

$$R^* \leqslant 2R - \log_\lambda f. \tag{4.3}$$

Definition 4.3 (Light and heavy)**.** Walls of rank lower than R^* are called *light*, the other ones are called *heavy*. ⌟
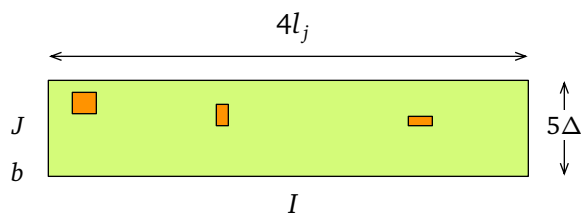
Figure 11: A correlated trap

Heavy walls of $\mathcal{M}$ will also be walls of $\mathcal{M}^*$. We will define walls only for either X or Y, but it is understood that they are also defined when the roles of X and Y are reversed.

The rest of the scale-up construction will be given in the following steps.

Step 1 (Cleanness). For an interval I, its right endpoint x will be called clean in I for $\mathcal{M}^*$ if

– It is clean in I for $\mathcal{M}$.

– The interval I contains no wall of $\mathcal{M}$ whose right end is closer to x than $f/3$.

We will say that a point is strongly clean in I for $\mathcal{M}^*$ if it is strongly clean in I for $\mathcal{M}$ and I contains no barrier of $\mathcal{M}$ whose right end is closer to it than $f/3$. Cleanness and strong cleanness of the left endpoint is defined similarly.

Let a point u be a starting point or endpoint of a rectangle Q. It will be called trap-clean in Q for $\mathcal{M}^*$ if

– It is trap-clean in Q for $\mathcal{M}$.

– Any trap contained in Q is at a distance $\geqslant g$ from u.

⌟

For the next definitions, Figure 4 may help.

Step 2 (Uncorrelated traps). A rectangle Q is called an *uncorrelated compound trap* if it contains two traps with disjoint projections, with a distance of their starting points at most f, and if it is minimal among the rectangles containing these traps. ⌟

Clearly, the size of an uncorrelated trap is bounded by $\Delta + f$.

Step 3 (Correlated trap). Let

$$g' = 2.2g, \ l_1 = 7\Delta, \ l_2 = g'. \tag{4.4}$$

(Choice motivated by the proof of Lemmas 4.7 and 8.1.) For a $j \in \{1, 2\}$ let I be a closed
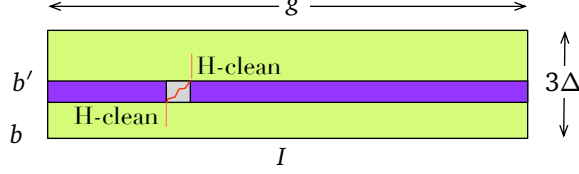
Figure 12: A trap of the missing-hole kind: good holes (see Definition 3.10, the type of hole shown in the picture with H-cleanness on the lower-left and upper-right) are missing

interval with length $|I| = 4l_j$, and $b \in \mathbb{Z}_+$, with $J = [b, b+5\Delta]$. Let $x(I), y(J)$ be given. We say that event

$$\mathscr{L}_j(x, y, I, b)$$

holds if for all right-closed intervals $\hat{I} \subseteq I$ of size l_j, the rectangle $\hat{I} \times J$ contains a trap. We will say that $I \times J$ is a *horizontal correlated trap* of type j if $\mathscr{L}_j(x, y, I, b)$ holds and for all $s \in \{1, \ldots, m\}$, we have the inequality

$$\mathbf{P}[\mathscr{L}_j(x, Y, I, b) \mid X(I) = x(I), Y(b-1) = s] \leqslant w^2. \tag{4.5}$$

Vertical correlated traps are defined analogously[4]. ⌟

Step 4 (Traps of the missing-hole kind). Let I be a closed interval of size g, let b be a site with $J = [b, b+3\Delta]$. Let $x(I), y(J)$ be fixed. We say that event

$$\mathscr{L}_3(x, y, I, b)$$

holds if, with there is a $b' > b + \Delta$ such that $(b + \Delta, b']$ is the body of a light horizontal potential wall W, and no good vertical hole (in the sense of Definition 3.10) $(a_1, a_2]$ with $(a_1 - \Delta, a_2 + \Delta] \subseteq I$ passes through W.

We say that $I \times J$ is a *horizontal trap of the missing-hole kind* if event $\mathscr{L}_3(x, y, I, b)$ holds and for all $s \in \{1, \ldots, m\}$ we have

$$\mathbf{P}[\mathscr{L}_3(x, Y, I, b) \mid X(I) = x(I), Y(b-1) = s] \leqslant w^2. \tag{4.6}$$

⌟

Inequalities (4.1) and (4.2) bound the size of all new traps by Δ^*.

Step 5 (Emerging walls). It is in this definition where the difference between barriers and walls first appears in the paper constructively. We define some objects as barriers, and then

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[4] The smallness of the conditional probability in the other direction will be proved in Lemma 5.5, without having to require it.
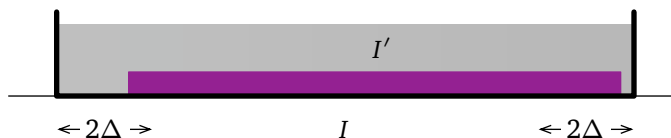
Figure 13: An emerging barrier

designate some of the barriers (but not all as walls).

A vertical emerging barrier is, essentially, a horizontal interval over which the conditional probability of a bad event $\mathscr{L}_j$ is not small (thus preventing a new trap). But in order to find enough barriers, the ends are allowed to be sligthly extended. Let x be a particular value of the sequence X over an interval $I = (u, v]$. For any $u' \in (u, u + 2\Delta]$, $v' \in (v - 2\Delta, v]$, let us define the interval $I' = [u', v']$. We say that interval I is the body of a vertical *barrier* of the *emerging kind*, of type $j \in \{1, 2, 3\}$ if the following inequality holds:

$$\sup_{I', k} \mathbf{P}[\mathscr{L}_j(x, Y, I', 1) \mid X(I') = x(I'), Y(0) = k] > w^2. \tag{4.7}$$

To make it more explicit, for example interval I is an emerging barrier of type 1 for the process X if it has a closed subinterval I' of size $4l_1$ within 2Δ of its two ends, such that conditionally over the value of $X(I')$ and $Y(0)$, with probability $> w^2$, for all right-closed subintervals $\hat{I}$ of I', the rectangle $\hat{I} \times [b, b + 5\Delta]$ contains a trap. Shortly, the value $X(I')$ makes not too improbable (in terms of a randomly chosen Y) a sequence of closely placed traps reaching horizontally across $I' \times [b, b + 5\Delta]$.

Note that emerging barriers of type 1 are smallest, and those of type 2 are largest. More precisely, let

$$L_1 = 4l_1, \quad L_2 = 4l_2, \quad L_3 = g.$$

Then emerging barriers of type j have length in $L_j + [0, 4\Delta]$.

We will designate some of the emerging barriers as walls. We will say that I is a *pre-wall* of the emerging kind if also the following properties hold:

(a) Either I is an external hop of $\mathscr{M}$ or it is the union of a dominant light wall and one or two external hops of $\mathscr{M}$, of size $\geqslant \Delta$, surrounding it.

(b) Each end of I is adjacent to either an external hop of size $\geqslant \Delta$ or a wall of $\mathscr{M}$.

Now, for $j = 1, 2, 3$, list all emerging pre-walls of type j in a sequence $(B_{j1}, B_{j2}, \ldots)$. First process pre-walls $B_{11}, B_{12}, \ldots$ one-by-one. Designate B_{1n} a wall if and only if it is disjoint of all emerging pre-walls designated as walls earlier. Next process the sequence $(B_{31}, B_{32}, \ldots)$. Designate B_{3n} a wall if and only if it is disjoint of all emerging pre-walls designated as walls earlier. Finally process the sequence $(B_{21}, B_{22}, \ldots)$. Designate B_{2n} a wall if and only if it is disjoint of all emerging pre-walls designated as walls earlier.

dominant light wall   wall

hop   hop   hop

I

dominant light wall

hop   hop   hop

I
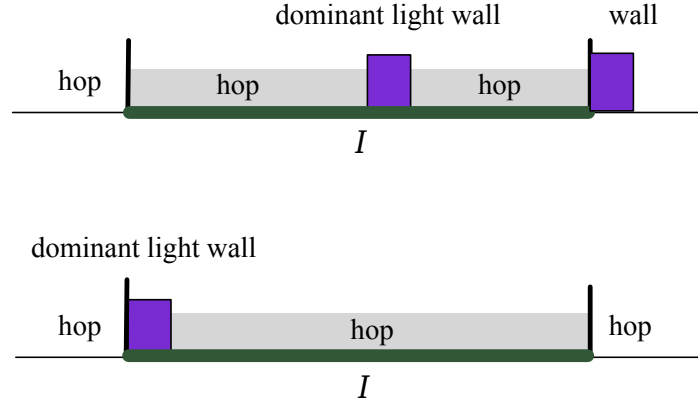
Figure 14: Some possibilities for a pre-wall

To emerging barriers and walls, we assign rank

$$\hat{R} > R^* \tag{4.8}$$

to be determined later.

Step 6 (Compound walls). We make use of a certain sequence of integers:

$$d_i = \begin{cases} i & \text{if } i = 0,1, \\ \lceil \lambda^i \rceil & \text{if } i \geqslant 2. \end{cases} \tag{4.9}$$

A *compound barrier* occurs in $\mathcal{M}^*$ for X wherever barriers W_1, W_2 occur (in this order) for X at a distance $d \in [d_i, d_{i+1})$, $d \leqslant f$, and W_1 is light. We will call this barrier a wall if W_1, W_2 are neighbor walls (that is, they are walls separated by a hop). We denote the new compound wall or barrier by

$$W_1 + W_2.$$

Its body is the smallest right-closed interval containing the bodies of W_j. For r_j the rank of W_j, we will say that the compound wall or barrier in question has *type*

$$\langle r_1, r_2, i \rangle.$$

Its rank is defined as

$$r = r_1 + r_2 - i. \tag{4.10}$$

Figure 15: Three (overlapping) types of compound barrier obtained: light-any, any-light, light-any-light. Here, "any" can also be a recently defined emerging barrier.

Thus, a shorter distance gives higher rank. This definition gives

$$r_1 + r_2 - \log_\lambda f \leqslant r \leqslant r_1 + r_2.$$

Inequality (4.3) will make sure that the rank of the compund walls is lower-bounded by R^*.

Now we repeat the whole compounding step, introducing compound walls and barriers in which now W_2 is required to be light. The barrier W_1 can be any barrier introduced until now, also a compound barrier introduced in the first compounding step.⌟

The walls that will occur as a result of the compounding operation are of the type L-$*$, $*$-L, or L-$*$-L, where L is a light wall of $\mathcal{M}$ and $*$ is any wall of $\mathcal{M}$ or an emerging wall of $\mathcal{M}^*$. Thus, the maximum size of a compound wall is

$$\Delta + f + (4g' + 4\Delta) + f + \Delta < \Delta^*,$$

where we used (4.1) and (4.2).

Step 7 (Finish). The graph $\mathcal{G}$ does not change in the scale-up: $\mathcal{G}^* = \mathcal{G}$. Remove all traps of $\mathcal{M}$.

Remove all light walls and barriers. If the removed light wall was dominant, remove also all other walls of $\mathcal{M}$ (even if not light) contained in it.⌟

The reader may miss the absence of an explanation for why we introduced exactly these higher-order traps and walls, and no others. The only explanation available is, however, that these are the only objects whose absence is used in the proof of the reachability property in $\mathcal{M}^*$ (Lemma 8.1).

4.2 Combinatorial properties

Let us prove some properties of $\mathcal{M}^*$ that can already be established. Note first that Condition 2.2 follows from Condition 3.8, Condition 2.3 follows from Conditions 3.6.2c-2d, and 2.5 for $\mathcal{M}^*$ follows from the definition of cleanness in $\mathcal{M}^*$ given in the present section.

Lemma 4.1. *The new mazery $\mathcal{M}^*$ satisfies Condition 3.6.1.*

Proof. We will see that all the properties in the condition follow essentially from the form of our definitions.

Condition 3.6.1a says that for any rectangle $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$. To check this, consider all possible traps of $\mathscr{M}^*$. We have the following kinds:

– Uncorrelated and correlated compound trap. The form of the definition shows that this event only depends on $X(I), Y(J)$.

– Trap of the missing-hole kind. Since the definition of good holes uses H-cleanness, this depends in $\mathscr{M}$ only on a Δ-neighborhood of a point. Therefore event $\mathscr{L}_3(x,y,I,b)$ depends only on the $x(I)$ part of x. Since W is required to be a potential wall, the event only depends on the $y(J)$ part of y. The conditional probability inequality also depends only on $x(I)$.

Condition 3.6.1b says that, say, for a vertical wall value E the event $\{E \in \mathscr{B}_0\}$ (that is the event that it is a vertical barrier) is a function of $X(\mathrm{Body}(E))$. There are two kinds of vertical barriers in $\mathscr{M}^*$: emerging and compound barriers. The definition of both of these refers only to $X(\mathrm{Body}(E))$.

Condition 3.6.1c says first that for every interval $I = (a,b]$, the strong cleanness of a or b in I are functions of $Z_d(I)$. The property that a or b is strongly clean in interval I in $\mathscr{M}^*$ is defined in terms of strong cleanness in $\mathscr{M}$ and the absence of barriers contained in I. Therefore strong cleanness of a or b in I for $\mathscr{M}^*$ is a function of $Z_d(I)$.

Since (strong) cleanness in I for $\mathscr{M}$ is a decreasing function of I, and the property stating the absence of walls (barriers) is a decreasing function of I, (strong) cleanness for $\mathscr{M}^*$ is also a decreasing function of I. The inequality $f/3 + \Delta < \Delta^*$, implies that these functions reach their minimum for $|I| = \Delta^*$.

Condition 3.6.1d says first that for any rectangle $Q = I \times J$, the event that its lower left corner is trap-clean in Q, is a function of the pair $X(I), Y(J)$. If u is this point then, our definition of its trap-cleanness for $\mathscr{M}^*$ in rectangle Q required the following:

– It is trap-clean in Q for $\mathscr{M}$;

– The starting point of any trap in Q is at a distance $\geqslant g$ from u.

All these requirements refer only to the projections of Q and depend therefore only on the pair $X(I), Y(J)$.

It can also be seen that, among rectangles with a fixed lower left corner, the event that this corner is trap-clean for $\mathscr{M}^*$ in Q is a decreasing function of Q (in the set of rectangles partially ordered by containment). And, since $g + \Delta$, $f/3 + \Delta < \Delta^*$, if point (x,y) is upper right trap-clean in a square of size Δ^*, then it is upper right trap-clean.

$\square$

Lemma 4.2. *The mazery $\mathscr{M}^*$ satisfies conditions 3.6.2a and 3.6.2b.*

Figure 16: An initial pure sequence. The light rectangles show the intervals K_j separated by maximal external intervals. The dark rectangles form the sequences of neighbor walls W_{jk} spanning the intervals K_j.

Proof. We will prove the statement only for vertical walls; it is proved for horizontal walls the same way. In what follows, "wall", "hop", and so on, mean vertical wall, horizontal hop, and so on.

Let $(U_1, U_2, \dots)$ be a (finite or infinite) sequence of disjoint walls of $\mathcal{M}$ and $\mathcal{M}^*$, and let $I_0, I_1, \dots$ be the (possibly empty) intervals separating them (interval I_0 is the interval preceding U_1). This sequence will be called *pure* if

(a) The intervals I_j are hops of $\mathcal{M}$.

(b) I_0 is an external interval of $\mathcal{M}$ starting at -1, while I_j for $j > 0$ is external if its size is $\geqslant 3\Delta$.

1. Let us build an initial pure sequence of $\mathcal{M}$ (where purity was defined at the beginning of the proof), which has also an additional property: *every dominant light wall of $\mathcal{M}$ belongs to it.*

   First we will use only elements of $\mathcal{M}$; however, later, walls of $\mathcal{M}^*$ will be added. Let $(E_1, E_2, \dots)$ be the (finite or infinite) sequence of maximal external intervals of size $\geqslant \Delta$, and let us add to it the maximal external interval starting at -1. Let $K_1, K_2, \dots$ be the intervals between them (or possibly after them, if there are only finitely many E_i). Clearly each dominant wall has one of the K_j as body. If there is both a dominant light wall and a dominant heavy wall with the same body, then we will take the light one as part of the sequence.

By Condition 3.6.2b of $\mathcal{M}$, each K_j that is not a wall can be spanned by a sequence of neighbors W_{jk}. Each pair of these neighbors will be closer than 3Δ to each other. Indeed, each point of the hop between them belongs either to a wall intersecting one of the neighbors, or to a maximal external interval of size $< \Delta$, so the distance between the neighbors is $< 2\Delta + \Delta = 3\Delta$. The union of these sequences is a single infinite pure sequence of neighbor walls

$$\mathbf{U} = (U_1, U_2, \dots), \qquad \mathrm{Body}(U_j) = (a_j, b_j]. \qquad (4.11)$$

Every wall of $\mathcal{M}$ intersects an element of $\mathbf{U}$.

A light wall in this sequence is called *isolated* if its distance from other elements of the sequence is greater than f. By our construction, all isolated light walls of the sequence $\mathbf{U}$ are dominant.

Figure 17: Adding an emerging wall to the pure sequence

Figure 18: Forming compound walls

Let us change the sequence $\mathbf{U}$ using the sequence $(W_1, W_2, \dots)$ of all emerging walls (disjoint by definition) as follows. For $n = 1, 2, \dots$, add W_n to $\mathbf{U}$. If W_n intersects an element U_i then delete U_i.

2.   (a) The result is a pure sequence $\mathbf{U}$ containing all the emerging walls.

      (b) When adding W_n, if W_n intersects an element U_i then U_i is a dominant wall of $\mathcal{M}$ contained in W_n, and W_n intersects no other element U_j.

Proof. The proof is by induction. Suppose that we have already processed $W_1, \dots, W_{n-1}$, and we are about to process $W = W_n$. The sequence will be called $\mathbf{U}$ before processing W and $\mathbf{U}'$ after it.

Let us show (b) first. By the requirement (a) on emerging walls, either W is an external hop of $\mathcal{M}$ or it is the union of a dominant light wall and one or two external hops of $\mathcal{M}$, of size $\geqslant \Delta$, surrounding it. If W is an external hop then it intersects no elements of $\mathbf{U}$. Otherwise, the dominant light wall inside it can only be one of the U_i.

Let us show now (a), namely that if $\mathbf{U}$ is pure then so is $\mathbf{U}'$. Property (b) of the definition of purity follows immediately, since the intervals between elements of $\mathbf{U}'$ are subintervals of the ones between elements of $\mathbf{U}$. For the same reason, these intervals do not contain walls of $\mathcal{M}$. It remains to show that if $I'_{j-1} = (b'_{j-1}, a'_j]$ and $I'_j = (b'_j, a'_{j+1}]$ are the intervals around W in $\mathbf{U}'$ then a'_j is clean in I'_{j-1} and b'_j is clean in I'_j. Let us show that, for example, a'_j is clean in I'_{j-1}.

By the requirement (b) on emerging walls, a'_j is adjacent to either an external hop of size $\geqslant \Delta$ or a wall W of $\mathcal{M}$. If the former case, it is left clean and therefore clean in I'_{j-1}. In the latter case, W by the definition of emerging walls, there is an external interval of size $\geqslant \Delta$ on the right of W. This implies the existence of a wall W' in the original sequence $\mathbf{U}$ whose right end is a'_j, and therefore I'_{j-1} is empty.

3. Let us break up the pure sequence $\mathbf{U}$ containing all the emerging walls into subsequences separated by its intervals I_j of size $> f$. Consider one of these (possibly infinite) sequences, call it $W_1, \dots, W_n$, which is not just a single isolated light wall.
   We will create a sequence of consecutive neighbor walls W'_i of $\mathcal{M}^*$ spanning the same interval as $W_1, \dots, W_n$. In the process, all non-isolated light walls of the sequence will be

incorporated into a compound wall.

Assume that W_i for $i < j$ have been processed already, and a sequence of neighbors W_i' for $i < j'$ has been created in such a way that

$$\bigcup_{i<j} W_i \subseteq \bigcup_{i<j'} W_i',$$

and W_j is not a light wall which is the last in the series. (This condition is satisfied when $j = 1$ since we assumed that our sequence is not an isolated light wall.) We show how to create $W_{j'}'$.

If W_j is the last element of the series then it is heavy, and we set $W_{j'}' = W_j$. Suppose now that W_j is not last.

Suppose that it is heavy. If W_{j+1} is also heavy, or light but not last then $W_{j'}' = W_j$. Else $W_{j'}' = W_j + W_{j+1}$, and W_j' replaces W_j, W_{j+1} in the sequence. In each later operation also, the introduced new wall will replace its components in the sequence.

Suppose now that W_j is light: then it is not last. If W_{j+1} is last or W_{j+2} is heavy then $W_{j'}' = W_j + W_{j+1}$.

Suppose that W_{j+2} is light. If it is last then $W_{j'}' = (W_j + W_{j+1}) + W_{j+2}$; otherwise, $W_{j'}' = W_j + W_{j+1}$.

Remove all isolated light walls from $\mathbf{U}$ and combine all the subsequences created in part 3 above into a single infinite sequence $\mathbf{U}$ again. Consider an interval I between its elements. Then I is inner clean for $\mathcal{M}$, and the only walls of $\mathcal{M}$ in I are covered by some isolated dominant light walls at distance at least $f/3$ from the endpoints. Thus, I is inner clean in $\mathcal{M}^*$. It does not contain any compound walls either (other than possibly those inside some dominant light wall that was removed), and by definition it does not contain emerging walls. Therefore it is a hop of $\mathcal{M}^*$.

4. Condition 3.6.2a holds for $\mathcal{M}^*$.

    Proof. Let J be a maximal external interval J of $\mathcal{M}^*$, of size $\geqslant \Delta^*$ or starting at -1. Since J has size $\geqslant \Delta^* > f$ it is an interval separating two elements of $\mathbf{U}$ and as such is a hop of $\mathcal{M}^*$. Otherwise it is the interval I_0. We have seen that this is also a hop of $\mathcal{M}^*$.

