A game between **Questioner** and **Responder**. Responder thinks of a number in \( \{1, \ldots, N\} \). Questioner asks yes/no questions, \( Q \) of them. In her answers, Responder may lie, \( rQ \) times, where the fraction \( r \) is given in advance.

Variants:

1. What kind of questions?
2. What other restrictions?
In our game, only **comparison questions** are allowed: is $x < y$?

Other possibilities:
- **general questions** of the sort $x \in S$ for sets $S$.
- questions asking one bit of a binary representation of $x$ (**bit questions**).

Questions are allowed to be **adaptive**.

Other possibilities:
- Questions must be submitted in advance (**batch questions**);
- Responder cannot lie in more than a fraction $r$ of **any starting segment**.
Batch questions: same as an error-correcting code. Indeed, for a number \( x \), let
\[
C(x) = (c_1, c_2, \ldots, c_Q)
\]
where \( c_i \) is the correct answer to the \( i \)-th question. Then the set
\[
\{ C(x) : x = 1, \ldots, N \}
\]
is a code correcting \( rQ \) errors, with rate \( Q^{-1} \log N \).

Adaptive questions: code with feedback.
First studied by Berlekamp. Exact solution is known for up to 3 lies (you do not want to see the algorithm!).
Theorem  For bit and comparison questions, there is a function $f(r)$ such that Responder wins unless $N < f(r)$.

Proof for the comparison questions. Since Responder sees all questions in advance, she knows which question of the form $x < k$ has been asked not more than $Q/N$ times.

- yes to questions $x < j$ for $j > k$.
- no to questions $x < j$ for $j < k$.
- yes to half of the questions $x < k$. This is allowed if $rQ > 0.5(Q/N)$, that is $N > 0.5/r$.

Then Paul cannot decide between $k - 1$ and $k$. □
After $t$ questions and answers, let $f_t(x)$ be the number of lies made by Responder, if $x$ was the number she thought of. All relevant information for the analysis is found in the numbers

$$V_t(i) = |S_t(i)| = |\{x : f_t(x) = i\}|.$$

If $k$ lies are allowed then the game ends when

$$\sum_{i \leq k} V_t(i) \leq 1.$$
Theorem (Winkler, Spencer)  In the adaptive game, if $N > 2$ and $r > 1/3$ then Questioner loses.

Proof. Winning strategy for Responder: it is sufficient to consider $N = 3$. Watch the three numbers $f_t(1), f_t(2), f_t(3)$. As long as all three are $< rQ$ choose the answer that increases at most one of them. Once there are only two numbers left, choose the answer that increases the smaller one. This way, it will take $\geq 3rQ - 1$ steps to drive two of the numbers beyond $rQ$. □
Adaptive, comparison questions

It does not seem easy to win no matter how small is $r$ and how large is $Q$.

A failed idea: repeat each question many times. This does not help since Responder can save up all lies to the end. Still:

**Theorem** With comparison questions, Questioner wins for all $r < 1/3$, asking

$$\left\lceil \frac{8 \log N}{(1 - 3r)^2} \right\rceil$$

questions.

Proof of $O(\log N)$ for the case $r < 1/4$. (The case $r < 1/3$ requires more sweat.)

Ideas: instead of trying to decide early the truth, count contradictions.

Try binary search but let Responder pay with contradiction every time when you have to abandon a cut-in-half.
Adaptive strategy, comparison questions, $N = 64$. Every line in the trashbox has $\leq 4$ questions containing a contradiction, so at least 1 lie.

For $1/4 \leq r < 1/3$, a similar strategy, but each nested interval (a pair of questions) must be repeated a certain number of times.
Consider general questions. Let $b$ be an upper bond on bounds $r$ needed for Questioner to win.

1. If the game is non-adaptive, $b = 1/4$.
2. If the game is adaptive, $b = 1/3$, the same as even with the special, comparison questions.
3. If $r$ bounds the fraction of lies in all beginning segments, then $b = 1/2$.

In all cases, for $r < b$ the number of questions needed is $O(\log n)$.

The proof analyses error-correcting codes.
Let $M$ be a $Q \times N$ 0-1 matrix showing all the questions in its rows. For Questioners to win, the Hamming distance between its columns must be more than $2\lfloor rQ \rfloor$. Let us ignore integer parts from now.
**Lower bound**  The sum of all distances must at least
\[ \frac{N(N-1)}{2} \cdot 2rQ \approx rQN^2. \]
Each row, containing \( k \) 1’s, contributes at most \( k(N - k) \leq N^2/4 \) to this sum, hence the total of \( Q \) rows is at most \( \frac{1}{4} QN^2 \).

**Upper bound**  Let \( 2^{QH(\rho)} \) be the volume of a Hamming ball of radius \( \rho Q \). Let us choose 0-1 vectors of length \( Q \) one-by-one, such that the distance of the next one is always at least \( 2rQ \) from the previous ones. If we found \( n \) and cannot continue then the balls of radius \( 2rQ \) around these vectors cover the space, so \( n \cdot 2^{QH(2r)} \geq 2^Q \). But then
\[ n \geq 2^{Q(1-H(2r))}. \]
Since \( H(2r) < 1 \) if \( r < 1/4 \), we will be done with \( O(\log N) \) questions in this case.