The game of “twenty questions” with a liar

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Joint paper (1991) with Aditi Dhagat and Peter Winkler.

A game between Questioner and Responder. Responder thinks of a number in $\{1, \ldots, N\}$. Questioner asks yes/no questions, Q of them. In her answers, Responder may lie, rQ times, where the fraction r is given in advance.

Variants:

1. What kind of questions?
2. What other restrictions?

1. In our game, only comparison questions are allowed: is $x < y$?
   Other possibilities:
   - general questions of the sort $x \in S$ for sets S.
   - questions asking one bit of a binary representation of x (bit questions).
2. Questions are allowed to be adaptive.
   Other possibilities:
   - Questions must be submitted in advance (batch questions);
   - Responder cannot lie in more than a fraction r of any starting segment.

Batch questions: same as an error-correcting code. Indeed, for a number x, let

$$C(x) = (c_1, c_2, \dots, c_Q)$$

where c_i is the correct answer to the i-th question. Then the set

$$\{\, C(x) : x = 1, \dots, N \,\}$$

is a code correcting rQ errors, with rate $Q^{-1} \log N$.

Adaptive questions: code with feedback.

First studied by Berlekamp. Exact solution is known for up to 3 lies (you do not want to see the algorithm!).

Batch game uninteresting for bit and comparison questions

Theorem For bit and comparison questions, there is a function $f(r)$ such that Responder wins unless $N < f(r)$.

Proof for the comparison questions. Since Responder sees all questions in advance, she knows which question of the form $x < k$ has been asked not more than Q/N times.

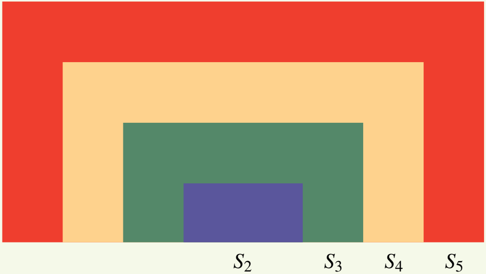
- yes to questions $x < j$ for $j > k$.
- no to questions $x < j$ for $j < k$.
- yes to half of the questions $x < k$. This is allowed if $rQ > 0.5(Q/N)$, that is$N > 0.5/r$.

Then Paul cannot decide between $k - 1$ and k. □

Lower bounds for general questions

After t questions and answers, let $f_t(x)$ be the number of lies made by Responder, if x was the number she thought of. All relevant information for the analysis is found in the numbers

$$V_t(i) = |S_t(i)| = |\{ x : f_t(x) = i \}|.$$



If k lies are allowed then the game ends when

$$\sum_{i \leqslant k} V_t(i) \leqslant 1.$$

Theorem (Winkler, Spencer) In the adaptive game, if $N > 2$ and $r > 1/3$ then Questioner loses.

Proof. Winning strategy for Responder: it is sufficient to consider $N = 3$. Watch the three numbers $f_t(1)$, $f_t(2)$, $f_t(3)$. As long as all three are $< rQ$ choose the answer that increases at most one of them. Once there are only two numbers left, choose the answer that increases the smaller one.
This way, it will take $\geqslant 3rQ - 1$ steps to drive two of the numbers beyond rQ. □

Adaptive, comparison questions

It does not seem easy to win no matter how small is r and how large is Q.

A failed idea: repeat each question many times. This does not help since Responder can save up all lies to the end. Still:

Theorem With comparison questions, Questioner wins for all $r < 1/3$, asking

$$\left\lceil \frac{8 \log N}{(1 - 3r)^2} \right\rceil$$

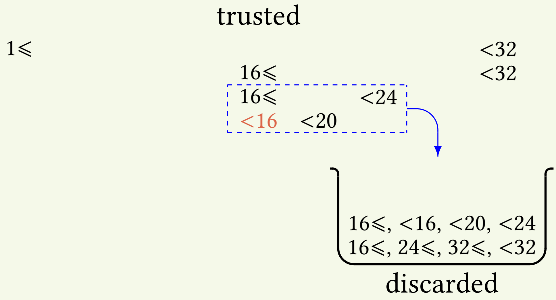
questions.

Proof of $O(\log N)$ for the case $r < 1/4$. (The case $r < 1/3$ requires more sweat.)

Ideas: instead of trying to decide early the truth, count contradictions.

Try binary search but let Responder pay with contradiction every time when you have to abandon a cut-in-half.

□



Adaptive strategy, comparison questions, $N = 64$. Every line in the thrashbox has $\leqslant 4$ questions containing a contradiction, so at least 1 lie.

For $1/4 \leqslant r < 1/3$, a similar strategy, but each nested interval (a pair of questions) must be repeated a certain number of times.

Theorem (Spencer, Winkler) Consider general questions. Let b be an upper bond on bounds r needed for Questioner to win.

1. If the game is non-adaptive, $b = 1/4$.
2. If the game is adaptive, $b = 1/3$, the same as even with the special, comparison questions.
3. If r bounds the fraction of lies in all beginning segments, then $b = 1/2$.

In all cases, for $r < b$ the number of questions needed is $O(\log n)$.

The proof analyses error-correcting codes.
Let M be a $Q \times N$ 0-1 matrix showing all the questions in its rows.
For Questioners to win, the Hamming distance between its columns must be more than $2\lfloor rQ \rfloor$. Let us ignore integer parts from now.

Lower bound The sum of all distances must at least
$\frac{N(N-1)}{2} \cdot 2rQ \approx rQN^2$.
Each row, containing k 1's, contributes at most $k(N-k) \leqslant N^2/4$ to this sum, hence the total of Q rows is at most $\frac{1}{4}QN^2$.

Upper bound Let $2^{QH(\rho)}$ be the volume of a Hamming ball of radius ρQ. Let us choose 0-1 vectors of length Q one-by-one, such that the distance of the next one is always at least $2rQ$ from the previous ones. If we found n and cannot continue then the balls of radius $2rQ$ around these vectors cover the space, so $n \cdot 2^{QH(2r)} \geqslant 2^Q$. But then

$$n \geqslant 2^{Q(1-H(2r))}.$$

Since $H(2r) < 1$ if $r < 1/4$, we will be done with $O(\log N)$ questions in this case.