5. Condition 3.6.2b holds for $\mathcal{M}^*$.

    Proof. By our construction, a maximal external interval of size $\geqslant \Delta^* > f$ is an interval separating two elements of $\mathbf{U}$. The segment between two such intervals (or one such and I_0) is spanned by elements of $\mathbf{U}$, separated by hops of $\mathcal{M}^*$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 4.3. *Suppose that interval I contains no walls of $\mathcal{M}^*$, and no wall of $\mathcal{M}$ closer to its ends than $f/3$ (these conditions are satisfied if it is a hop of $\mathcal{M}^*$). Then it either contains*

no walls of $\mathscr{M}$ or all walls of $\mathscr{M}$ in it are covered by a sequence $W_1,\ldots,W_n$ of dominant light neighbor walls of $\mathscr{M}$ separated from each other by external hops of $\mathscr{M}$ of size $> f$.

If I is a hop of $\mathscr{M}^*$ then either it is also a hop of $\mathscr{M}$ or the above end intervals are hops of $\mathscr{M}$.

Proof. If I contains no walls of $\mathscr{M}$ then there is nothing to prove. Otherwise, let U be the union of all walls of $\mathscr{M}$ in I. By assumption, it is not closer than $f/3$ to the ends of I. Let intervals J, K separate U from both ends, then they contain maximal external intervals J' and K' of size $\geqslant \Delta$ of $\mathscr{M}$, adjacent to U. Let $(E_1, E_2, \ldots, E_n)$ be the sequence of maximal external intervals of size $\geqslant \Delta$ in U, and let $E_0 = J'$, $E_{n+1} = K'$. Let F_i be the interval between E_{i-1} and E_i.

We claim that all intervals F_i are dominant light walls separated by a distance greater than f. Note that if F_i intersects a dominant light wall L then $F_i = L$. Indeed, L is surrounded by maximal external intervals of size $\geqslant \Delta$ which then must coincide with E_{i-1} and E_i. There is a sequence of neighbor walls spanning F_i. If any one is heavy then, since I contains no barriers of $\mathscr{M}^*$, it must be contained in a dominant light wall and then by the observation above, K_i is a dominant light wall. Assume therefore that all elements of the sequence are light. They cannot be farther than f from each other, since then the interval between them would contain an external interval of size $\geqslant \Delta$. If there is more than one then any two neighbors form a compound wall, which the absence of barriers of $\mathscr{M}^*$ forces to be part of a dominant light wall.

The hops E_j have size $> f$ since otherwise two neighbors would combine into a compound wall (which could not be covered by a dominant light wall).

Now suppose I is a hop of $\mathscr{M}^*$. If it contains no walls of $\mathscr{M}$ then it is clearly a hop of $\mathscr{M}$. Otherwise, look at an end interval, say J. It contains a maximal external subinterval J' on its right end, hence its right end is clean in J. Since I is a hop of $\mathscr{M}^*$, the left end of J is also clean in J. So, J is a hop of $\mathscr{M}$, and the same holds for K. $\qquad\square$

The following lemma shows that an emerging barrier in a "nice neighborhood" implies an emerging wall there.

Lemma 4.4. *Let us be given intervals $I' \subset I$, and also $x(I)$, with the following properties for some $j \in \{1, 2, 3\}$.*

(a) *All walls of $\mathscr{M}$ in I are covered by a sequence $W_1, \ldots, W_n$ of dominant light neighbor walls of $\mathscr{M}$ such that the W_i are at a distance $> f$ from each other and at a distance $\geqslant f/3$ from the ends of I.*

(b) *I' is an emerging barrier of type j.*

(c) *I' is at a distance $\geqslant L_j + 7\Delta$ from the ends of I.*

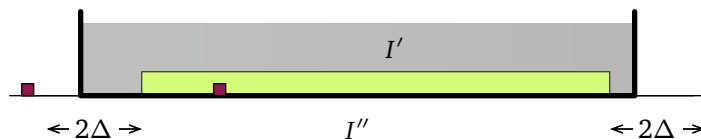*Then I contains an emerging wall.*

Figure 19: To Lemma 4.4: finding an emerging wall

Proof. By the definition of emerging barriers, I' contains an emerging barrier of size exactly L_j. From now on, we assume I' has this size.

Let $I = (a,b]$, $I' = (u',v']$. We will define an emerging wall $I'' = (u'',v'']$. The assumptions imply that the intervals between the walls W_i are external hops. However, the interval $(a,c]$ between the left end of I and W_1 may not be one. Let $(\hat{a},c]$ be a maximal external subinterval of $(a,c]$ ending at c. Then $\hat{a} - a \leqslant \Delta$. Let us define $\hat{b}$ similarly on the right end of I, and let $\hat{I} = (\hat{a},\hat{b}]$. We will find an emerging wall in $\hat{I}$, so let us simply redefine I to be $\hat{I}$. We now have the property that the interval between the left end of I and W_1 is an external hop of size $\geqslant f/3 - \Delta$, and similarly at the right end. Also, I' is at a distance $\geqslant L_j + 6\Delta$ from the ends of I.

Assume first that I is a hop of $\mathcal{M}$ (by the assumption, an external one). Let us define the interval I'' as follows. Assumption (c) implies $u' \geqslant a + 2\Delta$. Then, since no wall is contained in $(u' - 2\Delta, u' + \Delta]$, by Condition 3.6.2c, there is a point $u'' \in (u' - \Delta, u']$ clean in $\mathcal{M}$. (Since $|I'| > \Delta$, there is no problem with walls on the right of v' when finding clean points on the left of u'.) Similarly, $b - v' \geqslant 2\Delta$, and there is a point $v'' \in (v', v' + \Delta]$ clean in $\mathcal{M}$.

Assume now that I is not a hop of $\mathcal{M}$: then I is spanned by a nonempty extended sequence $W_1, \ldots, W_n$ of neighbor walls of $\mathcal{M}$ such that the W_i are at a distance $> f$ from each other and at a distance $> f/3 - \Delta$ from the ends of I.

Assume that I' falls into one of the hops, let us call this hop $(a',b']$. If $u' \geqslant a' + 2\Delta$ then, just as in the paragraph above in interval $I = (a,b]$, there is a point $u'' \in (u' - \Delta, u']$ clean in $\mathcal{M}$. Otherwise, set $u'' = a'$. Similarly, if $b' - v' \geqslant 2\Delta$, then there is a point $v'' \in (v', v' + \Delta]$ clean in $\mathcal{M}$, else we set $v'' = b'$.

Assume now that $I' = (u',v']$ intersects one of these walls, say $W_i = (c,d]$. Now, if $c \leqslant u' < d$ then take $u'' = c$. If $u' < c$ then there are no walls in the interval $(u' - 3\Delta, u']$, since it is in the hop on the left of W_i. Find a point u'' clean in $\mathcal{M}$ in the middle $(u' - 2\Delta, u' - \Delta]$ of this interval. The point v'' is defined similarly.

By this definition, interval I'' satisfies both requirements (a) and (b) of emerging pre-walls, and is at a distance $\geqslant 4\Delta + L_j$ from the ends of I.

If I contains no emerging walls then, in particular, it contains no walls of type i with $L_i \leqslant L_j$. Since I'' is at a distance $\geqslant 4\Delta + L_j$ (the bound on the size of emerging walls of type j) from the ends of I, it follows therefore that no wall of such type i intersects it. But

then the process of designating walls in Step 5 of the scale-up construction would designate I'' a wall, contrary to the assumption that I contains no emerging walls. □

Lemma 4.5. *Let the rectangle Q with X projection I contain no traps or vertical walls of $\mathscr{M}^*$, and no vertical wall of $\mathscr{M}$ closer than $f/3$ to its sides. Let $I' = [a, a+g]$, $J = [b, b+3\Delta]$ with $I' \times J \subseteq Q$ be such that I' is at a distance $\geqslant g + 7\Delta$ from the ends of I. Suppose that a light horizontal wall W starts at position $b + \Delta$. Then $[a + \Delta, a + g - \Delta]$ contains a vertical hole passing through W that is good in the sense of Definition 3.10. The same holds if we interchange horizontal and vertical.*

Proof. Suppose that this is not the case. Then event $\mathscr{L}_3(X, Y, I', b)$ holds, as defined in the introduction of missing-hole traps in Step 4 of the scale-up construction. Now, if inequality (4.6) holds then $I' \times J$ is a trap of the missing-hole kind; but this was excluded, since Q contains no traps of $\mathscr{M}^*$. On the other hand, if (4.6) does not hold then (due also to Lemma 4.3) Lemma 4.4 is applicable to the interval I' and the interval I that is the X projection of Q, and we can conclude that I contains a vertical emerging wall. But this was also excluded. □

Lemma 4.6. *Let rectangle Q with X projection I contain no traps or vertical walls of $\mathscr{M}^*$, and no vertical walls of $\mathscr{M}$ closer than $f/3$ to its sides. For $j \in \{1,2\}$, let l_j be as introduced in the definition of correlated traps and emerging walls in Steps 3 and 5 of the scale-up construction. Let $I' = [a, a + L_j]$, $J = [b, b + 5\Delta]$ with $I' \times J \subseteq Q$ be such that I' is at a distance $\geqslant L_j + 7\Delta$ from the ends of I. Then there is an interval $I'' \subseteq I'$ of size l_j, such that the rectangle $I'' \times J$ contains no trap of $\mathscr{M}$. The same holds if we interchange horizontal and vertical.*

Proof. The proof of this lemma is completely analogous to the proof of Lemma 4.5. □

Lemma 4.7. *The new mazery $\mathscr{M}^*$ defined by the above construction satisfies Conditions 3.6.2c and 3.6.2d.*

Proof. 1. Let us prove Condition 3.6.2c.

Consider an interval I of size $3\Delta^*$ containing no walls of $\mathscr{M}^*$. Condition 3.6.2b says that the real line is spanned by an extended sequence $(W_1, W_2, \dots)$ of neighbor walls of $\mathscr{M}$ separated from each other by hops of $\mathscr{M}$. As shown in the construction of part 1 of the proof of Lemma 4.2, we can also assume that every dominant light wall is an element of this sequence. Any of these walls W is contained in I then it is light. Indeed, I contains no wall of $\mathscr{M}^*$, so W can only be heavy if it is contained in a dominant light wall W', but then it is equal to W', as we assumed.

Since I contains no wall of $\mathscr{M}^*$, if two of these walls fall into I then they are separated by a hop of size $\geqslant f$.

Let I' be the middle third of I. Then $|I'| \geqslant 2f + \Delta$, and removing the W_i from I' leaves a subinterval $(a,b] \subseteq I'$ of size at least f. (If at least two W_i intersect I' take the interval between consecutive ones, otherwise I' is divided into at most two pieces of total length at least $2f$.) Now $K = (a + \Delta + f/3, b - \Delta - f/3]$ is an interval of length at least $f/3 - 2\Delta > 3\Delta$ which has distance at least $f/3$ from any wall of $\mathscr{M}$. There will be a clean point in the middle of K which will then be clean in $\mathscr{M}^*$.

2. Let us prove Condition 3.6.2d now for $\mathscr{M}^*$.

We will confine ourselves to the statement in which the point a is assumed clean and we find a b such that the point (a,b) is clean. The half clean cases are proved similarly. Let I, J be right-closed intervals of size $3\Delta^*$, suppose that the rectangle $I \times J$ contains no traps or horizontal walls of $\mathscr{M}^*$, and a is a point in the middle third of I that is clean in $\mathscr{M}^*$ for X. We need to prove that there is an integer b in the middle third of J such that the point (a,b) is clean in $\mathscr{M}^*$.

Just as in Part 1 above, we find K with $f/3 - 2\Delta \leqslant |K| \leqslant f$ in the middle of J which is at distance at least $f/3$ from any horizontal wall of $\mathscr{M}$. Let $I' = (a - g - \Delta, a + g + \Delta]$, then $I' \subseteq I$. We will find an interval $K'' \subseteq K$ with $|K''| \geqslant g'$ such that $I' \times K''$ contains no trap of $\mathscr{M}$. If there are no traps in $I' \times K$ let $K'' = K$. Assume now that $I' \times K$ contains a trap $T = \text{Rect}(u,v)$ of $\mathscr{M}$, where $u = (u_0, u_1)$, $v = (v_0, v_1)$. Since we assumed there are no traps of $\mathscr{M}^*$ and thus no uncorrelated traps, any trap must meet either $[u_0, v_0] \times K$ or $I' \times [u_1, v_1]$ or be at a distance at least f from T (and hence outside $I' \times K$). Let K' be a subinterval of $K \setminus [u_1, v_1]$ of size $4g'$ (which exists since $|K| \geqslant 2 \cdot (4g') + \Delta$). By Lemma 4.6, there must exist a subinterval K'' of K' of length $g' \geqslant 2g + 3\Delta$ such that $[u_0 - 2\Delta, u_0 + 3\Delta] \times K''$ contains no trap. Then also $I' \times K''$ contains no trap.

Now restrict K'' to size 3Δ and then find a clean point (a,b) in the middle third of K'' applying Condition 3.6.2d for $\mathscr{M}$. Then (a,b) has distance at least $g + \Delta$ from the boundary of $I' \times K''$ and so has distance at least g from any trap. Since b is at distance at least $f/3$ from any wall it is clean in $\mathscr{M}^*$. Hence (a,b) is clean in $\mathscr{M}^*$.

$\square$

5 Probability bounds

In this section, we derive all those bounds on probabilities in $\mathscr{M}^k$ that are possible to give without indicating the dependence on k.

5.1 General bounds

Recall the definitions needed for the hole lower bound condition, Condition 3.6.3d, in particular the definition of the event E. Since $(c-1)$ will be used often, we denote it by $\hat{c}$. Let $u \leqslant v < w$, and a be given with $v - u \leqslant 12\Delta$, and define $b = a + \lceil \frac{v-u}{2} \rceil$, $c = a + (v - u) + 1$.

We need to extend the lower bound condition in several ways. Since we will hold the sequence y of values of the sequence Y of random variables fixed in this subsection, we take the liberty and omit the condition $Y = y$ from the probabilities: it is always assumed to be there. For the following lemma, remember Condition 3.6.3c.

Lemma 5.1. *Let F_t be the event that the point (t,w) is upper right rightward H-clean. Let $\hat{E}$ be the event that E is realized with a hole $(d,t]$, and F_t holds (that is the hole is good as seen from $(a,u]$, in the sense of Definition 3.10). We have*

$$\mathbf{P}(\hat{E}) \geqslant (1-q)\mathbf{P}(E). \tag{5.1}$$

Proof. For $b \leqslant t \leqslant c + \Delta$, let E_t be the event that E is realized by a hole ending at t but is not realized by any hole ending at any $t' < t$. Then $E = \bigcup_t E_t$, $\hat{E} \supseteq \bigcup_t (E_t \cap F_t)$. Due to the Markov chain property of X and the form of E_t, the fact that E_t depends only on $X(0), \ldots, X(t)$ and using inequalities (3.7) and (3.8) of Condition 3.6.3c, we have

$$\mathbf{P}(E_t \cap F_t) = \mathbf{P}(E_t)\mathbf{P}(F_t \mid E_t) \geqslant \mathbf{P}(E_t)(1-q).$$

The events E_t are mutually disjoint. Hence

$$\mathbf{P}(\hat{E}) \geqslant \sum_t \mathbf{P}(E_t \cap F_t) \geqslant (1-q)\sum_t \mathbf{P}(E_t) = (1-q)\mathbf{P}(E).$$

$\square$

Recall Remark 3.7.2, referring to the most important special case of the hole lower bound: for any horizontal wall B of rank r, at any point b, the probability that there is a vertical hole passing through B at point b is at least $h(r)$. We strengthen this observation in a way similar to Lemma 5.1.

Lemma 5.2. *Let $v < w$, and let us fix the value y of the sequence of random variables Y in such a way that there is a horizontal wall B with body $(v,w]$. Let point b be given. Let E be the event that a good hole $(b,b']$ passes through B (this event still depends on the sequence $X = (X(1), X(2), \ldots)$ of random variables). Let $s \in \{1, \ldots, m\}$ then*

$$\mathbf{P}[E \mid X(b-\Delta) = s] \geqslant (1-q)^2 h(r).$$

Proof. Condition 3.6.3c implies that the probability that point b is lower left H-clean is lower-bounded by $(1-q)$. Conditioning on times during and before this event, Lemma 5.1 lower-bounds the probability that $(b,b']$ is an upper right rightward H-clean hole. The lower bound is then the product of these two lower bounds. $\square$

Now, we prove a version of the hole lower bound condition that will help proving the same bound for $\mathscr{M}^*$. This is probably the only part of the paper in which the probability estimates are somewhat tricky.

Definition 5.1. Recall the definition of event E in Condition 3.6.3d, and that it refers to a horizontal wall with body $(v, w]$ seen from a point (a, u). Take the situation described above, possibly without the bound on $(v - u)$. Let

$$E^* = E^*(u, v, w; a) \tag{5.2}$$

be the event (a function of the sequence X) that there is a $d \in [b, c - 1]$ with the following properties for $Q = \text{Rect}^{\rightarrow}((a, u), (d, v))$:

(i*) A vertical hole (of $\mathcal{M}$) fitting B starts at d.

(ii*) Q contains no traps or vertical barriers of $\mathcal{M}$ or $\mathcal{M}^*$ and is inner H-clean for $\mathcal{M}^*$.

The difference between $E^*(\cdot)$ and $E(\cdot)$ is only that E^* requires the H-cleanness for $\mathcal{M}^*$ and also absence of barriers and traps for $\mathcal{M}^*$ whenever possible. ⌟

Definition 5.2 (Wall and trap probability upper bounds)**.** Let

$$\overline{p} \tag{5.3}$$

be an upper bound of the probabilities over all possible points a of the line, and over all possible values of $X(a)$, that a barrier of $\mathcal{M}$ starts at a. Let it also bound similarly the probability that a barrier of $\mathcal{M}^*$ starts at a. Let

$$\overline{w} \tag{5.4}$$

be an upper bound of the conditional probabilities over X (with Y and $X(a-1)$ fixed in any possible way) over all possible points (a, b) of the plane, that a trap of $\mathcal{M}$ starts at (a, b). Let it also bound similarly the probability that a trap of $\mathcal{M}^*$ starts there. ⌟

Lemma 5.3. *Suppose that the requirement $v - u \leqslant 12\Delta$ in the definition of the event E^* is replaced with $v - u \leqslant 12\Delta^*$, while the rest of the requirements are the same. Let us fix $X(a) = s$ arbitrarily. Then the inequality*

$$\mathbf{P}[E^* \mid X(a) = s,\ Y = y] \geqslant 0.5 \wedge ((c - b)^{\chi} h(r)) - U \tag{5.5}$$

holds, with $U = 26\overline{p}\Delta^* + 338\overline{w}(\Delta^*)^2$. If $v - u > 12\Delta$ then we also have the somewhat stronger inequality

$$\mathbf{P}[E^* \mid X(a) = s,\ Y = y] \geqslant 0.5 \wedge (1.1(c - b)^{\chi} h(r)) - U. \tag{5.6}$$

Proof. For ease of reading, we will omit the conditions $X(a) = s$, $Y = y$ from the probabilities.

For the case $v - u \leqslant 12\Delta$, even the original stronger inequality holds, namely $\mathbf{P}(E^*) \geqslant (c - b)^{\chi} h(r)$. Condition 3.6.3d implies this already for $\mathbf{P}(E)$, so it is sufficient to show $E \subseteq E^*$ in this case.
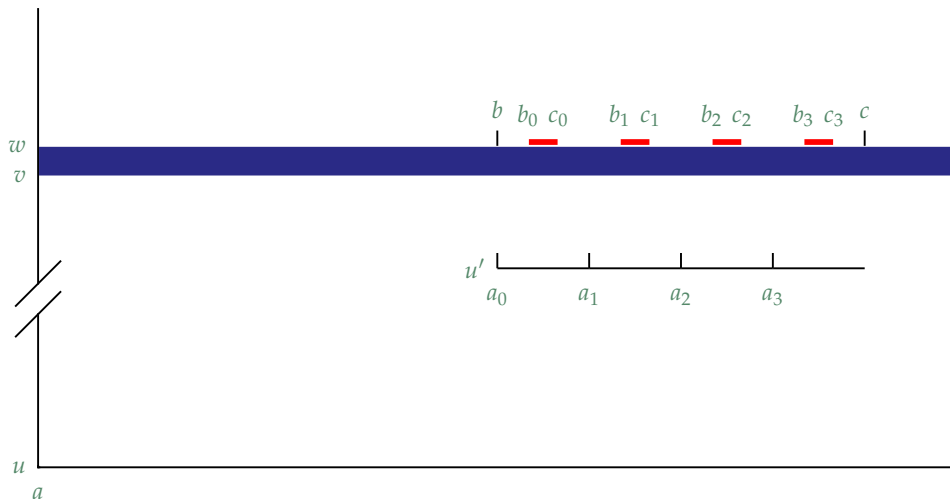
Figure 20: To the proof of Lemma 5.3.

As remarked after its definition, the event E^* differs from E only in requiring that rectangle Q contains no traps or vertical barriers of $\mathcal{M}^*$, not only for $\mathcal{M}$, and that points (a,u) and (d,v) are H-clean in Q for $\mathcal{M}^*$ also, not only for $\mathcal{M}$.

Consider a trap of $\mathcal{M}^*$ in Q: this cannot be an uncorrelated or correlated trap, since its components traps, which are traps of $\mathcal{M}$, are already excluded. It cannot be a trap of the missing-hole kind either, since that trap is too big for Q when $v - u \leqslant 12\Delta$. The same argument applies to vertical barriers of $\mathcal{M}^*$. The components of the compound barriers that belong to $\mathcal{M}$ are excluded, and the emerging barriers are too big.

These considerations take care also of the issue of H-cleanness for $\mathcal{M}^*$, since the latter also boils down to the absence of traps and barriers.

Lets move to the case of $v - u > 12\Delta$, which implies $\hat{c} - b \geqslant 6\Delta$. We will use the following inequality, which can be checked by direct calculation. Let $\alpha = 1 - 1/e = 0.632\ldots$, then for $x > 0$ we have

$$1 - e^{-x} \geqslant \alpha \wedge \alpha x. \tag{5.7}$$

Let $n = \lfloor (c - b)/(3\Delta) \rfloor$, then $n \geqslant 2$ and hence $(c - b)/(3\Delta) \leqslant n + 1 \leqslant 1.5n$, implying

$$n\Delta \geqslant (c - b)/4.5. \tag{5.8}$$

Let

$$u' = v - 2\Delta, \quad a_i = b + 3i\Delta,$$
$$E'_i = E(u', v, w; a_i) \quad \text{for } i = 0, \ldots, n-1,$$
$$E' = \bigcup_i E'_i.$$

Let C be the event that point (a, u) is upper right rightward H-clean in $\mathcal{M}$. Then by Conditions 3.6.3c

$$\mathbf{P}(\neg C) \leqslant 2(q/2) \leqslant 0.1. \tag{5.9}$$

Let D be the event that the rectangle $(a, c] \times [u, v]$ contains no trap or vertical barrier of $\mathcal{M}$ or $\mathcal{M}^*$. (Then $C \cap D$ implies that (a, u) is also upper right rightward H-clean in the rectangle $(a, c] \times [u, v]$ in $\mathcal{M}^*$.) By definition,

$$\mathbf{P}(\neg D) \leqslant 2\overline{p}(c - a) + 2\overline{w}(c - a)(v - u + 1)$$
$$\leqslant 2 \cdot 13\overline{p}\Delta^* + 2 \cdot 13 \cdot 13\overline{w}(\Delta^*)^2 = 26\overline{p}\Delta^* + 338\overline{w}(\Delta^*)^2.$$

1. Let us show $C \cap D \cap E' \subseteq E^*(u, v, w; a)$.
   Indeed, suppose that $C \cap D \cap E'_i$ holds with some hole starting at d. Then there is a rectangle $Q'_i = \text{Rect}^{\rightarrow}((a_i, u'), (d, v))$ containing no traps or vertical barriers of $\mathcal{M}$, such that (d, v) is H-clean in Q'_i. It follows from D that the rectangle

$$Q^*_i = \text{Rect}^{\rightarrow}((a, u), (d, v)) \supseteq Q'_i$$

   contains no traps or vertical barriers of $\mathcal{M}$ or $\mathcal{M}^*$. Since event C occurs, the point (a, u) is H-clean for $\mathcal{M}$ in Q^*_i. The event E'_i implies that (d, v) is H-clean in Q^*_i, and a hole passing through the potential wall starts at d in X. The event D implies that there is no trap or vertical barrier of $\mathcal{M}$ or $\mathcal{M}^*$ in Q^*_i. Hence Q^*_i is inner H-clean, and so E^* holds.
   We have $\mathbf{P}(E^*) \geqslant \mathbf{P}(C)\mathbf{P}(E' \mid C) - \mathbf{P}(\neg D)$.
2. It remains to estimate $\mathbf{P}(E' \mid C)$.
   Let us denote $s = \Delta^x h(r)$. Condition 3.6.3d is applicable to E'_i, and we have $\mathbf{P}(E'_i \mid C) \geqslant s$, hence for each $k \in \{1, \ldots, m\}$:

$$\mathbf{P}\left[\neg E'_i \mid C \cap \{X(a_i) = k\}\right] \leqslant 1 - s \leqslant e^{-s}.$$

Due to the Markov property of the sequence X, this implies $\mathbf{P}\left(\neg E'_i \mid C \cap \bigcap_{j<i} \neg E'_j\right) \leqslant e^{-s}$, and hence

$$\mathbf{P}(E' \mid C) = 1 - \mathbf{P}\left(\bigcap_i \neg E'_i \mid C\right) \geqslant 1 - e^{-ns} \geqslant \alpha \wedge (\alpha ns), \tag{5.10}$$

where in the last step we used (5.7). By (5.8):

$$\alpha n \Delta^{\chi} = \alpha n^{1-\chi}(\Delta n)^{\chi} \geqslant \alpha 2^{1-\chi}(\Delta n)^{\chi} \geqslant \alpha(2^{1-\chi}/4.5^{\chi})(c-b)^{\chi} \geqslant 1.223(c-b)^{\chi},$$

where we used the value of χ from (2.2). Substituting into (5.10):

$$\mathbf{P}(E' \mid C) \geqslant \alpha \wedge (1.223(c-b)^{\chi}h(r)),$$
$$\mathbf{P}(C)\mathbf{P}(E' \mid C) > 0.9 \cdot (\alpha \wedge (1.223(c-b)^{\chi}h(r))) > 0.5 \wedge (1.1(c-b)^{\chi}h(r)),$$

where we used (5.9).

$\square$

5.2 New traps

Recall the definition of uncorrelated compound traps in Step 2 of the scale-up construction in Section 4.

Lemma 5.4 (Uncorrelated Traps). *Given a string $x = (x(0), x(1), \dots)$, a point (a_1, b_1), let $\mathscr{F}$ be the event that an uncorrelated compound trap of $\mathscr{M}^*$ starts at (a_1, b_1). Let $s \in \{1, \dots, m\}$, then*

$$\mathbf{P}[\mathscr{F} \mid X = x, \, Y(b_1 - 1) = s] \leqslant 2f^2 w^2. \tag{5.11}$$

Proof. Let $\mathscr{G}(a, b)$ be the event that a trap of $\mathscr{M}$ starts at (a, b). Let $\mathscr{G}(a, b; a', b')$ be the event that a trap of $\mathscr{M}$ starts at (a, b), and is contained in $[a, a') \times [b, b')$. Since the new trap is the smallest rectangle containing two old traps, it must contain these in two of its opposite corners: let $\mathscr{E}$ be the event that one of these corners is (a_1, b_1).

Let $N = (a_1, b_1) + (0, f]^2$. Then

$$\mathscr{E} \subseteq \bigcup_{(a_2, b_2) \in N} \mathscr{G}(a_1, b_1; a_2, b_2) \cap \mathscr{G}(a_2, b_2).$$

The events $\mathscr{G}(a_1, b_1; a_2, b_2)$ and $\mathscr{G}(a_2, b_2)$ belong to rectangles whose projections are disjoint. Fixing $Y(b_1 - 1) = s$, by Condition 3.6.3a and the Markov property:

$$\mathbf{P}[\mathscr{G}(a_1, b_1; a_2, b_2) \mid X = x] \leqslant w,$$
$$\mathbf{P}[\mathscr{G}(a_2, b_2) \mid \mathscr{G}(a_1, b_1; a_2, b_2) \cap \{X = x\}] \leqslant w.$$

Hence by the union bound $\mathbf{P}[\mathscr{E} \mid X = x] \leqslant f^2 w^2$. If $\mathscr{F} \setminus \mathscr{E}$ holds then there is a random pair $(A, B) \in N$ such that $\mathscr{G}(a_1, B; A, \infty)$ and $\mathscr{G}(A, b_1; \infty, B)$ holds. A computation similar to the above one gives the upper bound $f^2 w^2$ for $\mathbf{P}[\mathscr{F} \setminus \mathscr{E} \mid X = x]$. $\square$

Recall the definition of correlated traps in Step 3 of the scale-up construction in Section 4.

Lemma 5.5 (Correlated Traps)**.** *Let a site* (a,b) *be given. For* $j = 1,2$*, let* $\mathscr{F}_j$ *be the event that a horizontal correlated trap of type* j *starts at* (a,b)*.*

(a) *Let us fix a string* $x = (x(0), x(1), \dots)$*, and also* $s \in \{1, \dots, m\}$ *arbitrarily. We have*

$$\mathbf{P}[\mathscr{F}_j \mid X = x,\ Y(b-1) = s] \leqslant w^2. \tag{5.12}$$

(b) *Let us fix a string* $y = (y(0), y(1), \dots)$*, and also* $s \in \{1, \dots, m\}$ *arbitrarily. We have*

$$\mathbf{P}[\mathscr{F}_j \mid Y = y,\ X(a) = s] \leqslant (5\Delta l_j w)^4. \tag{5.13}$$

Proof. Part (a) is an immediate consequence of requirement (4.5) of the definition of correlated traps. It remains to prove part (b). Note that this result implies the same bounds also if we fix $X(a-1)$ arbitrarily. If there is a correlated trap with X-projection starting at some a then there must be traps with X-projections in $(a + rl_j, a + (r+1)l_j]$ for $r = 0, 1, 2, 3$. Due to Condition 3.6.3a (the trap upper bound) and the Markov property, the probability of a trap in any one of these is at most $5\Delta l_j w$, even conditioned on the values of X before. Hence the probability of such a compound trap happening is at most $(5\Delta l_j w)^4$. $\qquad\square$

Recall the definition of traps of the missing-hole kind in Step 4 of the scale-up algorithm in Section 4.

Lemma 5.6 (Missing-hole traps)**.** *For* $a, b \in \mathbb{Z}_+$*, let* $\mathscr{F}$ *be the event that a horizontal trap of the missing-hole kind starts at* (a,b)*.*

(a) *Let us fix a string* $x = (x(0), x(1), \dots)$*, and also* $s \in \{1, \dots, m\}$ *arbitrarily. We have*

$$\mathbf{P}[\mathscr{F} \mid X = x,\ Y(b-1) = s] \leqslant w^2. \tag{5.14}$$

(b) *Let us fix a string* $y = (y(0), y(1), \dots)$*, and also* $s \in \{1, \dots, m\}$ *arbitrarily. Let* $n = \lfloor \frac{g}{3\Delta} \rfloor$*. We have*

$$\mathbf{P}[\mathscr{F} \mid Y = y,\ X(a) = s] \leqslant e^{-(1-q)^2 n h(R^*)}. \tag{5.15}$$

Proof. Part (a) is an immediate consequence of requirement (4.6) of the definition of missing-hole traps. It remains to prove part (b). Note that this result implies the same bounds also if we fix $X(a-1)$ arbitrarily. Let $J = [b, b + 3\Delta]$. According to the definition of missing-hole traps above, we can assume without loss of generality that, with $b_1 = b + \Delta$, there is a $b_2 > b_1$ such that $(b_1, b_2]$ is a potential light horizontal wall W. For $i = 0, \dots, n-1$, let $\mathscr{A}(d,i)$ be the event that no good hole $(a_1, a_2]$ with $a_1 = a + 3i\Delta + \Delta$ passes through W. All these events must hold if a horizontal trap of the missing-hole kind starts at (a,b). Using the Markov property and Lemma 5.2:

$$\mathbf{P}\big(\mathscr{A}(d,i) \mid \bigcap_{j<i} \mathscr{A}(d,j) \big) \leqslant 1 - (1-q)^2 h(R^*) \leqslant e^{-(1-q)^2 h(R^*)}.$$

Therefore $\mathbf{P}\big(\bigcap_i \mathscr{A}(d,i) \big) \leqslant e^{-n(1-q)^2 h(R^*)}$. $\qquad\square$

5.3 Emerging walls

Recall the definition of emerging walls in Step 5 of the scale-up algorithm in Section 4.

Lemma 5.7. *For any point u, let $\mathscr{F}(t)$ be the event that a barrier $(u,v]$ of X of the emerging kind, of length t starts at u. Let $k \in \{1,\dots,m\}$. We have, with $n = \lfloor \frac{g}{3\Delta} \rfloor$:*

$$\sum_t \mathbf{P}[\mathscr{F}(t) \mid X(u) = k] \leqslant 4m\Delta^2 w^2 \left(2 \cdot (5\Delta g')^4 + w^{-4} e^{-(1-q)^2 nh(R^*)} \right). \tag{5.16}$$

Proof. For interval $I' = [u',v']$ let event $\mathscr{L}_j(x,Y,I',1)$ be defined as in Steps 3 and 4 of the scale-up algorithm in Section 4. Let us fix an arbitrary $k' \in \{1,\dots,m\}$. By the proof of Lemma 5.5, for $j = 1,2$:

$$\mathbf{P}[\mathscr{L}_j(X,Y,I',1) \mid X(u) = k, \ Y(0) = k'] \leqslant (5\Delta l_j w)^4 =: U_j,$$

where we took inequality (5.13) and unconditioned on all $Y(b)$ for $b \geqslant 1$. Similarly by the proof of Lemma 5.6:

$$\mathbf{P}[\mathscr{L}_3(X,Y,I',1) \mid X(u) = k, \ Y(0) = k'] \leqslant e^{-(1-q)^2 nh(R^*)} =: U_3.$$

Let us define the function $\pi(x(I'))$ depending on the sequence $x(I')$ as $\pi(x(I')) = \mathbf{P}[X(I') = x(I') \mid X(u) = k]$. Then

$$\sum_{x(I')} \pi(x(I'))\mathbf{P}[\mathscr{L}_j(X,Y,I',1) \mid X(u) = k, \ X(I') = x(I'), \ Y(0) = k'] \leqslant U_j.$$

The Markov inequality implies that the probability (conditioned on $X(u) = k$) of those x for which the inequality

$$\mathbf{P}[\mathscr{L}_j(X,Y,I',1) \mid X(u) = k, \ X(I') = x(I'), \ Y(0) = k'] > w^2$$

holds will be upper-bounded by $w^{-2}U_j$.

The length of interval $I' = (u',v']$ is defined by the type, but not its starting point $u' \in u + [0,2\Delta)$, neither is the endpoint $v \in v' + [0,2\Delta)$ of the barrier. For every one of the $(2\Delta)^2$ possible choices of these, we obtain a particular length t of the barrier $(u,v]$. Multiplying by the number m of possible choices of k' we obtain an upper bound on desired sum for an emerging wall of type j. Adding up the three values and recalling $\max(l_1,l_2) = g'$ gives

$$4m\Delta^2 w^{-2}(U_1 + U_2 + U_3) < 4m\Delta^2 w^2 \left(2 \cdot (5\Delta g')^4 + w^{-4} e^{-(1-q)^2 nh(R^*)} \right).$$

$\square$

5.4 Compound walls

Let us use the definition of compound walls given in Step 6 of the scale-up algorithm of Section 4.

Lemma 5.8. *Consider ranks r_1, r_2 at any stage of the scale-up construction. Assume that Condition 3.6.3b already holds for rank values r_1, r_2. For a given point x_1, let us fix the $X(x_1) = k$ for some $k \in \{1, \dots, m\}$ arbitrarily. Then the sum, over all l, of the probabilities for the occurrence of a compound barrier of type $\langle r_1, r_2, i \rangle$ and width l at x_1 is bounded above by*

$$\lambda^i p(r_1) p(r_2). \tag{5.17}$$

Proof. Noting $d_{i+1} - d_i \leqslant \lambda^i$ for all i, we will prove an upper bound $(d_{i+1} - d_i) p(r_1) p(r_2)$. For fixed r_1, r_2, x_1, d, let $B(d, l)$ be the event that a compound barrier of any type $\langle r_1, r_2, i \rangle$ with distance d between the component barriers, and size l appears at x_1. For any l, let $A(x, r, l)$ be the event that a barrier of rank r and size l starts at x. We can write

$$B(d, l) = \bigcup_{l_1 + d + l_2 = l} A(x_1, r_1, l_1) \cap A(x_1 + l_1 + d, r_2, l_2).$$

where events $A(x_1, r_1, l_1)$, $A(x_1 + l_1 + d, r_2, l_2)$ belong to disjoint intervals. Recall the definition of $p(r, l)$ in (3.3). By the Markov property,

$$\mathbf{P}(B(d, l)) \leqslant \sum_{l_1 + d + l_2 = l} p(r_1, l_1) p(r_2, l_2).$$

Hence Condition 3.6.3b implies $\sum_l \mathbf{P}(B(d, l)) \leqslant \sum_{l_1} p(r_1, l_1) \sum_{l_2} p(r_2, l_2) \leqslant p(r_1) p(r_2)$, which completes the proof. □

In the lemma below, we use w_1, w_2: please note that these are integer coordinates, and have nothing to do with the trap probability upper bound w: we will never have these two uses of w in a place where they can be confused.

Lemma 5.9. *Let $u \leqslant v_1 < w_2$, and a be given with $v_1 - u \leqslant 12\Delta^*$, and let*

$$b = a + \lceil (v_1 - u)/2 \rceil, \qquad c = b + (v_1 - u) + 1.$$

Assume that $Y = y$ is fixed in such a way that W is a compound horizontal wall with body $(v_1, w_2]$, and type $\langle r_1, r_2, i \rangle$, with rank r as given in (4.10). Assume also that the component walls W_1, W_2 already satisfy the hole lower bound, Condition 3.6.3d. Let

$$E_2 = E_2(u, v_1, w_2; a) = E^*(u, v_1, w_2; a)$$

where E^* was defined in (5.2). Assume

$$(\Delta^*)^\chi h(r_j) \leqslant 0.07, \text{ for } j = 1, 2. \tag{5.18}$$
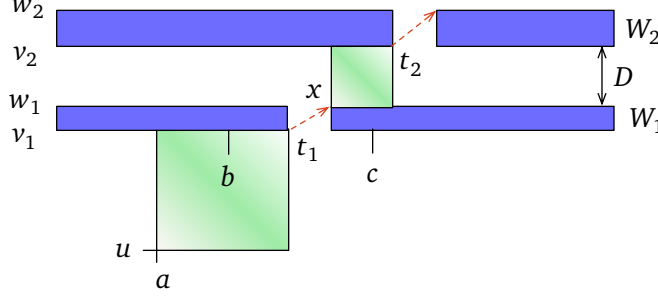
Figure 21: Hole through a compound wall. The light rectangles are inner H-clean.

Let $k \in \{1,\dots,m\}$. Then

$$\mathbf{P}[E_2 \mid X(a) = k,\ Y = y] \geqslant (c - b)^X (\lambda^i/2)^X h(r_1) h(r_2) \cdot (1 - V) \tag{5.19}$$

with $V = 2 \cdot (26\overline{p}\Delta^* + 338\overline{w}(\Delta^*)^2)/h(r_1 \vee r_2)$.

Proof. Figure 21 shows the role of the various coordinates in the proof. Let D be the distance between the component walls W_1, W_2 of the wall W, where the body of W_i is $(v_i, w_i]$. Consider first passing through W_1. For each $x \in [b, c + \Delta - 1]$, let A_x be the event that $E^*(u, v_1, w_1; a)$ holds with the vertical projection of the hole ending at x, and that x is the smallest possible number with this property. Let $B_x = E^*(w_1, v_2, w_2; x)$.

1. We have $E_2 \supseteq \bigcup_x (A_x \cap B_x)$.

   Proof. If for some x we have A_x, then there is a rectangle $\mathrm{Rect}((a, u), (t_1, v_1))$ satisfying the requirements of $E^*(u, v_1, w_1; a)$ and also a hole $\mathrm{Rect}((t_1, v_1), (x, w_1))$ through the first wall. If also B_x holds, then there is a rectangle $\mathrm{Rect}((x, w_1), (t_2, v_2))$ satisfying the requirements of $E^*(w_1, v_2, w_2; x)$, and also a hole $\mathrm{Rect}((t_2, v_2), (x', w_2))$ through the second wall that is good as seen from point (x, w_1).

   Let us show $(t_1, v_1) \rightsquigarrow (x', w_2)$, meaning that the interval $(t_1, x']$ is a hole that passes through the compound wall W. We already know $(t_1, v_1) \rightsquigarrow (x, w_1)$ and $(t_2, v_2) \rightsquigarrow (x', w_2)$; we still need to prove $(x, w_1) \rightsquigarrow (t_2, v_2)$.

   The requirements imply that $\mathrm{Rect}((x, w_1), (t_2, v_2))$ is a hop of $\mathcal{M}$. Indeed, the inner H-cleanness of $(x, t_2]$ in the process X follows from B_x. The inner cleanness of $(w_1, v_2]$ in the process Y is implied by the fact that $(v_1, w_2]$ is a compound wall. The fact that W is a compound wall also implies that the interval $(w_1, v_2]$ contains no horizontal walls.

   According to B_x, this rectangle has the necessary slope constraints, hence by the reachability condition of $\mathcal{M}$, its endpoint is reachable from its starting point.

It remains to lower-bound $\mathbf{P}\big(\bigcup_x (A_x \cap B_x)\big)$. For each x, the events A_x, B_x belong to disjoint intervals, and the events A_x are disjoint of each other.

2. Let us lower-bound $\sum_x \mathbf{P}(A_x)$.

   We have, using the notation of Lemma 5.3: $\sum_x \mathbf{P}(A_x) = \mathbf{P}(E^*(u, v_1, w_1; a))$. Lemma 5.3 is applicable and we get $\mathbf{P}(E^*(u, v_1, w_1; a)) \geqslant F_1 - U$ with

$$F_1 = 0.5 \wedge ((c-b)^x h(r_1)),$$
$$U = 26\overline{p}\Delta^* + 338\overline{w}(\Delta^*)^2. \tag{5.20}$$

   By the assumption (5.18): $(c-b)^x h(r_1) \leqslant (7\Delta^*)^x h(r_1) \leqslant 0.5$, hence the operation $0.5 \wedge$ can be deleted from F_1:

$$F_1 = G_1 := (c-b)^x h(r_1). \tag{5.21}$$

3. Let us now lower-bound $\mathbf{P}(B_x)$, for an arbitrary condition $X(x) = k$ for $k \in \{1, \ldots, m\}$.
   We have $B_x = E^*(w_1, v_2, w_2; x)$. The conditions of Lemma 5.3 are satisfied for $u = w_1$, $v = v_2$, $w = w_2$, $a = x$. It follows that $\mathbf{P}(B_x) \geqslant F_2 - U$ with

$$F_2 = 0.5 \wedge ((\lfloor D/2 \rfloor + 1)^x h(r_2)),$$

   which can again be simplified using assumption (5.18) and $D \leqslant f \leqslant \Delta^*$:

$$F_2 = G_2 := (\lfloor D/2 \rfloor + 1)^x h(r_2).$$

4. Let us combine these estimates, using $G = G_1 \wedge G_2 > h(r_1 \vee r_2)$.
   By the Markov property, we find that the lower bound on $\mathbf{P}(B_x)$ (for arbitrary $X(x) = k$) is also a lower bound on $\mathbf{P}(B_x \mid A_x)$:

$$\begin{aligned}
\mathbf{P}(E_2) &\geqslant \sum_x \mathbf{P}(A_x)\mathbf{P}(B_x \mid A_x) \geqslant (G_1 - U)(G_2 - U) \\
&\geqslant G_1 G_2 (1 - U(1/G_1 + 1/G_2)) \geqslant G_1 G_2 (1 - 2U/G) \\
&= (c-b)^x (\lfloor D/2 \rfloor + 1)^x h(r_1) h(r_2)(1 - 2U/G) \\
&\geqslant (c-b)^x (\lfloor D/2 \rfloor + 1)^x h(r_1) h(r_2)(1 - 2U/h(r_1 \vee r_2)).
\end{aligned}$$

5. We conclude by showing $\lfloor D/2 \rfloor + 1 \geqslant \lambda^i/2$.
   Recall $d_i \leqslant D < d_{i+1}$ where d_i was defined in (4.9). For $i = 0, 1$, we have $\lfloor D/2 \rfloor + 1 = 1 > \lambda^1/2$. For $i \geqslant 2$, we have $\lfloor D/2 \rfloor + 1 \geqslant D/2 \geqslant \lambda^i/2$.

$\square$

6 The scale-up functions

Lemma 2.6 says that there is an m_0 such that if $m > m_0$ then the sequence $\mathscr{M}^k$ can be constructed in such a way that the claim (2.3) of the main lemma holds. If we computed m_0 explicitly then all parameters of the construction could be turned into constants: but this is unrewarding work and it would only make the relationships between the parameters less intelligible. We prefer to name all these parameters, to point out the necessary inequalities among them, and finally to show that if m is sufficiently large then all these inequalities can be satisfied simultaneously.

Instead of m, it we will be more convenient to use the parameter R_0 introduced below. Here is how increasing R_0 increases m. Mazery $\mathscr{M}^1$ is defined in Example 3.9, with a parameter $w = w_1$ that gets smaller as m gets larger. more precisely, one needs $m \geqslant 1/w_1 + 1$. So, the following is an upper bound on the least acceptable value of m:

$$\lceil 1/w_1 + 1 \rceil \leqslant 2/w_1 = 2\lambda^{R_0 \omega \tau}, \tag{6.1}$$

where we used definitions 6.1 and 6.3 below.

The following definition introduces some of the parameters needed for scale-up. Recall that the slope lower bound σ must satisfy $\sigma < 1/2$.

Definition 6.1. We set

$$\sigma_1 = 0. \tag{6.2}$$

To obtain the new rank lower bound, we multiply R by a constant:

$$R = R_k = R_0 \tau^k, \quad R_{k+1} = R^* = R\tau, \quad 1 < \tau < 2, \quad 1 < R_0. \tag{6.3}$$

The rank of emerging walls, introduced in (4.8), is defined using a new parameter τ':

$$\hat{R} = \tau' R.$$

⌐

We require

$$\tau < \tau' < \tau^2. \tag{6.4}$$

We need some bounds on the possible rank values.

Definition 6.2. Let $\bar{\tau} = 2\tau/(\tau - 1)$. ⌐

Lemma 6.1 (Rank upper bound). *In a mazery, all ranks are upper-bounded by $\bar{\tau} R$.*

Proof. Let $\overline{R}$ denote the maximum of all ranks. Since $\tau' < \tau^2 < \overline{\tau}$, emerging walls got a rank equal to $\tau' R < \overline{\tau} R$, and the largest rank produced by the compound operation is at most $\overline{R} + 2R^*$ (since the compound operation is applied twice):

$$\overline{R}^* \leqslant \overline{R} + 2R^*, \quad \overline{R}_k \leqslant 2 \sum_{i=1}^{k} R_i = 2R_0 \tau \frac{\tau^k - 1}{\tau - 1} \leqslant R_k \frac{2\tau}{\tau - 1}. \tag{6.5}$$

$\square$

Corollary 6.2. *Every rank exists in $\mathcal{M}^k$ for at most $\lceil \log_\tau \frac{2\tau}{\tau-1} \rceil$ values of k.*

Recall $\lambda = 2^{1/2}$, as defined in (3.4). It can be seen from the definition of compound ranks in (4.10) and from Lemma 5.8 that the probability bound $p(r)$ of a wall should be approximately λ^{-r}. The actual definition makes the bound a little smaller:

It is convenient to express several other parameters of $\mathcal{M}$ and the scale-up in terms of a single one, T:

Definition 6.3 (Exponential relations)**.** Let $T = \lambda^R$,

$$\Delta = T^\delta, \quad f = T^\varphi, \quad g = T^\gamma, \quad w = T^{-\omega}.$$

As we will see, ω just needs to be sufficiently large with respect to the other constants. On the other hand, we require

$$0 < \delta < \gamma < \varphi < 1. \tag{6.6}$$

The values $\delta, \varphi, \gamma, \tau, \tau'$ will be chosen independent of the mazery level. A bound on φ has been indicated in the requirement (4.3) which will be satisfied by

$$\tau \leqslant 2 - \varphi. \tag{6.7}$$

We turn this into equality: $\tau = 2 - \varphi$.

Let us estimate Δ^*. Emerging walls can have size as large as $4g' + 4\Delta$, and at the time of their creation, they are the largest existing ones. We get the largest new walls when the compound operation combines these with light walls on both sides, leaving the largest gap possible, so the largest new wall size is

$$4g' + 2f + 6\Delta < 3f,$$

where we used (4.1). Hence any value larger than $3f$ can be chosen as $\Delta^* = \Delta^\tau$. With R large enough, we always get this if

$$\varphi < \tau\delta. \tag{6.8}$$

As a reformulation of one of the inequalities of (4.1), we require

$$\gamma \geqslant \frac{\delta + \varphi}{2}. \tag{6.9}$$

We also need

$$4(\gamma + \delta) < \omega(4 - \tau), \tag{6.10}$$

$$\tau < 2, \tag{6.11}$$

$$4\gamma + 6\delta + \tau' < \omega, \tag{6.12}$$

$$\tau(\delta + 1) < \tau'. \tag{6.13}$$

Using the exponent χ introduced in (2.2), we require

$$\tau\chi < \gamma - \delta, \tag{6.14}$$

$$\overline{\tau}\chi < 1 - \tau\delta, \tag{6.15}$$

$$\overline{\tau}\chi < \omega - 2\tau\delta. \tag{6.16}$$

Note that all these inequalities require χ to be just sufficiently small.

Lemma 6.3. *The exponents $\delta, \varphi, \gamma, \tau, \tau', \chi$ can be chosen to satisfy the inequalities (6.3),(6.4), (6.6)-(6.16).*

Proof. It can be checked that the choices $\delta = 0.15$, $\gamma = 0.2$, $\varphi = 0.25$, $\tau = 1.75$, $\tau' = 2.5$, $\omega = 4.5$, $\overline{\tau} = 4.66\ldots$ satisfy all the inequalities in question. □

Definition 6.4. Let us fix now the exponents $\delta, \varphi, \gamma, \tau, \chi$ as chosen in the lemma. In order to satisfy all our requirements also for small k, we will fix c_2 sufficiently small, then c_1 sufficiently large, then c_3 sufficiently large, and finally R_0 sufficiently large. ⌐

We need to specify some additional parameters.

Definition 6.5. We define

$$\overline{p} = T^{-1}, \qquad \overline{w} = w, \tag{6.17}$$

$$q^* = q + \Delta^* \overline{p}. \tag{6.18}$$

⌐

7 Probability bounds after scale-up

7.1 Bounds on traps

The structures $\mathcal{M}^k$ are now defined but we have not proved that they are mazeries, since not all inequalities required in the definition of mazeries have been verified yet.

Lemma 7.1. *For any value of the constant c_3, if R_0 is sufficiently large then the following holds: if $\mathcal{M} = \mathcal{M}^k$ is a mazery then $\mathcal{M}^*$ satisfies the trap upper bound 3.6.3a.*

Proof. For some string $x = (x(0), x(1), \ldots)$, for a point (a, b), let $\mathcal{E}$ be the event that a trap starts at (a, b). We assume $Y(b-1) = s$ fixed arbitrarily. We need to bound $\mathbf{P}[\mathcal{E} \mid X = x, Y(b-1) = s]$. There are three kinds of trap in $\mathcal{M}^*$: uncorrelated and correlated compound traps, and traps of the missing-hole kind. Let $\mathcal{E}_1$ be the event that an uncorrelated trap occurs. According to (5.11), using $\tau = 2 - \varphi$ (and recalling that w^* plays the role of the trap probability upper bound *tub* for $\mathcal{M}^*$):

$$\mathbf{P}[\mathcal{E}_1 \mid X = x, Y(b-1) = s] \leqslant 2f^2 w^2 = 2T^{2\varphi - 2\omega}$$
$$= 2T^{-\tau\omega - (2-\tau)\omega + 2\varphi} = w^* \cdot 2/f^{\omega - 2}.$$

This can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

Let $\mathcal{E}_2$ be the event that a vertical correlated trap appears. By Lemma 5.5, using (4.4):

$$\mathbf{P}[\mathcal{E}_2 \mid X = x, Y(b-1) = s] \leqslant \sum_{j=1}^{2}(5\Delta l_j w)^4 \leqslant 2 \cdot (5\Delta g' w)^4$$
$$= 2 \cdot 11^4 T^{4\gamma + 4\delta - 4\omega - \tau\omega + \tau\omega} = 2w^* \cdot 11^4 T^{4(\gamma + \delta) - \omega(4 - \tau)}.$$

Due to (6.10), this can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

Let $\mathcal{E}_3$ be the event that a vertical trap of the missing-hole kind appears at (a, b). Lemma 5.6 implies for $n = \lfloor \frac{g}{3\Delta} \rfloor$:

$$\mathbf{P}[\mathcal{E}_3 \mid X = x, Y(b-1) = s] \leqslant e^{-(1-q)^2 nh(R^*)}.$$

Further, using inequality (4.1) and the largeness of R_0:

$$n > g/(3\Delta) - 1 > g/(4\Delta) = T^{\gamma - \delta}/4.$$

Now,

$$h(R^*) = c_3 T^{-\tau\chi},$$
$$(1 - q)^2 nh(R^*) > 0.8nh(R^*) > 0.2c_3 T^{\gamma - \delta - \tau\chi},$$
$$\mathbf{P}[\mathcal{E}_3 \mid X = x, Y(b-1) = s] \leqslant e^{-0.2c_3 T^{\gamma - \delta - \tau\chi}}.$$

Due to (6.14), this can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

For $j = 1, 2$, let $\mathcal{E}_{4,j}$ be the event that a horizontal trap of the correlated kind of type j starts at (a,b). Let $\mathcal{E}_{4,3}$ be the event that a horizontal trap of missing-hole kind starts at (a,b). Lemmas 5.5 and 5.6 imply

$$\mathbf{P}[\mathcal{E}_{4,j} \mid X = x,\ Y(b-1) = s] \leqslant w^2 = w^* T^{-\omega(2-\tau)}.$$

Due to (6.11), this can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

Thus, if R_0 is sufficiently large then the sum of these six probabilities is still less than w^*. $\qquad\square$

7.2 Bounds on walls

Recall the definition of $p(r)$ in (3.5).

Lemma 7.2. *For every possible value of c_1, c_2, c_3, if R_0 is sufficiently large then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. Fixing any point a and fixing $X(a)$ in any way, the sum of the probabilities over l that a barrier of the emerging kind of size l starts at a is at most $p(\hat{R})/2 = p(\tau' R)/2$.*

Proof. We use the result and notation of Lemma 5.7, and also the estimate of $\mathbf{P}(\mathcal{E}_3)$ in the proof of Lemma 7.1, replacing m with the upper bound from (6.1)

$$\sum_l \mathbf{P}(\mathcal{F}(l)) \leqslant 8 w_1^{-1} \Delta^2 w^2 \left(2 \cdot (5\Delta g')^4 + w^{-4} e^{-(1-q)^2 nh(R^*)} \right),$$
$$8 w_1^{-1} \Delta^2 w^{-2} e^{-(1-q)^2 nh(R^*)} \leqslant 8 w_1^{-1} T^{2\omega+2\delta} e^{-0.2 c_3 T^{\gamma-\delta-\tau\chi}}.$$

Due to (6.14), the last expression decreases exponentially in T, so for sufficiently large R_0 it is less than $p(\tau' R)$ by an arbitrarily large factor. On the other hand, using (4.4):

$$8 w_1^{-1} \Delta^2 w^2 \cdot 2 \cdot (5\Delta g')^4 \leqslant 16 \cdot 11^4 T^{-\omega+4\gamma+6\delta} = 16 T^{-\tau'} \cdot 11^4 T^{4\gamma+6\delta+\tau'-\omega}.$$

If R_0 is sufficiently large then, due to (6.12), this is less than $p(\tau' R)$ by an arbitrarily large factor. $\qquad\square$

Lemma 7.3. *For a given value of c_2, if we choose the constants c_1, R_0 sufficiently large in this order then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. After one operation of forming compound barriers, fixing any point a and fixing $X(a)$ in any way, for any rank r, the sum, over all widths w, of the probability that a compound barrier of rank r and width w starts at a is at most $p(r) R^{-c_1/2}$.*

Proof. Let $r_1 \leqslant r_2$ be two ranks, and assume that r_1 is light: $r_1 < R^* = \tau R$. With these, we can form compound barriers of type $\langle r_1, r_2, i \rangle$. The bound (5.17) and the definition of $p(r)$ in (3.5) shows that the contribution by this term to the sum of probabilities, over all sizes w, that a barrier of rank $r = r_1 + r_2 - i$ and size w starts at x is at most

$$\lambda^i p(r_1) p(r_2) = c_2^2 \lambda^{-r} (r_1 r_2)^{-c_1} = c_2 (r/r_1 r_2)^{c_1} p(r).$$

Now $r_1 r_2 \geqslant R r_2 \geqslant (R/2)(r_1 + r_2) \geqslant rR/2$, hence the above bound reduces to $c_2 (R/2)^{-c_1} p(r)$. The same rank r can be obtained by the compound operation at most the following number of times:

$$|\{ (i, r_1) : i \leqslant R\varphi, \ r_1 < \tau R \}| \leqslant (\varphi R + 1)\tau R.$$

The total probability contributed to rank r is therefore at most

$$c_2 (R/2)^{-c_1} p(r)(\varphi R + 1)\tau R < p(r) R^{-c_1/2}$$

if R_0 and c_1 are sufficiently large. $\qquad\square$

Lemma 7.4. *For every choice of c_1, c_2, c_3 if we choose R_0 sufficiently large then the following holds. Suppose that each structure $\mathcal{M}^i$ for $i \leqslant k$ is a mazery. Then Condition 3.6.3b holds for $\mathcal{M}^{k+1}$.*

Proof. By Corollary 6.2, each rank r occurs for at most a constant number $n = \lceil \log_\tau \frac{2\tau}{\tau - 1} \rceil$ values of $i \leqslant k$. For any rank, a barrier can be formed only as an emerging barrier or compound barrier. The first can happen for one i only, and Lemma 7.2 bounds the probability contribution by $p(r)/2$. A compound barrier can be formed for at most n values of i, and for each value in at most two steps. Lemma 7.3 bounds each contribution by $p(r)R^{-c_1/2}$. After these increases, the probability becomes at most $p(r)(1/2 + 2nR^{-c_1/2}) < p(r)$ if R_0 is sufficiently large. $\qquad\square$

7.3 Auxiliary bounds

The next two lemmas show that the choices made in Definition 6.5 satisfy the requirements imposed in Definition (5.3). Recall the introduction of the wall probability upper bound $\overline{p}$ in Definition 5.2 and its value assignment T^{-1} in (6.17).

Lemma 7.5. *For small enough c_2, the probability of a barrier of $\mathcal{M}$ starting at a given point b is bounded by $\overline{p}$.*

Proof. We have $\sum_{r \geqslant R} p(r) < c_2 \sum_{r \geqslant R} \lambda^{-r} = \lambda^{-R} c_2 (1 - 1/\lambda)^{-1} < \lambda^{-R}$ if $c_2 < 1 - 1/\lambda$. $\qquad\square$

Lemma 7.6. *If R_0 is sufficiently large then $\sum_k (2\Delta_{k+1} \overline{p}_k + \Delta_{k+1}^2 w_k) < 0.5$.*

Proof. Substituting the definitions of $\overline{p}$ in (6.17), further the values of all other parameters given in Definitions 6.1 and 6.3:

$$\sum_k (2\Delta_{k+1}\overline{p}_k + \Delta_{k+1}^2 w_k) \leqslant 2\sum_k \lambda^{-R_0\tau^k(1-\delta\tau)} + \sum_k \lambda^{-R_0\tau^k(\omega-2\delta\tau)}$$

which because of (6.15) and (6.16), is less than 0.5 if R_0 is large. $\qquad\square$

Note that for R_0 large enough, the relations

$$\Delta^*\overline{p} < 0.5(0.1-q), \tag{7.1}$$

$$\Lambda g/f < 0.5(0.5-\sigma) \tag{7.2}$$

hold for $\mathcal{M} = \mathcal{M}^1$ and $\sigma = \sigma_1$. This is clear for (7.1). For (7.2), since $\sigma_1 = 0$ according to (6.2), we only need $0.25 > \Lambda g/f = \Lambda T^{-(\varphi-\gamma)}$, which is satisfied if R_0 is large enough.

Lemma 7.7. *Suppose that the structure $\mathcal{M} = \mathcal{M}^k$ is a mazery and it satisfies (7.1) and (7.2). Then $\mathcal{M}^* = \mathcal{M}^{k+1}$ also satisfies these inequalities if R_0 is chosen sufficiently large (independently of k), and also satisfies Condition 3.6.3c.*

Proof. Let us show first that $\mathcal{M}^*$ also satisfies the inequalities if R_0 is chosen sufficiently large.

For sufficiently large R_0, we have $\Delta^{**}\overline{p}^* < 0.5\Delta^*\overline{p}$. Indeed, this says $T^{(\tau\delta-1)(\tau-1)} < 0.5$. Hence using (7.1) and the definition of q^* in (6.18):

$$\Delta^{**}\overline{p}^* \leqslant 0.5\Delta^*\overline{p} \leqslant 0.5(0.1-q) - 0.5\Delta^*\overline{p}$$
$$\leqslant 0.5(0.1-q) - 0.5(q^*-q) = 0.5(0.1-q^*).$$

This is inequality (7.1) for $\mathcal{M}^*$.

For inequality (7.2), the scale-up definition Definition 4.2 says $\sigma^* - \sigma = \Lambda g/f$. The inequality $g^*/f^* < 0.5g/f$ is guaranteed if R_0 is large. From here, we can conclude the proof as for q.

To verify Condition 3.6.3c for $\mathcal{M}^*$, recall the definition in (6.18) of

$$q^* = q + \Delta^*\overline{p}.$$

For the first inequality of Condition 3.6.3c, we upper-bound the conditional probability that a point a of the line is strongly clean in $\mathcal{M}$ but not in $\mathcal{M}^*$ by

$$(2f/3 + \Delta)\overline{p}, \tag{7.3}$$

which upper-bounds the probability that a horizontal barrier of $\mathcal{M}$ starts in $(a - f/3 - \Delta, a + f/3]$. This can be upper-bounded by

$$f\overline{p} < T^{\varphi-1} < T^{\delta\tau-1}/2 = \Delta^*\overline{p}/2,$$

where the last inequality holds due to (6.8) if R_0 is sufficiently large. Hence an upper bound on the conditional probability of not strong cleanness in $\mathscr{M}^*$ is $q/2 + \Delta^* \overline{p}/2 = q^*/2$ as required.

For the other inequalities in Condition 3.6.3c, consider a rectangle $Q = \text{Rect}^{\rightarrow}(u,v)$ and fix $X(u_0) = k$, and $Y = y$. The conditional probability that a point u is not trap-clean in Q for $\mathscr{M}$ but not for $\mathscr{M}^*$ is upper-bounded by the probability of the appearance of a trap of $\mathscr{M}$ within a distance g of point u in Q. There are at most g^2 positions for the trap, so a bound is

$$g^2 w = T^{2\gamma - \omega} < 0.5 T^{\tau\delta - 1},$$

if R_0 is sufficiently large, due to the inequality (6.12). From here, we conclude the same way for the first inequality. The argument for the other inequalities in Condition 3.6.3c is identical. □

7.4 Lower bounds on holes

We will make use of the following estimate.

Lemma 7.8. *Let* $(a_0, b_0]$, $[a_1, b_1]$ *be intervals with length* $\leqslant 12\Delta^*$. *Suppose that the sequence Y and the value $s \in \{1, \ldots, m\}$ are fixed arbitrarily. If c_3 and then R_0 are chosen sufficiently large then the following event holds with probability at least* 0.75 *even if conditioned on $X(a_0) = s$: The rectangle $Q = (a_0, b_0] \times [a_1, b_1]$ is inner H-clean for $\mathscr{M}^*$, and contains no traps or vertical barriers of $\mathscr{M}$ or $\mathscr{M}^*$.*

Proof. We will just write "probability" but will understand conditional probability, when $Y = y$ and $X(a_0)$ is fixed. According to Lemma 7.7, the probability that one of the two cleanness conditions is not satisfied is at most 0.2. Using Lemmas 7.1, 7.4 and 7.5, the probability that a vertical barrier of $\mathscr{M}$ or $\mathscr{M}^*$ is contained in Q is at most

$$12\Delta^* (\overline{p} + \overline{p}^*) \leqslant 24\Delta^* \overline{p} = 24 T^{\tau\delta - 1}.$$

The probability that a trap of $\mathscr{M}$ or $\mathscr{M}^*$ is contained in Q is at most

$$(12\Delta^*)^2 (w + w^*) \leqslant 2 \cdot 144 (\Delta^*)^2 w = 288 T^{2\tau\delta - \omega}.$$

If R_0 is sufficiently large then the sum of these two terms is at most 0.05. □

Lemma 7.9. *For emerging walls, the fitting holes satisfy Condition 3.6.3d if R_0 is sufficiently large.*

Proof. Recall Condition 3.6.3d applied to the present case. Let $u \leqslant v < w$, a be given with $v - u \leqslant 12\Delta^*$, and define $b = a + \lceil (v - u)/2 \rceil$, $c = a + (v - u) + 1$. Assume that $Y = y$ is
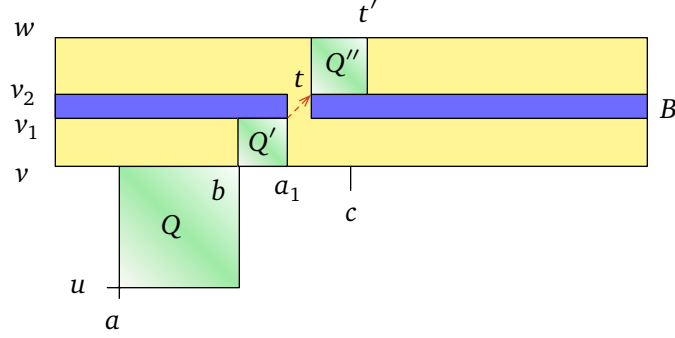
Figure 22: Hole through an emerging wall: to the proof of Lemma 7.9

fixed in such a way that B is a horizontal wall of the emerging kind with body $(v,w]$. Let $E^* = E^*(u,v,w;a)$ be as given in Definition 5.1. Let $s \in \{1,\dots,m\}$. We will prove

$$\mathbf{P}[E^* \mid X(a) = s,\ Y = y] \geqslant (c-b)^{\times} h(\hat{R}).$$

Recall the definition of emerging walls in Step 5 of the scale-up construction. The condition at the end says, in our case, that either $(v,w]$ is a hop of Y or it can be partitioned into a light (horizontal) wall $(v_1,v_2]$ of some rank r, and two (possibly empty) hops surrounding it: so, $v \leqslant v_1 < v_2 \leqslant w$. Without loss of generality, assume this latter possibility. Let

$$a_1 = b + (v_1 - v).$$

Let $\mathscr{F}$ be the event that

(a) Rectangle $Q = \mathrm{Rect}^{\rightarrow}((a,u),(b,v))$ contains no vertical barriers or traps of $\mathscr{M}$ or $\mathscr{M}^*$, further is inner H-clean for $\mathscr{M}^*$. Rectangle $Q' = \mathrm{Rect}^{\rightarrow}((b,v),(a_1,v_1))$ contains no vertical barriers or traps of $\mathscr{M}$, and is inner H-clean for $\mathscr{M}$.

(b) For an arbitrary $t \in (a_1, a_1 + \Delta]$, let

$$t' = t + (w - v_2).$$

We require that event $E(v_1,v_1,v_2;a_1)$ is realized with some hole $(a_1,t]$, and the rectangle $Q'' = \mathrm{Rect}^{\rightarrow}((t,v_2),(t',w))$ contains no vertical barriers or traps, and is inner H-clean for $\mathscr{M}$.

1. Event $\mathscr{F}$ implies event E^*.
   Proof. Assume that $\mathscr{F}$ holds. Rectangle $\mathrm{Rect}^{\rightarrow}((a,u),(b,v))$ has the necessary inner cleanness properties, the absence of barriers and traps, and the slope lower bound to have $(a,u) \rightsquigarrow (b,v)$: it remains to show $(b,v) \rightsquigarrow (t',w)$. We have $(b,v) \rightsquigarrow (a_1,v_1)$ since

$\text{Rect}^{\rightarrow}((b,v),(a_1,v_1))$ is a hop that is also a square, hence satisfies the slope lower bound condition. For similar reasons, $(t,v_2) \rightsquigarrow (t',w)$. Also, since $(a_1,t]$ is a hole through $(v_1,v_2]$, we have $(a_1,v_1) \rightsquigarrow (t,v_2)$.

2. We have $\mathbf{P}[\mathscr{F} \mid X(a) = s,\ Y = y] \geqslant 0.75^3 c_3 T^{-\chi\tau}$.

   Proof. Condition (a) in the definition of $\mathscr{F}$ is coming from two rectangles with disjoint projections, therefore by the method used throughout the paper, we can multiply their probability lower bounds, which are given as 0.75 by Lemma 7.8.

   Condition (b) also refers to an event with a projection disjoint from the previous ones. The probability of the existence of a hole is lower-bounded via Condition 3.6.3d, by $h(r) \geqslant h(R^*) = c_3 T^{-\chi\tau}$. A reasoning similar to the proof of Lemma 5.2 shows that the whole condition (b) is satisfied at the expense of another factor 0.75 via Lemma 7.8.

Recall the definition of $h(r)$ in (3.6). The required lower bound of Condition 3.6.3d is

$$(c-b)^{\chi} h(\hat{R}) \leqslant (6\Delta^* + 1)^{\chi} h(\hat{R}) \leqslant (7T^{\tau\delta})^{\chi} h(\tau'R) = c_3 7^{\chi} T^{\chi(\tau\delta - \tau')}$$
$$< 0.75^3 c_3 T^{-\chi\tau}$$

if R_0 is sufficiently large, due to (6.13). $\qquad\square$

Lemma 7.10. *After choosing c_1, c_3, R_0 sufficiently large in this order, the following holds. Assume that $\mathscr{M} = \mathscr{M}^k$ is a mazery: then every compound wall satisfies the hole lower bound, Condition 3.6.3d, provided its components satisfy it.*

Proof. The proof starts from the setup in Lemma 5.9, making the appropriate substitutions in the estimates.

1. Recall what is required.

   Let $u \leqslant v_1 < w_2$, a be given with $v_1 - u \leqslant 12\Delta^*$, and define

$$b = a + \lceil (v_1 - u)/2 \rceil, \qquad c = a + (v_1 - u) + 1.$$

   Assume that $Y = y$ is fixed in such a way that there is a compound horizontal wall W with body $(v_1, w_2]$, and type $\langle r_1, r_2, i \rangle$, with rank

$$r = r_1 + r_2 - i \tag{7.4}$$

   as in (4.10). Also, let $X(a) = s$ be fixed in an arbitrary way. Let $E_2 = E^*(u, v_1, w_2; a)$ as defined in (5.2). We need to prove $\mathbf{P}[E_2 \mid X(a) = s] \geqslant (c-b)^{\chi} h(r)$.

2. Let us apply Lemma 5.9.

   Recall the definition of $h(r)$ in (3.6). The assumption $(\Delta^*)^{\chi} h(r_i) \leqslant 0.07$ of the lemma holds since

$$h(r_i) = c_3 \lambda^{-\chi r_i} \leqslant c_3 T^{-\chi}, \quad (\Delta^*)^{\chi} h(r_i) \leqslant c_3 T^{-\chi(1-\delta\tau)} \tag{7.5}$$

which, due to (6.15), is always smaller than 0.07 if R_0 is sufficiently large. We conclude

$$\mathbf{P}[E_2 \mid X(a) = s] \geqslant (c-b)^{\chi} (\lambda^i/2)^{\chi} h(r_1) h(r_2) \cdot (1-V)$$

with $V = 2 \cdot (26\overline{p}\Delta^* + 338\overline{w}(\Delta^*)^2)/h(r_1 \vee r_2)$.

3. Let us estimate the part of this expression before $1-V$.
   Using and the formula for $h(r_i)$ in (7.5) and the definition of r in (7.4):

$$(\lambda^i/2)^{\chi} h(r_1) h(r_2) = 2^{-\chi} c_3^2 \lambda^{-\chi(r_1+r_2-i)} = 2^{-\chi} c_3^2 \lambda^{-r\chi},$$
$$(c-b)^{\chi} (\lambda^i/2)^{\chi} h(r_1) h(r_2) \geqslant 2^{-\chi} c_3 (c-b)^{\chi} h(r) > 2(c-b)^{\chi} h(r)$$

if c_3 is sufficiently large.

4. To complete the proof, we show $1 - V \geqslant 0.5$ for large enough R_0.
   Lemma 6.1 gives $r_1 \vee r_2 \leqslant \overline{\tau} R$, hence

$$h(r_1 \vee r_2) = c_3 \lambda^{-(r_1 \vee r_2)\chi} \geqslant c_3 \lambda^{-\chi \overline{\tau} R} = c_3 T^{-\chi \overline{\tau}}.$$

Let us estimate both parts of V:

$$\overline{p}\Delta^*/h(r_1 \vee r_2) \leqslant c_3^{-1} T^{\chi \overline{\tau} + \tau \delta - 1}, \quad \overline{w}(\Delta^*)^2/h(r_1 \vee r_2) \leqslant c_3^{-1} T^{\chi \overline{\tau} + 2\tau \delta - \omega}.$$

Conditions (6.15)-(6.16) imply that V can be made arbitrarily small if R_0 is sufficiently large.

$\square$

For the hole lower bound condition for $\mathcal{M}^*$, there is one more case to consider.

Lemma 7.11. *After choosing c_1, c_3, R_0 sufficiently large in this order, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery: then every wall of $\mathcal{M}^{k+1}$ that is also a heavy wall of $\mathcal{M}^k$ satisfies the hole lower bound, Condition 3.6.3d.*

Proof. Recall Condition 3.6.3d applied to the present case. Let $u \leqslant v < w$, a be given with $v - u \leqslant 12\Delta^*$, and define $b = a + \lceil (v-u)/2 \rceil$, $c = a + (v-u) + 1$. Assume that $Y = y$ is fixed in such a way that B is a horizontal wall of $\mathcal{M}$ with body $(v, w]$, with rank $r \geqslant R^*$ (since it is a heavy wall). Assume also that $X(a) = s$ is fixed arbitrarily. Let $E^* = E^*(u, v, w; a)$ be defined as after (5.2). We will prove

$$\mathbf{P}[E^* \mid X(a) = s, \ Y = y] \geqslant (c-b)^{\chi} h(r).$$

Suppose first $v - u \leqslant 12\Delta$. Then the fact that $\mathcal{M}^k$ is a mazery implies the same inequality with E in place of E^*. In our case, however, the event E implies E^*, as shown already at the beginning of the proof of Lemma 5.3.

It remains to check the case $v - u > 12\Delta$: for this, Lemma 5.3 says

$$\mathbf{P}(E^*) \geqslant 0.5 \wedge (1.1(c-b)^{\chi} h(r)) - U$$

with $U = 26\overline{p}\Delta^* + 338\overline{w}(\Delta^*)^2$. The operation $0.5\wedge$ can be omitted since $1.1(c-b)^{\chi}h(r) \leqslant 0.5$. Indeed, $c - b \leqslant 7\Delta^*$ implies

$$1.1(c-b)^{\chi} h(r) \leqslant 7.7c_3\lambda^{R\tau\delta\chi}\lambda^{-r\chi} = 7.7c_3\lambda^{\chi(R\tau\delta - r)}.$$

It follows from (6.15) that $\tau\delta < 1$. Since $r \geqslant R$, the right-hand side can be made < 0.5 for large enough R_0. Now

$$1.1(c-b)^{\chi} h(r) - U \geqslant (c-b)^{\chi} h(r)(1.1 - U/h(r)).$$

The part subtracted from 1.1 is less than 0.1 if R_0 is sufficiently large, by the same argument as the estimate of V at the end of the proof of Lemma 7.10. $\qquad\square$

8 The approximation lemma

The crucial combinatorial step in proving the main lemma is the following.

Lemma 8.1 (Approximation). *The reachability condition, Condition 3.8, holds for $\mathcal{M}^*$ if R_0 is sufficiently large.*

The name suggest to view our renormalization method as successive approximations: the lemma shows reachability in the absence of some less likely events (traps, walls, and uncleanness in the corners of the rectangle). The present section is taken up by the proof of this lemma. Recall that we are considering a bottom-open or left-open or closed rectangle Q with starting point u and endpoint v with

$$\text{minslope}(u,v) \geqslant \sigma^* = \sigma + \Lambda g/f.$$

Denote $u = (u_0, u_1)$, $v = (v_0, v_1)$. We require Q to be a hop of $\mathcal{M}^*$. Thus, the points u, v are clean for $\mathcal{M}^*$ in Q, and Q contains no traps or walls of $\mathcal{M}^*$. We have to show $u \rightsquigarrow v$. Without loss of generality, assume

$$Q = I_0 \times I_1 = \text{Rect}^{\varepsilon}(u, v)$$

with $|I_1| \leqslant |I_0|$, where $\varepsilon = \rightarrow, \uparrow$ or nothing.

8.1 Walls and trap covers

Let us determine the properties of the set of walls in Q.

Lemma 8.2. *Under conditions of Lemma 8.1, with the notation given in the discussion after the lemma, the following holds.*

(a) *For $d = 0, 1$, for some $n_d \geqslant 0$, there is a sequence $W_{d,1}, \ldots, W_{d,n}$ of dominant light neighbor walls of $\mathcal{M}$ separated from each other by external hops of $\mathcal{M}$ of size $\geqslant f$, and from the ends of I_d (if $n_d > 0$) by hops of $\mathcal{M}$ of size $\geqslant f/3$.*

(b) *For every (horizontal) wall W of $\mathcal{M}$ occurring in I_1, for every subinterval J of I_0 of size g such that J is at a distance $\geqslant g + 7\Delta$ from the ends of I_0, there is an outer rightward- or upward-clean hole fitting W, its endpoints at a distance of at least Δ from the endpoints of J. The same holds if we interchange vertical and horizontal.*

Proof. This is a direct consequence of Lemmas 4.3 and 4.5. □

From now on, in this proof, whenever we mention a *wall* we mean one of the walls $W_{d,i}$, and whenever we mention a trap then, unless said otherwise, we mean only traps of $\mathcal{M}$ not intersecting any of these walls. Let us limit the places where traps can appear in Q.

Definition 8.1 (Trap cover). A set of the form $I_0 \times J$ with $|J| \leqslant 4\Delta$ containing the starting point of a trap of $\mathcal{M}$ will be called a *horizontal trap cover*. Vertical trap covers are defined similarly. ⌟

In the following lemma, when we talk about the distance between two traps, we mean the distance between their starting points.

Lemma 8.3 (Trap cover). *Let T_1 be a trap of $\mathcal{M}$ contained in Q. Then there is a horizontal or vertical trap cover $U \supseteq T_1$ such that the starting point of every other trap in Q is either contained in U or is at least at a distance $f - \Delta$ from T_1. If the trap cover is vertical, it intersects none of the vertical walls $W_{0,i}$; if it is horizontal, it intersects none of the horizontal walls $W_{1,j}$.*

Proof. Let (a_1, b_1) be the starting point of T_1. If there is no trap $T_2 \subseteq Q$, with starting point (a_2, b_2), closer than $f - \Delta$ to T_1, such that $|a_2 - a_1| \geqslant 2\Delta$, then $U = [a_1 - 2\Delta, a_1 + 2\Delta] \times I_1$ will do. Otherwise, let T_2 be such a trap and let $U = I_0 \times [b_1 - 2\Delta, b_1 + 2\Delta]$. We have $|b_2 - b_1| < \Delta$, since otherwise T_1 and T_2 would form together an uncorrelated compound trap, which was excluded.

Consider now a trap $T_3 \subseteq Q$, with starting point (a_3, b_3), at a distance $< f - \Delta$ from (a_1, b_1). We will show $(a_3, b_3) \in U$. Suppose it is not so: then $|a_3 - a_1| < \Delta$, otherwise T_1 and T_3 would form an uncorrelated compound trap. Also, the distance of (a_2, b_2) and (a_3, b_3) must be at least f, since otherwise they would form an uncorrelated compound

trap. Since $|a_2 - a_1| < f - \Delta$ and $|a_3 - a_1| < \Delta$, we have $|a_2 - a_3| < f$. Therefore necessarily $|b_2 - b_3| \geqslant f$. Since $|b_2 - b_1| < \Delta$, it follows $|b_3 - b_1| > f - \Delta$, so T_3 is at a distance at least $f - \Delta$ from T_1, contrary to our assumption.

If the trap cover thus constructed is vertical and intersects some vertical wall, just decrease it so that it does not intersect any such walls. Similarly with horizontal trap covers. □

Let us measure distances from the diagonal.

Definition 8.2 (Relations to the diagonal)**.** Define, for a point $a = (a_0, a_1)$:

$$d(a) = (a_1 - u_1) - \text{slope}(u, v)(a_0 - u_0)$$

to be the distance of a above the diagonal of Q, then for $w = (x, y)$, $w' = (x', y')$:

$$\begin{aligned} d(w') - d(w) &= y' - y - \text{slope}(u, v)(x' - x), \\ |d(w') - d(w)| &\leqslant |y' - y| + |x' - x|, \end{aligned} \tag{8.1}$$

as $\text{slope}(u, v) \leqslant 1$. We define the strip

$$C^\varepsilon(u, v, h_1, h_2) = \{ w \in \text{Rect}^\varepsilon(u, v) : h_1 < d(w) \leqslant h_2 \},$$

a channel of vertical width $h_2 - h_1$ parallel to the diagonal of $\text{Rect}^\varepsilon(u, v)$. ⌟

Lemma 8.4. *Assume that points u, v are clean for $\mathcal{M}$ in $Q = \text{Rect}^\varepsilon(u, v)$, with*

$$1 \geqslant \text{slope}(u, v) \geqslant \sigma + 4g/f.$$

If $C = C^\varepsilon(u, v, -g, g)$ contains no traps or walls of $\mathcal{M}$ then $u \rightsquigarrow v$.

Proof. If $|I_0| < g$ then there is no trap or wall in Q, therefore Q is a hop and we are done via Condition 3.8 for $\mathcal{M}$. Suppose $|I_0| \geqslant g$. Let

$$n = \left\lceil \frac{|I_0|}{0.9g} \right\rceil, \quad h = \frac{|I_0|}{n}.$$

Then $g/2 \leqslant h \leqslant 0.9g$. Indeed, the second inequality is immediate. For the first one, if $n \leqslant 2$, we have $g \leqslant |I_0| = nh \leqslant 2h$, and for $n \geqslant 3$:

$$\frac{|I_0|}{0.9g} \geqslant n - 1,$$
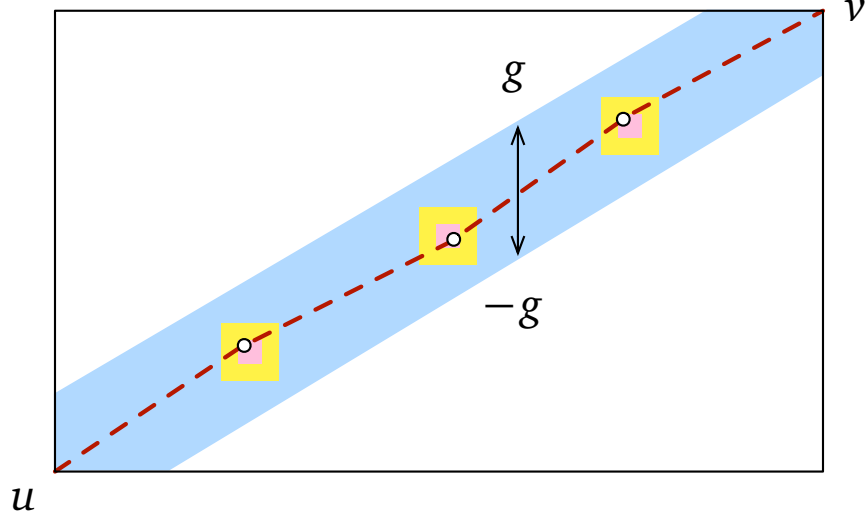$$|I_0|/n \geqslant (1 - 1/n)0.9g \geqslant 0.6g.$$

Figure 23: To Lemma 8.4. The shaded area is $C^{\varepsilon}(u,v,-g,g)$.

For $i = 1, 2, \ldots, n-1$, let

$$a_i = u_0 + ih, \quad b_i = u_1 + ih \cdot \text{slope}(u,v), \quad w_i = (a_i, b_i), \quad S_i = w_i + [-\Delta, 2\Delta]^2.$$

Let us show $S_i \subseteq C$. For all elements w of S_i, we have $|d(w)| \leqslant 3\Delta$, and we know $3\Delta < g$ from (4.1). To see $S_i \subseteq \text{Rect}^{\varepsilon}(u,v)$, we need (from the worst case $i = n-1$) $\text{slope}(u,v)h > 2\Delta$. Using (4.1) and the assumptions of the lemma:

$$\frac{2\Delta}{h} \leqslant \frac{2\Delta}{g/2} = 4\Delta/g \leqslant 4g/f \leqslant \text{slope}(u,v).$$

By Remark 3.7.1, there is a clean point $w'_i = (a'_i, b'_i)$ in the middle third $w_i + [0,\Delta]^2$ of S_i. Let $w'_0 = u$, $w'_n = v$. By their definition, each rectangle $\text{Rect}^{\varepsilon}(w'_i, w'_{i+1})$ has size $< 0.9g + \Delta < g - \Delta$, hence falls into the channel C and is consequently trap-free.

Let us show $\text{minslope}(w'_i, w'_{i+1}) \geqslant \sigma$: this will imply $w'_i \rightsquigarrow w'_{i+1}$. It is sufficient to show $\text{slope}(w'_i, w'_{i+1}) \geqslant \sigma$. Note that, using $s = \text{slope}(u,v)$, the distance from w'_i to w'_{i+1} is at most $h + \Delta$ in the x coordinate and at least $sh - \Delta$ in the y coordinate. We have

$$\text{slope}(w'_i, w'_{i+1}) \geqslant \frac{sh - \Delta}{h + \Delta} = s - \frac{\Delta + s\Delta}{h + \Delta} \geqslant s - \frac{2\Delta}{g/2} \geqslant s - 4g/f \geqslant \sigma,$$

where the last inequality used again (4.1). $\qquad\square$

We introduce particular strips around the diagonal.

Definition 8.3. Let

$$H = 12, \tag{8.2}$$

$$C = C^{\varepsilon}(u, v, -3Hg, 3Hg). \tag{8.3}$$

⌟

Inequalities (4.1) imply

$$\Lambda \geqslant 33H + 7. \tag{8.4}$$

Let us introduce the system of walls and trap covers we will have to overcome.

Definition 8.4. Let us define a sequence of trap covers $U_1, U_2, \ldots$ as follows. If some trap T_1 is in C, then let U_1 be a (horizontal or vertical) trap cover covering it according to Lemma 8.3. If U_i has been defined already and there is a trap T_{i+1} in C not covered by $\bigcup_{j \leqslant i} U_j$ then let U_{i+1} be a trap cover covering this new trap. To each trap cover U_i we assign a real number a_i as follows. Let (a_i, a_i') be the intersection of the diagonal of Q and the left or bottom edge of U_i (if U_i is vertical or horizontal respectively). Let (b_i, b_i') be the intersection of the diagonal and the left edge of the vertical wall $W_{0,i}$ introduced in Lemma 8.2, and let (c_i', c_i) be the intersection of the diagonal and the bottom edge of the horizontal wall $W_{1,i}$. Let us define the finite set

$$\{s_1, s_2, \ldots\} = \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\} \cup \{c_1', c_2', \ldots\}$$

where $s_i \leqslant s_{i+1}$.

We will call the objects (trap covers or walls) belonging to the points s_i our *obstacles*.

⌟

Lemma 8.5. *If s_i, s_j belong to the same obstacle category among the three (horizontal wall, vertical wall, trap cover), then $|s_i - s_j| \geqslant 3f/4$.*

It follows that for every i at least one of the three numbers $(s_{i+1} - s_i)$, $(s_{i+2} - s_{i+1})$, $(s_{i+3} - s_{i+2})$ is larger than $f/4$.

Proof. If both s_i and s_j belong to walls of the same orientation then they are farther than f from each other, since the walls from which they come are at least f apart. (For the numbers c_i', this uses slope$(u, v) \leqslant 1$.)

Suppose that both belong to the set $\{a_1, a_2, \ldots\}$, say they are $a_1 \leqslant a_2$, coming from U_1 and U_2. Let (x_j, y_j) be the starting point of some trap T_j in $U_j \cap C$ (with C defined in (8.3)). If U_j is vertical then $|x_j - a_j| \leqslant 4\Delta$, and $|y_j - a_j'| \leqslant 3Hg + 4\Delta$. If U_j is horizontal then $|x_j - a_j| \leqslant (3Hg + 4\Delta)/\text{slope}(u, v)$, and $|y_j - a_j'| \leqslant 4\Delta$.

Suppose that $a_2 - a_1 \leqslant 0.75f$, then also $a'_2 - a'_1 \leqslant 0.75f$. From the above estimates it follows that

$$|x_2 - x_1| \vee |y_2 - y_1| \leqslant 0.75f + (2 \cdot 3Hg + 8\Delta)/\text{slope}(u,v) \leqslant 0.75f + 2.1 \cdot 3Hf/\Lambda$$
$$= f - 0.05f - (0.2 - 2.1 \cdot 3H/\Lambda)f \leqslant f - 0.05f < f - \Delta,$$

where we used $\text{slope}(u,v) \geqslant \sigma^* \geqslant \Lambda g/f$, (8.4) and $\Delta < 0.05f$ which follows from (4.1). But this would mean that the starting points of the traps T_j are closer than $f - \Delta$, in contradiction to Lemma 8.3. $\qquad\square$

8.2 Passing through the obstacles

The remark after Lemma 8.5 allows us to break up the sequence of obstacles into groups of size at most three, which can be dealt with separately. This leads to the following, weaker, form of the Approximation Lemma, which still takes a lot of sweat (case distinctions) to prove:

Lemma 8.6. *Assume* $\text{slope}(u,v) \leqslant 1$, $(\sigma^* =)\sigma + \Lambda g/f < 1/2$, *and let* u,v *be points with*

$$\sigma + (\Lambda - 1)g/f \leqslant \text{slope}(u,v). \tag{8.5}$$

Assume that the set $\{s_1, s_2, \ldots\}$ *defined above consists of at most three elements, with the consecutive elements less than* $f/4$ *apart. Assume also*

$$v_0 - s_i, \ s_i - u_0 \geqslant 0.1f. \tag{8.6}$$

Then if $\text{Rect}^{\rightarrow}(u,v)$ *or* $\text{Rect}^{\uparrow}(u,v)$ *is a hop of* $\mathcal{M}^*$ *then* $u \rightsquigarrow v$.

Proof. We can assume without loss of generality that there are indeed three points s_1, s_2, s_3. By Lemma 8.5, they must then come from three obstacles of different categories: $\{s_1, s_2, s_3\} = \{a, b, c'\}$ where b comes from a vertical wall, c' from a horizontal wall, and a from a trap cover. There is a number of cases: we illustrated the most complex one in Figure 24.

If the index $i \in \{1, 2, 3\}$ of a trap cover is adjacent to the index of a wall of the same orientation, then this pair will be called a *parallel pair*. A parallel pair is either horizontal or vertical. It will be called a *trap-wall pair* if the trap cover comes first, and the *wall-trap pair* if the wall comes first. If $s_i - s_{i-1} < 1.1g$ for a vertical pair or $(s_i - s_{i-1})\text{slope}(u,v) < 1.1g$ for a horizontal pair then we say that the pair is *bound*. Thus, a pair is bound if the distance between the starting edges of its obstacles is less than $1.1g$. A bound pair (if exists) is more difficult to pass, therefore its crossing points will be chosen in a coordinated way, starting from the trap cover side.

We will call an obstacle i *free*, if it is not part of a bound pair. Consider the three disjoint channels

$$C(u,v,K,K+2Hg), \text{ for } K = -3Hg, -Hg, Hg.$$

The three lines (bottom or left edges) of the trap covers or walls corresponding to s_1, s_2, s_3 can intersect in at most two places, so at least one of the above channels does not contain such an intersection. Let K belong to such a channel. For $i \in \{1,2,3\}$, we shall choose points

$$w_i = (x_i, y_i), \quad w'_i = (x'_i, y'_i), \quad w''_i = (x''_i, y''_i)$$

in the channel $C(u, v, K + 2g, K + (2H - 2)g)$ in such a way that w_i is on the (horizontal or vertical) line corresponding to s_i. Not all these points will be defined. The points w_i, w'_i, w''_i will always be defined if i is free. Their role in this case is the following: w'_i and w''_i are points on the two sides of the trap cover or wall with $w'_i \rightsquigarrow w''_i$. Point w_i will be on the starting edge of the obstacle, and it will direct us in locating w'_i, w''_i. However, w_i by itself will not determine w'_i, w''_i: other factors are involved. Correspondingly, for each free obstacle, we will distinguish a *forward* way of crossing (when $d(w_i)$ will be made equal to $d(w''_j)$ for some $j < i$) and a *backward* way of crossing (when $d(w_i)$ will be made equal to $d(w'_j)$ for some $j > i$).

Part 1 of the proof collects some estimates on crossing free obstacles. Part 2 proves the lemma in the cases when all obstacles are free. In the case when there is a horizontal trap-wall pair, finding the wall crossing point requires more freedom than finding the trap cover crossing point. To be able do it first, all obstacles are crossed in the backward direction. Part 3 collects estimates for different cases of crossing of a bound pair. Finally, part 4 proves the lemma also for the cases where a bound pair exists.

1. Consider crossing a free obstacle s_i, assuming that w_i has been defined already.

   There are cases corresponding to whether the obstacle is a trap cover or a wall, and whether it is vertical or horizontal. Backward crossings are quite similar to forward ones.

   1.1. Consider crossing a trap cover s_i.

      1.1.1. Assume that the trap cover is vertical.

      Consider crossing a vertical trap cover forwards. Recall

      $$l_1 = 7\Delta, \ L_1 = 4l_1.$$

      Let us apply Lemma 4.6 to vertical correlated traps $J \times I'$, with $J = [x_i, x_i + 5\Delta]$, $I' = [y_i, y_i + L_1]$. The lemma is applicable since $w_i \in C(u, v, K + 2g, K + (2H - 2)g)$ implies $u_1 < y_i - L_1 - l_1 < y_i + 2L_1 + l_1 < v_1$. Indeed, formula (8.6) implies, using (8.5):

      $$y_i > u_1 + 0.1 f \operatorname{slope}(u, v) - 3Hg \geqslant u_1 + (0.1(\Lambda - 1) - 3H)g \geqslant u_1 + 13g.$$

This implies $u_1 < y_i - L_1 - l_1$, using $\Delta \ll g$ from (4.1). The inequality about v_1 is similar, using the other inequality of (8.6). It implies that there is a region $[x_i, x_i + 5\Delta] \times [y, y + l_1]$ containing no traps, with $[y, y + l_1) \subseteq [y_i, y_i + L_1)$. Thus, there is a y in $[y_i, y_i + L_1 - l_1)$ such that $[x_i, x_i + 5\Delta] \times [y, y + l_1]$ contains no traps. (In the present proof, all other arguments finding a region with no traps in trap covers are analogous, so we will not mention Lemma 4.6 explicitly again.) However, all traps must start in a trap cover, so the region $[x_i - 2\Delta, x_i + 6\Delta] \times [y, y + l_1]$ contains no trap either. Thus there are clean points $w_i' \in (x_i - \Delta, y + \Delta) + [0, \Delta]^2$ and $w_i'' \in (x_i + 4\Delta, y + 5\Delta) + [0, \Delta]^2$. Note that $\text{minslope}(w_i', w_i'') \geqslant 1/2$, giving $w_i' \rightsquigarrow w_i''$. We have, using (8.1) and $\text{slope}(u, v) \leqslant 1$:

$$
\begin{aligned}
-\Delta \leqslant x_i' - x_i \leqslant 0, && 4\Delta \leqslant x_i'' - x_i \leqslant 5\Delta, \\
\Delta \leqslant y_i' - y_i \leqslant 23\Delta, && 5\Delta \leqslant y_i'' - y_i \leqslant 27\Delta, \\
\Delta \leqslant d(w_i') - d(w_i) \leqslant 24\Delta, && 0 \leqslant d(w_i'') - d(w_i) \leqslant 27\Delta. && (8.7)
\end{aligned}
$$

Consider crossing a vertical trap cover backwards. There is a y in $[y_i - L_1, y_i - l_1)$ such that the region $[x_i - 2\Delta, x_i + 6\Delta] \times [y, y + l_1]$ contains no trap. There are clean points $w_i' \in (x_i - \Delta, y + \Delta) + [0, \Delta]^2$ and $w_i'' \in (x_i + 4\Delta, y + 5\Delta) + [0, \Delta]^2$ with $\text{minslope}(w_i', w_i'') \geqslant 1/2$, giving $w_i' \rightsquigarrow w_i''$. We have

$$
\begin{aligned}
-\Delta \leqslant x_i' - x_i \leqslant 0, && 4\Delta \leqslant x_i'' - x_i \leqslant 5\Delta, \\
-27\Delta \leqslant y_i' - y_i \leqslant -5\Delta, && -23\Delta \leqslant y_i'' - y_i \leqslant -\Delta, \\
-27\Delta \leqslant d(w_i') - d(w_i) \leqslant -4\Delta, && -28\Delta \leqslant d(w_i'') - d(w_i) \leqslant -\Delta.
\end{aligned}
$$

1.1.2. Assume that the trap cover is horizontal.

Consider crossing a horizontal trap cover forwards. There is an x in $[x_i - L_1, x_i - l_1)$ such that $[x, x + l_1] \times [y_i - 2\Delta, y_i + 6\Delta]$ contains no trap. Thus there are clean points $w_i' \in (x + \Delta, y_i - \Delta) + [0, \Delta]^2$ and $w_i'' \in (x + 5\Delta, y_i + 4\Delta) + [0, \Delta]^2$ with $w_i' \rightsquigarrow w_i''$. We have similarly to the above, the inequalities

$$
\begin{aligned}
-27\Delta \leqslant x_i' - x_i \leqslant -5\Delta, && -23\Delta \leqslant x_i'' - x_i \leqslant -\Delta, \\
-\Delta \leqslant y_i' - y_i \leqslant 0, && 4\Delta \leqslant y_i'' - y_i \leqslant 5\Delta, \\
-\Delta \leqslant d(w_i') - d(w_i) \leqslant 27\Delta, && 4\Delta \leqslant d(w_i'') - d(w_i) \leqslant 28\Delta.
\end{aligned}
$$

Consider crossing a horizontal trap cover backwards. There is an x in $[x_i, x_i + L_1 - l_1)$ such that $[x, x + l_1] \times [y_i - 2\Delta, y_i + 6\Delta]$ contains no trap. Thus there are clean points $w_i' \in (x + \Delta, y_i - \Delta) + [0, \Delta]^2$ and $w_i'' \in (x + 5\Delta, y_i + 4\Delta) + [0, \Delta]^2$ with $w_i' \rightsquigarrow w_i''$.

We again have

$$\Delta \leqslant x_i' - x_i \leqslant 23\Delta, \qquad\qquad 5\Delta \leqslant x_i'' - x_i \leqslant 27\Delta,$$
$$-\Delta \leqslant y_i' - y_i \leqslant 0, \qquad\qquad 4\Delta \leqslant y_i'' - y_i \leqslant 5\Delta,$$
$$-24\Delta \leqslant d(w_i') - d(w_i) \leqslant 0, \qquad -23\Delta \leqslant d(w_i'') - d(w_i) \leqslant 5\Delta.$$

1.2. Consider crossing a wall.

1.2.1. Assume that the wall is vertical.

Consider crossing a vertical wall forwards. Let us apply Lemma 8.2(b), with $I' = [y_i, y_i + g]$. The lemma is applicable since $w_i \in C(u, v, K + 2g, K + (2H - 2)g)$ implies $u_1 \leqslant y_i - g - 7\Delta < y_i + 2g + 7\Delta < v_1$. It implies that our wall contains an outer upward-clean hole $(y_i', y_i''] \subseteq y_i + (\Delta, g - \Delta]$ passing through it. (In the present proof, all other arguments finding a hole through walls are analogous, so we will not mention Lemma 8.2(b) explicitly again.) Let $w_i' = (x_i, y_i')$, and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$x_i' = x_i, \qquad\qquad 0 \leqslant x_i'' - x_i \leqslant \Delta,$$
$$\Delta \leqslant y_i' - y_i \leqslant y_i'' - y_i \leqslant g - \Delta, \qquad\qquad (8.8)$$
$$\Delta \leqslant d(w_i') - d(w_i) \leqslant g - \Delta, \qquad 0 \leqslant d(w_i'') - d(w_i) \leqslant g - \Delta.$$

Consider crossing a vertical wall backwards. This wall contains an outer upward-clean hole $(y_i', y_i''] \subseteq y_i + (-g + \Delta, -\Delta]$ passing through it. Let $w_i' = (x_i, y_i')$, and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$x_i' = x_i, \qquad\qquad 0 \leqslant x_i'' - x_i \leqslant \Delta,$$
$$-g + \Delta \leqslant y_i' - y_i \leqslant y_i'' - y_i \leqslant -\Delta,$$
$$-g + \Delta \leqslant d(w_i') - d(w_i) \leqslant -\Delta, \qquad -g \leqslant d(w_i'') - d(w_i) \leqslant -\Delta.$$

1.2.2. Assume that the wall is horizontal.

Consider crossing a horizontal wall forwards. Similarly to above, this wall contains an outer rightwards clean hole $(x_i', x_i''] \subseteq x_i + (-g + \Delta, -\Delta]$ passing through it. Let $w_i' = (x_i', y_i)$ and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$-g + \Delta \leqslant x_i' - x_i \leqslant x_i'' - x_i \leqslant -\Delta,$$
$$y_i' = y_i, \qquad\qquad 0 \leqslant y_i'' - y_i \leqslant \Delta,$$
$$0 \leqslant d(w_i') - d(w_i) \leqslant g - \Delta, \qquad 0 \leqslant d(w_i'') - d(w_i) \leqslant g.$$

Consider crossing a horizontal wall backwards. This wall contains an outer rightward-clean hole $(x_i', x_i''] \subseteq x_i + (\Delta, g - \Delta]$ passing through it. Let $w_i' = (x_i', y_i)$ and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$\Delta \leqslant x_i' - x_i \leqslant x_i'' - x_i \leqslant g - \Delta,$$

$$y_i' = y_i, \qquad\qquad\qquad 0 \leqslant y_i'' - y_i \leqslant \Delta,$$

$$-g + \Delta \leqslant d(w_i') - d(w_i) \leqslant 0, \qquad -g + \Delta \leqslant d(w_i'') - d(w_i) \leqslant \Delta.$$

1.3. Let us summarize some of the inequalities proved above, with

$$D = d(w) - d(w_i),$$

where w is equal to any one of the defined w_i', w_i''.

$$
\begin{aligned}
\text{trap covers going forwards:} &\quad -\Delta \leqslant D \leqslant 28\Delta,\\
\text{trap covers going backwards:} &\quad -28\Delta \leqslant D \leqslant 5\Delta,\\
\text{walls going forwards:} &\quad 0 \leqslant D \leqslant g,\\
\text{walls going backwards:} &\quad -g \leqslant D \leqslant \Delta.
\end{aligned}
\tag{8.9}
$$

Further

$$
\begin{aligned}
\text{vertical obstacles:} &\quad -\Delta \leqslant x_i' - x_i \leqslant x_i'' - x_i \leqslant 5\Delta,\\
\text{horizontal obstacles:} &\quad -\Delta \leqslant y_i' - y_i \leqslant y_i'' - y_i \leqslant 5\Delta,\\
\text{horizontal trap covers:} &\quad -27\Delta \leqslant x_i' - x_i \leqslant x_i'' - x_i \leqslant 27\Delta,\\
\text{horizontal walls:} &\quad -g + \Delta \leqslant x_i' - x_i \leqslant x_i'' - x_i \leqslant g - \Delta.
\end{aligned}
\tag{8.10}
$$

Let $\pi_x w, \pi_y w \in \mathbb{R}$ be the X and Y projections of point w, and let $\pi_i w \in \mathbb{R}^2$ be the projection of point w onto the (horizontal or vertical) line corresponding to s_i. Then the above inequalities and (8.1) imply, with $\hat{w} = \pi_i(w) - w$ where $w = w_i', w_i''$:

$$-5\Delta \leqslant d(\hat{w}), \pi_x \hat{w}, \pi_y \hat{w} \leqslant 5\Delta. \tag{8.11}$$

Indeed, for example for $\pi_x \hat{w}$, for a horizontal wall or trap cover, the difference between the projection of w and the projection of the projection onto the wall is 0. For a vertical wall or trap cover, one of w_i', w_i'' is at a distance at most Δ from the line corresponding to s_i, the other one is inside the trap cover or within Δ of the other side of the wall, therefore is at most at a distance 5Δ. The inequality for $d(\hat{w})$ follows from this, (8.1) and that only one of the coordinates changes.

For crossing a wall we have

$$-\Delta \leqslant d(w_i'') - d(w_i') \leqslant \Delta. \tag{8.12}$$

Indeed, w_i' and w_i'' are the two opposite corners of a hole through a wall of width $\leqslant \Delta$.

2. Assume that there is no bound pair: then $u \rightsquigarrow v$.

Proof.

   2.1. Assume that there is no horizontal trap-wall pair.

We choose w_1 with $d(w_1) = K + 3g$. For each $i > 1$ we choose w_i with $d(w_i) = d(w''_{i-1})$, and we cross each obstacle in the forward direction.

   2.1.1. For all i, the points we created are inside a certain channel:

$$d(w), d(\pi_i w) \in K + [2g, (2H-2)g], \tag{8.13}$$

where w is any one of w_i, w'_i, w''_i.

Proof. It follows from (8.9) that, for $w \in \{w'_i, w''_i\}$:

$$-\Delta \leqslant d(w) - d(w_i) \leqslant 28\Delta \qquad \text{for trap covers,} \tag{8.14}$$
$$0 \leqslant d(w) - d(w_i) \leqslant g \qquad \text{for walls.} \tag{8.15}$$

To estimate for example $d(w_3)$, we write

$$d(w''_3) = d(w_1) + d(w_2) - d(w_1) + d(w_3) - d(w_2) + d(w''_3) - d(w_3)$$
$$= K + 3g + (d(w''_1) - d(w_1)) + (d(w''_2) - d(w_2)) + (d(w''_3) - d(w_3)).$$

Since we have two walls and a trap cover, adding up $d(w_1) = K + 3g$, then (8.14) once and (8.15) twice gives, similarly also for w_i, w'_i, w''_i:

$$K + 3g - \Delta \leqslant d(w_i), d(w'_i), d(w''_i) \leqslant K + 5g + 28\Delta.$$

Then (8.11) implies $K + 3g - 6\Delta \leqslant d(w), d(\pi_i w) \leqslant K + 5g + 33\Delta$, where w is any one of w_i, w'_i, w''_i.

   2.1.2. For $i = 2, 3$ the inequality $x'_i - x''_{i-1} \geqslant g$ holds.

Proof. If s_{i-1}, s_i come from trap covers or walls in different orientations, then the intersection of their lines lies outside $C(u, v, K, K + 2Hg)$. Part 2.1.1 above says

$$\pi_i w'_i, \pi_{i-1} w''_{i-1} \in C(u, v, K + 2g, K + (2H-2)g).$$

Now if two segments A, B of different orientation (horizontal and vertical) are contained in $C(u, v, K + 2g, K + (2H-2)g)$ and are such that A is to the left of B and their lines intersect outside $C(u, v, K, K + 2Hg)$, then for any points $a \in A$, $b \in B$ we have $\pi_x(b - a) \geqslant 2g$. In particular, $\pi_x \pi_i w'_i - \pi_x \pi_{i-1} w''_{i-1} \geqslant 2g$. Using (8.11) we get:

$$x'_i - x''_{i-1} = \pi_x \pi_i w'_i - \pi_x \pi_{i-1} w''_{i-1} + (x'_i - \pi_x \pi_i w'_i) - (x''_{i-1} - \pi_x \pi_{i-1} w''_{i-1})$$
$$\geqslant 2g - 10\Delta.$$

If s_{i-1}, s_i come from a vertical trap-wall or wall-trap pair, then freeness implies that elements of this pair are farther than $1.1g$ from each other. Then we get $x_i' - x_{i-1}'' \geqslant 1.1g - 5\Delta > g$, as can be seen by considering the two possible orders: trap-wall, wall-trap separately.

If s_{i-1}, s_i come from a horizontal wall-trap pair then, using $\text{slope}(u,v) \leqslant 1$ and (8.10):

$$x_i - x_{i-1}'' = (y_i - y_{i-1}'')/\text{slope}(u,v) \geqslant y_i - y_{i-1}'' = y_i - y_{i-1} - (y_{i-1}'' - y_{i-1})$$
$$\geqslant 1.1g - 5\Delta.$$

By (8.10) we have $x_i' - x_i \geqslant -27\Delta$. Combination with the above estimate and (4.1) gives $x_i' - x_{i-1}'' \geqslant 1.1g - 32\Delta > g$ due to (4.1).

2.1.3. Let us show $u \rightsquigarrow v$.

Proof. We defined all w_i', w_i'' as clean points with $w_i' \rightsquigarrow w_i''$ and the sets $C^{\varepsilon_i}(w_i'', w_{i+1}', -g, g)$ are trap-free, where $\varepsilon_i = \rightarrow$ for horizontal walls, $\uparrow$ for vertical walls and nothing for trap covers. If we are able to show that the minslopes between the endpoints of these sets are lower-bounded by $\sigma + 4g/f$ then Lemma 8.4 will imply $u \rightsquigarrow v$. For this, it will be sufficient to show that the slopes are lower-bounded by $\sigma + 4g/f$ and upper-bounded by 2, since (4.1) implies $1/(\sigma + 4g/f) > 2$. We will make use of the following relation for arbitrary $a = (a_0, a_1)$, $b = (b_0, b_1)$:

$$\text{slope}(a,b) = \text{slope}(u,v) + \frac{d(b) - d(a)}{b_0 - a_0}. \tag{8.16}$$

Let us bound the end slopes first. We have

$$\text{slope}(u, w_1') = \text{slope}(u,v) + \frac{d(w_1')}{x_1' - u_0}, \quad \text{slope}(w_3'', v) = \text{slope}(u,v) - \frac{d(w_3'')}{v_0 - x_3''}. \tag{8.17}$$

Inequalities (8.13) yield the bounds $K + 2g \leqslant d(w_1'), d(w_3'') \leqslant K + (2H-2)g$, hence the restriction of K to $[-3Hg, Hg]$ implies $|d(w_1')|, |d(w_3'')| \leqslant (3H-2)g$. By (8.6) and (4.1), using also (8.10):

$$(x_1' - u_0), \ (v_0 - x_3'') \geqslant 0.1f - g + \Delta \geqslant f/11.$$

This shows

$$|\text{slope}(u, w_1') - \text{slope}(u,v)|, \ |\text{slope}(w_3'', v) - \text{slope}(u,v)| \leqslant 33Hg/f. \tag{8.18}$$

Using $1 \geqslant \text{slope}(u,v) \geqslant \sigma + (\Lambda - 1)g/f$:

$$2 > 1 + 33Hg/f \geqslant \text{slope}(u, w_1'), \ \text{slope}(w_3'', v)$$
$$\geqslant \sigma + (\Lambda - 1 - 33H)g/f \geqslant \sigma + 4g/f$$

by (4.1) and (8.4).

Let us proceed to lower-bounding $\text{minslope}(w''_{i-1}, w'_i)$ for $i = 2, 3$. Using (8.16):

$$\text{slope}(w''_{i-1}, w'_i) = \text{slope}(u, v) + \frac{d(w'_i) - d(w''_{i-1})}{x'_i - x''_{i-1}}. \tag{8.19}$$

Using (8.9) and Part 2.1.2 above:

$$-5\Delta/g \leqslant \text{slope}(w''_{i-1}, w'_i) - \text{slope}(u, v) \leqslant 1.$$

(For just the case of crossing forwards, we could have used the lower bound $-\Delta$, but a similar application below to backward crossing requires -5Δ.) By the conditions of the lemma and (4.1):

$$\text{slope}(w''_{i-1}, w'_i) \geqslant \text{slope}(u, v) - \Delta/g \geqslant \text{slope}(u, v) - g/f$$
$$\geqslant \sigma + (\Lambda - 1)g/f \geqslant \sigma + 4g/f,$$
$$\text{slope}(w''_{i-1}, w'_i) \leqslant \text{slope}(u, v) + 1 < 2.$$

2.2. Assume now that there is a horizontal trap-wall pair.

What has been done in part 2.1 will be repeated, going backwards through $i = 3, 2, 1$ rather than forwards. Thus, we choose w_3 with $d(w_3) = d(v) + K + (2H - 3)g$. Assuming that w'_{i+1} has been chosen already, we choose w_i with $d(w_i) = d(w'_{i+1})$, and we cross each obstacle in the backward direction.

It follows from (8.9), that for all i we have (8.13) again.

2.2.1. The inequality $x'_i - x''_{i-1} \geqslant g$ holds.

Proof. If s_{i-1} and s_i come from trap covers or walls in different orientations, then we can reason as in Part 2.1.2. If s_{i-1}, s_i come from a horizontal trap-wall pair then $d(w'_i) = d(w_{i-1})$ and $y'_i = y_i$ imply

$$x'_i - x_{i-1} = (y_i - y_{i-1})/\text{slope}(u, v) \geqslant 1.1g.$$

By (8.10), $x_{i-1} - x''_{i-1} \geqslant -27\Delta$, hence $x'_i - x''_{i-1} \geqslant 1.1g - 27\Delta \geqslant g$.

2.2.2. We have $u \rightsquigarrow v$.

Proof. Let us estimate the minslopes as in part 2.1.3, using (8.17) again. The estimates for $\text{slope}(u, w'_1)$ and $\text{slope}(w''_3, v)$ are as before. We conclude for $\text{minslope}(w''_{i-1}, w'_i)$ using (8.19) and Part 2.2.1 just as we did in Part 2.1.3 above.

3. Consider crossing a bound pair.

A bound trap-wall or wall-trap pair will be crossed with an approximate slope 1 rather than slope(u,v). We first find a big enough (size $l_2 = g'$) hole in the trap cover, and then locate a hole in the wall that allows to pass, with slope 1, through the big hole of the trap cover. There are cases according to whether we have a trap-wall pair or a wall-trap pair, and whether it is vertical or horizontal.

We will prove

$$-1.2g \leqslant d(w) - d(w_i) \leqslant 10g, \tag{8.20}$$

for w equal to any one of the defined w'_j, w''_j where $j \in \{i-1, i\}$ (wall-trap) and $j \in \{i, i+1\}$ (trap-wall). The inequalities (8.10) and (8.11) will hold also if the obstacle is within a bound pair.

3.1. Consider crossing a trap-wall pair $(i, i+1)$, assuming that w_i has been defined already.

    3.1.1. Assume that the trap-wall pair is vertical.

    Let us apply Lemma 4.6 with $j = 2$, so taking $l_2 = g' = 2.2g$, $L_2 = 4l_2$ similarly to the forward crossing in Part 1.1.1. We find a $y^{(1)}$ in $[y_i, y_i + L_2 - l_2)$ such that the region $[x_i - 3\Delta, x_{i+1}] \times [y^{(1)}, y^{(1)} + l_2]$ contains no trap.

    Let w_{i+1} be defined by $y_{i+1} = y^{(1)} + (s_{i+1} - s_i) + 2\Delta$. Thus, it is the point on the left edge of the wall if we intersect it with a slope 1 line from $(s_i, y^{(1)})$ and then move up 2Δ. Similarly to the forward crossing in Part 1.2.1, the wall starting at s_{i+1} is passed through by an outer upward-clean hole $(y'_{i+1}, y''_{i+1}] \subseteq y_{i+1} + (\Delta, g - \Delta]$. Let $w'_{i+1} = (x_{i+1}, y'_{i+1})$, and let $w''_{i+1} = (x''_{i+1}, y''_{i+1})$ be the point on the other side of the wall reachable from w'_{i+1}.

    Let $w = (x_i, y^{(2)})$ be defined by $y^{(2)} = y'_{i+1} - (s_{i+1} - s_i)$. Thus, it is the point on the left edge of the trap cover if we intersect it with a slope 1 line from w'_{i+1}. Then $3\Delta \leqslant y^{(2)} - y^{(1)} \leqslant g + \Delta$, therefore $w + [-3\Delta, 0]^2$ contains no trap, and there is a clean point $w'_i \in w + [-2\Delta, -\Delta]^2$. (Point w''_i is not needed.)

    Let us estimate $d(w'_i) - d(w_i)$ and $d(w''_{i+1}) - d(w_i)$. Recalling $l_2 = g'$, $L_2 = 4g'$ gives

$$3\Delta \leqslant y^{(2)} - y_i = d(w) - d(w_i) \leqslant 3g' + g + \Delta, \tag{8.21}$$
$$-2\Delta \leqslant d(w'_i) - d(w) \leqslant 2\Delta,$$
$$0 \leqslant d(w'_{i+1}) - d(w) \leqslant 1.1g.$$

Combining the last inequalities with (8.21) gives

$$\Delta \leqslant d(w'_i) - d(w_i) \leqslant 3g' + g + 3\Delta,$$
$$3\Delta \leqslant d(w'_{i+1}) - d(w_i) \leqslant 3g' + 2.1g + \Delta.$$

These prove (8.20) for our case if we also invoke (8.12) to infer about w''_{i+1}. Let us show $w'_i \rightsquigarrow w'_{i+1}$. We apply Condition 3.8. It is easy to see that the rectangle $\mathrm{Rect}^{\varepsilon_i}(w'_i, w'_{i+1})$ is trap-free. Consider the slope condition. We have $1/2 \leqslant \mathrm{slope}(w'_i, w) \leqslant 2$, and $\mathrm{slope}(w, w'_{i+1}) = 1$. Hence, $1/2 \leqslant \mathrm{slope}(w'_i, w'_{i+1}) \leqslant 2$, which implies $\mathrm{minslope}(w'_i, w'_{i+1}) \geqslant \sigma + 4g/f$ as needed.

3.1.2. Assume that the trap-wall pair is horizontal.

There is an $x^{(1)}$ in $[x_i - L_2, x_i - l_2)$ such that the region $[x^{(1)}, x^{(1)} + l_2] \times [y_i - 3\Delta, y_{i+1}]$ contains no trap. Let w_{i+1} be defined by $x_{i+1} = x^{(1)} + (s_{i+1} - s_i)\mathrm{slope}(u, v) + 2\Delta$. The wall starting at s_{i+1} contains an outer rightward-clean hole $(x'_{i+1}, x''_{i+1}] \subseteq x_{i+1} + (\Delta, g - \Delta]$ passing through it. Let $w'_{i+1} = (x'_{i+1}, y_{i+1})$, and let $w''_{i+1} = (x''_{i+1}, y''_{i+1})$ be the point on the other side of the wall reachable from w'_{i+1}. Let $w = (x^{(2)}, y_i)$ be defined by $x^{(2)} = x'_{i+1} - (s_{i+1} - s_i)\mathrm{slope}(u, v)$. Then there is a clean point $w'_i \in w + [-2\Delta, -\Delta]^2$ as before. We have

$$3\Delta \leqslant x^{(2)} - x^{(1)} \leqslant g + \Delta, \quad -4g' + 3\Delta \leqslant x^{(2)} - x_i \leqslant -g' + g + \Delta.$$

Using this, $d(w) - d(w_i) = -(x^{(2)} - x_i)\mathrm{slope}(u, v)$ and $\mathrm{slope}(u, v) \leqslant 1$ gives

$$0 \leqslant d(w) - d(w_i) \leqslant 4g' - 3\Delta.$$

As in Part 3.1.1, this gives

$$-2\Delta \leqslant d(w'_i) - d(w_i) \leqslant 4g' - \Delta,$$
$$0 \leqslant d(w'_{i+1}) - d(w_i) \leqslant 4g' + 1.1g - 3\Delta.$$

These and (8.12) prove (8.20) for our case. We show $w'_i \rightsquigarrow w'_{i+1}$ similarly to Part 3.1.1.

3.2. Consider crossing a wall-trap pair $(i - 1, i)$, assuming that w_i has been defined already.

3.2.1. Assume that the wall-trap pair is vertical.

This part is somewhat similar to Part 3.1.1, and is illustrated in Figure 24. There is a $y^{(1)}$ in $[y_i + l_2, y_i + L_2)$ such that the region $[x_i, x_i + 6\Delta] \times [y^{(1)} - l_2, y^{(1)}]$ contains no trap. Let w_{i-1} be defined by $y_{i-1} = y^{(1)} - (s_i - s_{i-1}) - 5\Delta$. The wall starting at s_{i-1} contains an outer upward-clean hole $(y'_{i-1}, y''_{i-1}] \subseteq y_{i-1} + (-g + \Delta, -\Delta]$ passing through it. We define w'_{i-1}, and w''_{i-1} accordingly. Let $w = (x_i, y^{(2)})$ where $y^{(2)} =$

$y'_{i-1} + (s_i - s_{i-1})$. There is a clean point $w''_i \in w + (4\Delta, 4\Delta) + [0, \Delta]^2$. We have

$$-g - 4\Delta \leqslant y^{(2)} - y^{(1)} \leqslant -6\Delta,$$
$$g' - g - 4\Delta \leqslant y^{(2)} - y_i = d(w) - d(w_i) \leqslant 4g' - 6\Delta. \tag{8.22}$$
$$-\Delta \leqslant d(w''_i) - d(w) \leqslant 5\Delta, \tag{8.23}$$
$$-1.1g \leqslant d(w'_{i-1}) - d(w) \leqslant 0. \tag{8.24}$$

Combining the last two inequalities with (8.22) gives

$$g' - 2.1g - 4\Delta \leqslant d(w'_{i-1}) - d(w_i) \leqslant 4g' - 6\Delta,$$
$$g' - g - 5\Delta \leqslant d(w''_i) - d(w_i) \leqslant 4g' - \Delta.$$

These and (8.12) prove (8.20) for our case. The reachability $w''_{i-1} \rightsquigarrow w''_i$ is shown similarly to Part 3.1.1. For this note

$$y_{i-1} \geqslant y^{(1)} - 1.1g - 5\Delta,$$
$$y''_{i-1} \geqslant y^{(1)} - 2.1g - 4\Delta \geqslant y^{(1)} - 2.2g = y^{(1)} - g'.$$

This shows that the rectangle $\mathrm{Rect}^\varepsilon(w''_{i-1}, w''_i)$ is trap-free. The bound $1/2 \leqslant$ minslope(w''_{i-1}, w''_i) is easy to check.

3.2.2. Assume that the wall-trap pair is horizontal.

This part is somewhat similar to Parts 3.1.2 and 3.2.1. There is an $x^{(1)}$ in $[x_i - L_2 + l_2, x_i)$ such that the region $[x^{(1)} - l_2, x^{(1)}] \times [y_i, y_i + 6\Delta]$ contains no trap. Let w_{i-1} be defined by $x_{i-1} = x^{(1)} - (s_i - s_{i-1})$slope$(u, v) - 5\Delta$. The wall starting at s_{i-1} contains an outer rightward-clean hole $(x'_{i-1}, x''_{i-1}] \subseteq x_{i-1} + (-g + \Delta, -\Delta]$ passing through it. We define w'_{i-1}, w''_{i-1} accordingly. Let $w = (x^{(2)}, y_i)$ where $x^{(2)} = x'_{i-1} + (s_i - s_{i-1})$slope$(u, v)$. There is a clean point $w''_i \in w + (4\Delta, 4\Delta) + [0, \Delta]^2$. We have

$$x^{(2)} \in x^{(1)} - 5\Delta + (-g + \Delta, -\Delta], \quad -3g' - g - 4\Delta \leqslant x^{(2)} - x_i \leqslant -6\Delta.$$

This gives $0 \leqslant d(w) - d(w_i) \leqslant 3g' + g + 4\Delta$. Combining with (8.23) and (8.24) which holds just as in Part 3.2.1, we get

$$-1.1g \leqslant d(w'_{i-1}) - d(w_i) \leqslant 3g' + g + 4\Delta,$$
$$-\Delta \leqslant d(w''_i) - d(w_i) \leqslant 3g' + g + 9\Delta.$$

These and (8.12) prove (8.20) for our case. The reachability $w''_{i-1} \rightsquigarrow w''_i$ is shown similarly to Part 3.2.1.

4. Assume that there is a bound pair: then $u \rightsquigarrow v$.
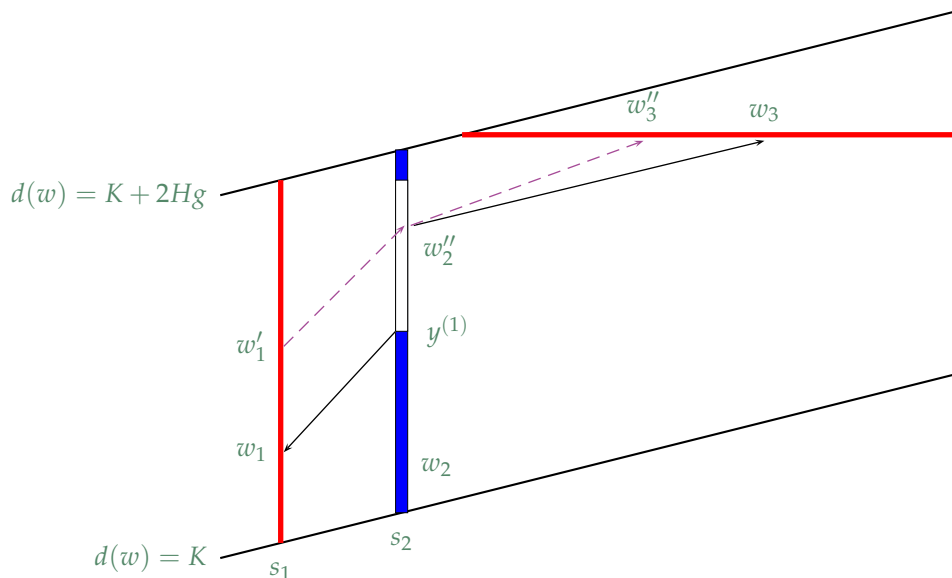
Figure 24: Approximation Lemma: the case of a bound wall-trap pair $(1,2)$. The arrows show the order of selection. First w_2 is defined. Then the trap-free segment of size g' above w_2 is found. Its starting point is projected back by a slope 1 line onto the vertical wall to find w_1 after moving up by 2Δ. The hole starting with w_1' is found within g above w_1. Then w_2'' is found near the back-projection of w_1''. Then w_2'' is projected forward, by a slope(u,v) line onto the horizontal wall, to find w_3. Finally, the hole ending in w_3'' is found within g backwards from w_3.

Proof. We define w_i with

$$d(w_i) = d(u) + K + 5g \tag{8.25}$$

if we have a trap-wall pair $(i, i+1)$ or a wall-trap pair $(i-1, i)$. As follows from the starting discussion of the proof of the lemma, the third obstacle, outside the bound pair, is a wall.

4.1. Assume that the bound pair is $(1,2)$.

4.1.1. Assume that we have a trap-wall pair.

We defined $d(w_1) = d(u) + K + 5g$, further define w_1', w_2'' as in Part 3.1, and w_3, w_3', w_3'' as in Part 2.1. Let us show that these points do not leave $C(u, v, K +$

$2g, K + (2H-2)g)$: for all i, we have (8.13). Inequalities (8.20) imply

$$-1.2g \leqslant d(w_1') - d(w_1), \ d(w_2'') - d(w_1) = d(w_3) - d(w_1) \leqslant 10g,$$

while inequalities (8.9) imply $0 \leqslant d(w_3') - d(w_3), \ d(w_3'') - d(w_3) \leqslant g$. Combining with (8.25) gives for $w \in \{w_1', w_2'', w_3', w_3''\}$:

$$K + 3.8g \leqslant d(w) - d(u) \leqslant K + 16g < K + (2H-2)g$$

according to (8.2).

We have shown $w_1' \rightsquigarrow w_2''$ and $w_3' \rightsquigarrow w_3''$, further such that the sets $C^{\varepsilon_1}(u, w_1', -g, g)$, $C^{\varepsilon_2}(w_2'', w_3', -g, g)$ and $C^{\varepsilon_3}(w_3'', v, -g, g)$ for the chosen ε_i are trap-free. It remains to show that the minslopes between the endpoints of these sets are lower-bounded by $\sigma + 4g/f$: then a reference to Lemma 8.4 will imply $u \rightsquigarrow v$. This is done for all three pairs (u, w_1'), (w_3'', v) and (w_2'', w_3') just as in Part 2.1.3.

4.1.2. Assume that we have a wall-trap pair.

We defined $d(w_2) = d(u) + K + 5g$; we further define w_1', w_2'' as in Part 3.2, and w_3, w_3', w_3'' as in Part 2.1. The proof is finished similarly to Part 4.1.1.

4.2. Assume now that the bound pair is $(2, 3)$.

4.2.1. Assume that we have a trap-wall pair.

We defined $d(w_2) = d(u) + K + 5g$; we further define w_2', w_3'' as in Part 3.1, and w_1, w_1', w_1'' as in Part 2.2. Let us show that these points do not leave $C(u, v, K + 2g, K + (2H-2)g)$. Inequalities (8.20) imply

$$-1.2g \leqslant d(w_2') - d(w_2) = d(w_1) - d(w_2), \ d(w_3'') - d(w_2) \leqslant 10g,$$

while inequalities (8.9) imply $-g \leqslant d(w_1') - d(w_1), \ d(w_1'') - d(w_1) \leqslant \Delta$. Combining with (8.25) gives for $w \in \{w_1', w_1'', w_2', w_3''\}$:

$$K + 2.8g \leqslant d(w) - d(u) \leqslant K + 15g + \Delta < K + (2H-2)g.$$

Reachability is proved as in Part 4.1.1.

4.2.2. Assume that we have a wall-trap pair.

We defined $d(w_3) = d(u) + K + 5g$; we further define w_2', w_3'' as in Part 3.2, and w_1, w_1', w_1'' as in Part 2.2. The proof is finished as in Part 4.2.1.

$\square$

Proof of Lemma 8.1(Approximation). Recall that the lemma says that if a rectangle $Q = \text{Rect}^{\varepsilon}(u,v)$ contains no walls or traps of $\mathscr{M}^*$, is inner clean in $\mathscr{M}^*$ and satisfies the slope lower bound condition $\text{minslope}(u,v) \geqslant \sigma^*$ for $\mathscr{M}^*$, then $u \rightsquigarrow v$.

The proof started by recalling, in Lemma 8.2, that walls of $\mathscr{M}$ in Q can be grouped to a horizontal and a vertical sequence, whose members are well separated from each other and from the sides of Q. Then it showed, in Lemma 8.3, that all traps of $\mathscr{M}$ are covered by certain horizontal and vertical stripes called trap covers. Walls of $\mathscr{M}$ and trap covers were called obstacles.

Next it showed, in Lemma 8.4, that in case there are no traps or walls of $\mathscr{M}$ in Q then there is a path through Q that stays close to the diagonal.

Next, a series of obstacles (walls or trap covers) was defined, along with the points $s_1, s_2, \ldots$ that are obtained by the intersection points of the obstacle with the diagonal, and projected to the x axis. It was shown in Lemma 8.5 that these obstacles are well separated into groups of up to three. The most laborious lemma of the paper, Lemma 8.6, showed how to pass each triple of obstacles. It remains to conclude the proof.

For each pair of numbers s_i, s_{i+1} with $s_{i+1} - s_i \geqslant f/4$, define its midpoint $(s_i + s_{i+1})/2$. Let $t_1 < t_2 < \cdots < t_n$ be the sequence of all these midpoints. Let $t_0 = u_0$, $t_{n+1} = v_0$. Let us define the square

$$S_i = (t_i, u_1 + \text{slope}(u,v)(t_i - u_0)) + [0, \Delta] \times [-\Delta, 0].$$

By Remark 3.7.1, each of these squares contains a clean point p_i.

1. For $1 \leqslant i < n$, the rectangle $\text{Rect}(p_i, p_{i+1})$ satisfies the conditions of Lemma 8.6, and therefore $p_i \rightsquigarrow p_{i+1}$. The same holds also for $i = 0$ if the first obstacle is a wall, and for $i = n$ if the last obstacle is a wall.

   Proof. By Lemma 8.5, there are at most three points of $\{s_1, s_2, \ldots\}$ between t_i and t_{i+1}. Let these be $s_{j_i}, s_{j_i+1}, s_{j_i+2}$. Let t'_i be the x coordinate of p_i, then $0 \leqslant t'_i - t_i \leqslant \Delta$. The distance of each t'_i from the closest point s_j is at most $f/8 - \Delta \geqslant 0.1f$. It is also easy to check that p_i, p_{i+1} satisfy (8.5), so Lemma 8.6 is indeed applicable.

2. We have $u \rightsquigarrow p_1$ and $p_n \rightsquigarrow v$.

   Proof. If $s_1 \geqslant 0.1f$, then the statement is proved by an application Lemma 8.6, so suppose $s_1 < 0.1f$. Then s_1 belongs to a trap cover. By the same reasoning used in Lemma 8.5, we then find that $s_2 - s_1 > f/4$, and therefore there is only s_1 between u and t_1, and also $t_1 - s_1 > 0.1f$.

   If the trap cover belonging to s_1 is at a distance $\geqslant g - 6\Delta$ from u then we can pass through it, going from u to p_1 just like in Part 2 of the proof of Lemma 8.6. If it is closer than $g - 6\Delta$ then the fact that u is clean in $\mathscr{M}^*$ implies that it contains a large trap-free region where it is easy to get through.

   The relation $p_n \rightsquigarrow v$ is shown similarly.

   $\square$

9   Proof of Lemma 2.6 (Main)

Lemma 2.6 says that if m is sufficiently large then the sequence $\mathscr{M}^k$ can be constructed, in such a way that it satisfies all the above conditions and also $\sum_k \mathbf{P}(\mathscr{F}'_k) < 1$.

The construction of $\mathscr{M}^k$ is complete by the definition of $\mathscr{M}^1$ in Example 3.9 and the scale-up algorithm of Section 4, after fixing all parameters in Section 6.

We will prove, by induction, that every structure $\mathscr{M}^k$ is a mazery. We already know that the statement is true for $k = 1$. Assuming that it is true for all $i \leqslant k$, we prove it for $k + 1$. The dependency properties in Condition 3.6.1 are satisfied according to Lemma 4.1. The combinatorial properties in Condition 3.6.2 have been proved in Lemmas 4.2 and 4.7.

The trap probability upper bound in Condition 3.6.3a has been proved in Lemma 7.1. The wall probability upper bound in Condition 3.6.3b has been proved in Lemma 7.4. The cleanness probability lower bounds in Condition 3.6.3c have been proved in Lemma 7.7. The hole probability lower bound in Condition 3.6.3d has been proved in Lemmas 7.9, 7.10 and 7.11.

The reachability property in Condition 3.8 is satisfied via Lemma 8.1 (the Approximation Lemma). There are some restrictions on the parameters f, g, Δ used in this lemma. Of these, condition (4.1) holds if R_0 is sufficiently large; the rest follows from our choice of parameters and Lemma 7.7.

Finally, $\sum_k \mathbf{P}(\mathscr{F}'_k) < 1$ follows from Lemma 7.6.

The main result, Theorem 1, has been proved in Section 2, using Lemma 2.6 and some conditions. Of these, the reachability property in Condition 2.2 is implied by Condition 3.8. The property saying that the set of clean points is sufficiently dense, Condition 2.3, is implied by Remark 3.7.1. The property saying that the absence of lower-level bad events near the origin imply k-level cleanness of the origin, Condition 2.5, follows immediately from the definition of cleanness.

10   Conclusions

It was pointed out in [4] that the clairvoyant demon does not really have to look into the infinite future, it is sufficient for it to look as far ahead as maybe n^3 when scheduling $X(n), Y(n)$. This is also true for the present paper.

Another natural question is: how about three independent random walks? The methods of the present paper make it very likely that three independent random walks on a very large complete graph can also be synchronized, but it would be nice to have a very simple, elegant reduction.

It seems possible to give a common generalization of the model of the paper [4] and the present paper. Let us also mention that we have not used about the independent Markov processes X, Y the fact that they are homogenous: the transition matrix could depend on i.

We only used the fact that for some small constant w, the inequality $\mathbf{P}[X(i+1) = j \mid X(i) = k] \leqslant w$ holds for all i, j, k (and similarly for Y).

What will strike most readers as the most pressing open question is how to decrease the number of elements of the smallest graph for which scheduling is provably possible from super-astronomical to, say, 5. Doing some obvious optimizations on the present renormalization method is unlikely to yield impressive improvement: new ideas are needed.

Maybe computational work can find the better probability thresholds needed for renormalization even on the graph K_5, introducing supersteps consisting of several single steps.

Acknowledgement

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Glossary

Concepts

Approximation lemma. This is the lemma with the longest proof, though the fact that something like it can be proven is not surprising. The lemma says that the reachability condition will hold in the higher-order model $\mathscr{M}^{k+1} = \mathscr{M}^{k*}$, and is part of the project to prove that $\mathscr{M}^{k+1}$ is a mazery. The statement and the proof is elementary geometrical: it can be translated as saying that one can pass through a network of walls that are well-separated and densely perforated by holes, with only well-separated traps between them.

The proof is long only because of the awkwardly large number of cases to consider, though these cases differ from each other only slightly.

Barrier. Barriers are introduced in Section 3. Barrier is the same type of object as a wall. A mazery has a random set of barriers just as a random set of walls. The set $\mathscr{W}_d$ of walls is a subset of the set $\mathscr{B}_d$ of barriers. The simpler definition of barriers in the renormalization operation of Section 4 allows the computation of probability upper bounds. The definition of walls guarantees certain combinatorial properties allowing to show how to pass through and between them.

Barrier probability upper bound $p(r)$**.** Part of the definition of a mazery, this function upper-bounds the probability of a barrier of rank r arising at a specific point.

Moreover, as said in Condition 3.6.3b, it will be an upper bound over $\sum_l p(r,l)$, where $p(r,l)$ is the probability of a barrier of rank r and length l arising.

The actual size of $p(r)$ will depend negative exponentially on r, as defined in Section 6.

Body of a wall. A right-closed interval $\mathrm{Body}(E)$, part of the definition of a wall or barrier.

Cleanness. Introduced informally in Subsection 2.2, and formally in Section 3. There are several kinds of cleanness, but they all express the same idea. For example, informally, the fact that a point u is upper right trap-clean in mazery $\mathscr{M}^k$ stands for the property that for all $i < k$, there is no i-level trap closer than approximately Δ_i to u in the upper right quadrant starting from u. Formally, the random sets $\mathscr{S}_d$ for $d = 0, 1, 2$ and $\mathscr{C}_d$ for $d = 0, 1$ describe cleanness.

The combination of several kinds of cleanness can result in some complex versions, introduced in Definition 3.6. Of these, only H-cleanness will be much used, it is suited for proving probability lower bounds on holes.

Cleanness must appear with a certain density: this is spelled out in Conditions 3.6.2c and 3.6.2d. Actually every point must be clean (in all kinds of sense) with rather large probability. This is spelled out in Condition 3.6.3c.

Closed and open points. Defined in Subsection 1.3.

Collision. Only used in the introduction.

Compound trap. Introduced informally in Subsection 2.2, and formally in the scale-up operation of Section 4. The event that some traps are too close to each other. It can be *uncorrelated*, when the projections of the traps are disjoint, and *correlated* otherwise.

Compound wall. Introduced informally in Subsection 2.2, and formally in the scale-up operation of Section 4. The event that some walls are too close to each other. The *rank* increases with the rank of its components but decreases with their distance.

Correlated trap. Introduced formally in the scale-up operation of Section 4. The event that some traps are too close to each other when the projections of the traps are not disjoint, but still *both conditional probabilities* are small. When one of the conditional probabilities is not small, this will give rise to an emerging wall.

Compatible. Only used in the introduction.

Delay sequence. Only used in the introduction.

Dependencies and monotonicities. The probability estimates require to control which random objects depend on what parts of the original random process Z. The requirements are formulated in Condition 3.6.1.

Dominant wall. Introduced in Definition 3.2, it is a wall that is surrounded by external intervals of size $\geqslant \Delta$.

Emerging barriers and walls. Introduced informally in Subsection 2.2, and formally in Section 4. Suppose that an event $\mathscr{A}$ appears that would lead to a trap of the correlated or missing-hole kind. If the conditional probability of $\mathscr{A}$ (with respect to, say, the process X) is not small, then we do not define a trap, but then this is an improbable event as a function of the process X and defines an emerging (vertical) barrier.

Before deciding which of the emerging barriers become walls, we introduce some cleanness requirements in the notion of *pre-wall*, and then apply a selection process to make walls disjoint.

Exponents. The construction deals with several exponents, all of which will be fixed, but chosen appropriately to satisfy certain inequalities.

$\chi = 0.015$: Informally, if the probability of the occurrence of a certain kind of wall is upper-bounded by p then the probability, at a given site, to find a hole through the wall is lower-bounded by p^χ.

The other exponents are defined by their role in the expression of the parameters already seen elsewhere, as set in Definition 6.3:

$$T = \lambda^R \text{ with } \lambda = 2^{1/2},$$
$$\Delta = T^\delta, \quad f = T^\varphi, \quad g = T^\gamma, \quad w = T^{-\omega} \text{ with } \omega = 4.5,$$
$$R_k = R_0 \tau^k \text{ with } \tau = 2 - \varphi.$$

The choice of these parameters is made in Lemma 6.3, where the necessary inequalities are spelled out.

External interval. Introduced in Definition 3.2, a right-closed interval that intersects no walls.

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. This random graph is the same throughout the proof. We have $\mathcal{V} = \mathbb{Z}_+^2$, and $\mathcal{E}$ consists of all rightward and upward edges between open lattice points.

Hole. Formally defined in Definition 3.10, a vertical hole is defined by a pair of points on the opposite sides of a horizontal wall, and an event that it is possible to pass from one to the other.

The hole is called *good* if it is lower-left and upper-right H-clean (see the reference to partial cleanness). This partial cleanness requirement is useful in the proof of the hole probability lower bound. There is also a qualified version of goodness, as "seen from" a point u.

Hole probability lower bound. As said in the item on the exponent, the lower bound of the probability of a hole through a wall of rank r is essentially $p(r)^\chi$. For technical reasons, it is larger by a polynomial factor, as given in (3.6). In the interests of the inductive proof (in order to handle compound walls), the actual condition on the hole probability lower bound is rather complex, and is spelled out in Condition 3.6.3d.

Hop. A hop, as defined in Definition 3.7, is an inner clean right-closed interval that contains no walls. The role of a hop is that it is an interval that is manifestly possible to pass through.

Isolated light wall. This concept is only used in the proof of Lemma 4.2.

Light and heavy walls. Those of rank $< R^*$ and $\geqslant R^*$.

Mazery $\mathcal{M}^k$. Introduced in Subsection 2.1 in an abstract way just as a k-level "model". Mazeries are defined in Section 3, and the operation defining $\mathcal{M}^{k+1}$ in terms of $\mathcal{M}^k$ is defined in Section 4. A mazery is a random process containing a random set of traps, random sets of barriers and walls, and a random assignment of some cleanness properties to points. These random processes must obey certain conditions, spelled out in Section 3.

Number of colors m. Introduced in the Introduction, it is the size of the complete graph, and the number of colors in the color-percolation model. It does not appear directly as a parameter in the mazeries, but Example 3.9 defines it with the help of the trap probability upper bound w of model $\mathcal{M}^1$ as $m = 1 + \lceil 1/w \rceil$.

Potential wall. Whether a segment is a barrier or not depends only on symbols in the segment itself. However, whether a barrier is a wall depends on the environment. As introduced in Definition 3.3, a segment is a *potential wall* when there is an environment making it a wall.

Power law convergence. Only used in the introduction.

Pure sequence. This concept is used only in the proof of Lemma 4.2.

Rank of a wall. This is a value that classifies walls to help in the definition of the renormalization operation. Higher rank will mean lower bound on the probability. When forming mazery $\mathcal{M}^{k+1}$ we will not delete all walls of mazery $\mathcal{M}^k$, only the isolated walls of rank $\leqslant R^*$ (the so-called *light walls*). Also, in forming compound walls one of the components will be light.

Rank lower bound R. Introduced in 3, it is part of the definition of a mazery. In our series of mazeries $\mathcal{M}^k$, we will have $R_1 < R_2 < \cdots$. Actually, R_k grows exponentially with k, as defined in Section 6 and above in the description of exponents.

(The rank computation for compound walls suggests to view rank as analogous to "free energy" in statistical physics (energy minus entropy), but ignore the analogy if it only confuses you.)

Rank upper bound. No apriori rank upper bound is given, but the construction guarantees a that all ranks of a mazery are between lower bound R and upper bound $\overline{\tau}R$ for a certain constant $\overline{\tau}$. This will guarantee that each rank value is present in a mazery $\mathcal{M}^k$ for only a constant number of values of k.

Rectangle. Notation for rectangles is introduced in Subsection 3.1.

Reachability. The notation $\rightsquigarrow$ is introduced in Subsection 3.1. In a mazery, the reachability condition, Condition 3.8 says, essentially, that one can pass from a clean point to another one, if there are no obstacles (walls, holes) in between, and the slope between the two points is not too small or too large.

Renormalization, or scale-up. The word is used only informally, as the operation $\mathcal{M} \mapsto \mathcal{M}^*$ that leads from mazery $\mathcal{M}^k$ to mazery $\mathcal{M}^{k+1}$.

Scale parameter Δ_k. Upper bound on the size of traps and walls in the model of level k. The parameter Δ is part of the definition of a mazery.

Scale-up parameters. The parameters $g \ll f$ play a special role in the scale-up operation. We will have $\Delta \ll g$. They play several roles, but the most important is this: We try not to permit walls closer than f to each other, and we try not to permit intervals larger than g without holes. We will also use f in the definition of one-dimensional cleanness and g in the definition of trap-cleanness. (The value $g' = 2.2g$ plays a subsidiary role.)

Sequence of neighbor walls. A sequence of neighbor walls, as introduced in Definition 3.8, is a sequence of walls separated by hops. Such a sequence is useful since it allows the analysis of passage through it. It is a crucial part of the combinatorial construction to establish the existence of such sequences (under the appropriate conditions) in $\mathcal{M}^{k+1}$. The existence requirement is given in Conditions 3.6.2a and 3.6.2b. Of course, the proof uses the corresponding property of $\mathcal{M}^k$.

Sequences of random variables. The basic sequences we consider are the two independent random walks $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$. But for notational convenience, we sometimes denote $X = Z_0$, $Y = Z_1$ and $Z = (X, Y)$. These are the only source of randomness, all other random structures introduced later (mazeries $\mathscr{M}^k$) are functions of these.

Slope constraint. For a pair of points u, v the requirement that the slope s of the segment between them be neither too horizontal nor too vertical. This is expressed by saying $\sigma \leqslant s \leqslant 1/\sigma$. The value of the bound σ belongs to the mazery. It is σ_k in mazery $\mathscr{M}^k$. We will have $\sigma_1 < \sigma_2 < \cdots < 1/2$. The notation $\text{minslope}(u, v)$ is the smaller of the two slopes of the segment between points u and v, as defined in Subsection 3.1.

Strong cleanness. One-dimensional cleanness has a strong version, introduced in Definition 3.4. The motivation is similar to introduction of barriers, and in fact in the renormalization operation of Section 4, strong cleanness will come from the absence of lower-level barriers.

Trap. Introduced informally in Subsection 2.2, and formally in Section 3, where the set of traps is denoted by $\mathscr{T}$. On the lowest level, this is where two equal colors collide. On level $k + 1$, it is either the event that two close k-level traps occur, or that some local bad event of a new kind (emerging on this level) occurs. An example is a trap of the *missing hole* kind: when on a certain wall of level k, there is a long segment without a hole.

Trap cover. In the proof of the Approximation Lemma, a strip that covers a trap. The idea is that under the conditions of that lemma, after covering all traps with trap covers, all the trap covers and the walls still form a rather sparse network, which it will be possible to pass through.

Trap probability bound w. Introduced in Section 3, it is part of the definition of a mazery. It bounds the *conditional probability* of a trap, as spelled out in Condition 3.6.3b. The value w_k decreases super-exponentially in k, as defined in Section 6.

Trap-cleanness. Introduced in Definition 3.5. See the item on cleanness.

Ultimate bad event $\mathscr{F}_k$. Introduced in Subsection 2.1.

Wall. Introduced informally in Subsection 2.2, and formally in Section 3 as an object having a *body* and a *rank*. On level $k + 1$, it is either the event that two close k-level walls occur (*compound wall*), or that some bad event of a new kind (emerging on this level) occurs in one of the projections (*emerging wall*). An example of such an event: when the conditional probability that two close traps occur is too high.

Winkler percolation. Only used in the introduction.

Symbols

$\mathscr{B}_0$ Set of vertical barriers.

$\mathscr{C}$ Structure defining the various kinds of one-dimensional cleanness.

c_1, c_2 Constants used in the definition of $p(r)$. c_1 is fixed at the end of the proof of Lemma 7.3. c_2 can be fixed as anything $< 1 - 1/\lambda$, as said at the end of the proof of Lemma 7.5.

c_3 Constant used in the definition of $h(r)$. Fixed at the end of the proof of Lemma 7.10.

d Frequently used to denote an index 0 or 1, with $Z_0 = X$, $Z_1 = Y$.

Δ Upper bound on the size of walls and traps. Fixed as T^δ in Definition 6.3.

δ Exponent used in defining Δ. Chosen 0.15 in the proof of Lemma 6.3.

$\mathscr{E}$ Set of edges of the random graph $\mathscr{G}$ in $\mathbb{Z}_+^2$ defined by the processes X, Y.

f Lower bound on the distance between walls. Fixed as T^φ in Definition 6.3.

φ Exponent used in defining f. Chosen 0.25 in the proof of Lemma 6.3.

$\mathscr{G}$ The random graph $(\mathscr{V}, \mathscr{E})$ in $\mathbb{Z}_+^2$ defined by the processes X, Y.

g Upper bound on the length of a wall segment without holes (has also some other roles). Fixed as T^γ in Definition 6.3.

g' Plays a subsidiary role in defining correlated traps. Defined as $2.2g$ in (4.4).

γ Exponent used in defining g. Fixed as 0.2 in the proof of Lemma 6.3.

$h(r)$ Hole lower bound probability. Fixed as $c_3\lambda^{-\chi r}$ in (3.6).

χ Exponent used in lower-bounding hole probabilities. Fixed as 0.015 in (2.2).

H Constant defined as 12 in (8.2) and used in the proof of the Approximation Lemma.

k Frequently denotes the level of the mazery $\mathscr{M}^k$.

l_j We defined $l_1 = 7\Delta$, $l_2 = g'$ in connection with the definition of correlated traps in the scale-up.

L_j We defined $L_1 = 4l_1$, $L_2 = 4l_2$, $L_3 = g$. These parameters determine approximately the widths of emerging barriers of three types.

$\mathscr{L}_j$ In the scale-up operation, for $j = 1,2,3$, the "bad event" $\mathscr{L}_3(x,y,I,b)$ triggers the occurrence of a new trap provided its conditional probability (with condition $X = x$) is small. The event (a function of x) that its conditional probability is large triggers the occurrence of an emerging vertical barrier of the corresponding type.

λ The lower base of our double exponents. Fixed as $2^{1/2}$ in Definition 3.12.

Λ Constant used in the definition of σ_{k+1} and in some bounds. Fixed as 500 in Definition 4.1.

m Number of elements of the complete graph where the random walk takes place, or equivalently, the number of colors in the color percolation model. Lower-bounded by $1/w_1 + 1$ in Example 3.9 which serves as the base case $\mathscr{M}^1$.

$\mathscr{M}^k$ Mazery of level k.

$p(r)$ Probability upper bound of a barrier of rank r. Fixed as $p(r) = c_2 r^{-c_1} \lambda^{-r}$ in (3.5).

$p(r,l)$ Supremum of the probabilities of any barrier of rank r and length l starting at a given point. Depends on the mazery, but the conditions require $\sum_l p(r,l) \leqslant p(r)$, where the function $p(r)$ is fixed.

r Frequently denotes a rank.

R Rank lower bound. Fixed as $R_k = R_0 \tau^k$ in Definition 6.1.

$\hat{R}$ Rank of emerging walls. Fixed as $\tau' R$ in Definition 6.1.

R_0 There are several lemmas that hold when it is chosen sufficiently large, and there are no other conditions on it.

$\mathscr{S}$ Structure describing the one-dimensional kinds of strong cleanness and also trap cleanness.

σ Slope lower bound. It is defined by $\sigma_1 = 0$ in (6.2), and by $\sigma_{k+1} = \sigma_k + \Lambda g_k / f_k$ in Definition 4.2.

T Auxiliary parameter for defining several other parameters. Fixed as λ^R in Definition 6.3.

$\mathscr{T}$ The set of traps.

τ Used in the definition of R_k. Fixed as $2 - \varphi$ in Definition 6.3.

τ' Coefficient used in the definition of $\hat{R}$. Fixed as 2.5 in the proof of Lemma 6.3.

$\mathscr{V}$ Set of points of the random graph $\mathscr{G}$ in $\mathbb{Z}_+^2$ defined by the processes X, Y.

$\mathscr{W}_0$ Set of vertical walls.

w Upper bound on the probability of traps. Fixed as $T^{-\omega}$ in Definition 6.3.

ω Exponent in the definition of w. Fixed as 4.5 in Definition 6.3.

X The sequence $X(1), X(2), \ldots$ is a random walk over the complete graph K_m.

Y The sequence $Y(1), Y(2), \ldots$ is a random walk over the complete graph K_m.

Z We defined $Z_0(i) = X(i)$, $Z_1(i) = Y(i)$.