Clairvoyant embedding in one dimension

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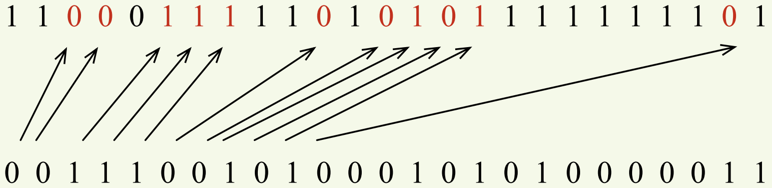
The embedding problem

Given $m > 0$ and infinite 0-1 sequences x, y we say y is m-embeddable in x, if there exists an increasing sequence $(n_i : i \geqslant 1)$ of positive integers such that $y(i) = x(n_i)$, and $1 \leqslant n_i - n_{i-1} \leqslant m$ for all $i \geqslant 1$ $(n_0 = 0)$.



Let $X(1), X(2), \ldots$ and $Y(1), Y(2), \ldots$ be independent Bernoulli(1/2) sequences.

Theorem There is an m with the property that Y is m-embeddable into X with positive probability.

Why clairvoyant? Because choosing the embedding without seeing the future is not going to work.

What is it good for? I do not know.

Why interesting?

- Simple question with (so far only) complex solution.
- Built-in power-law behavior, like other Winkler-type problems (see below).
- A nail to which I had a hammer.

Attracted some attention after Grimmett posed the question. By now three simultaneous, independent proofs: the others by Bashu-Sly, and Sidoravicius.

The compatible sequences problem

In two infinite 0-1 sequences x, y, we have collision at i if $x(i) = y(i)$. We call x, y compatible if we can delete some 0's (or, equivalently, insert 1's), so that the resulting sequences x', y', have no collision.

Example The following two sequences are not compatible:

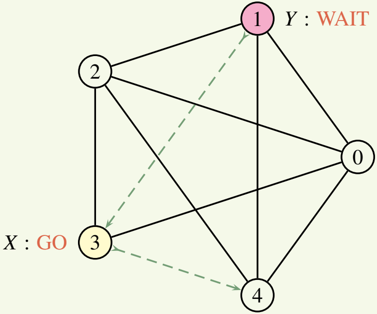
$$x = \texttt{0001100100001111}\ldots,$$
$$y = \texttt{1101010001011001}\ldots.$$

The x, y below are.

$$x = \texttt{0000100100001111001001001001001}\ldots,$$
$$y = \texttt{0101010001011000000000101011010 10}\ldots,$$
$$x' = \texttt{00001001}\underline{\texttt{1}}\texttt{000011110010}\underline{\texttt{1}}\texttt{01001001001}\ldots,$$
$$y' = \texttt{0101010001011000000000010101101}\underline{\texttt{1}}\texttt{010}\ldots.$$

Theorem For two independent, Bernoulli(p) sequences X, Y, if p is sufficiently small then X, Y are compatible with positive probability.

So, there is some critical value p_c. Computer simulations suggest $p_c \approx 0.3$. My lower bound is about 10^{-300}.



X, Y are walks on the same graph: say, the complete graph K_m on m nodes. In each instant, either X or Y will move. A demon knows both (infinite) walks completely in advance. She decides every time, whose turn it is and wants to prevent collision. Say:

$$X = \texttt{233334002}\ldots,$$
$$Y = \texttt{001211144 3}\ldots.$$

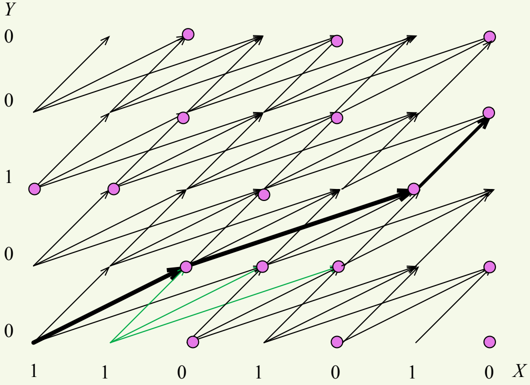
The repetitions are the demon's insertions.

The walks are called compatible if the demon can succeed.

Theorem If m is sufficiently large then in the complete graph K_m, two independent random walks X, Y are compatible with positive probability.

Computer simulations suggest $m = 5$ suffices, maybe even $m = 4$. The bound coming from the proof is $> 10^{500}$.

The three problems are similar: in each of them, we want to fit one random sequence to another, by some non-sequential algorithm. Each of them benefit from a 2-dimensional picture.



Variation

The two other problems also have a formulation involving directed, dependent percolation. They also allow a variation: undirected percolation.

- For the clairvoyant demon (scheduling of random walks), the undirected version was solved by Winkler and, independently, by Balister, Bollobás, Stacey.
- The above undirected percolations have exponential convergence; the three presented models have power-law convergence (see next), so they need new methods.

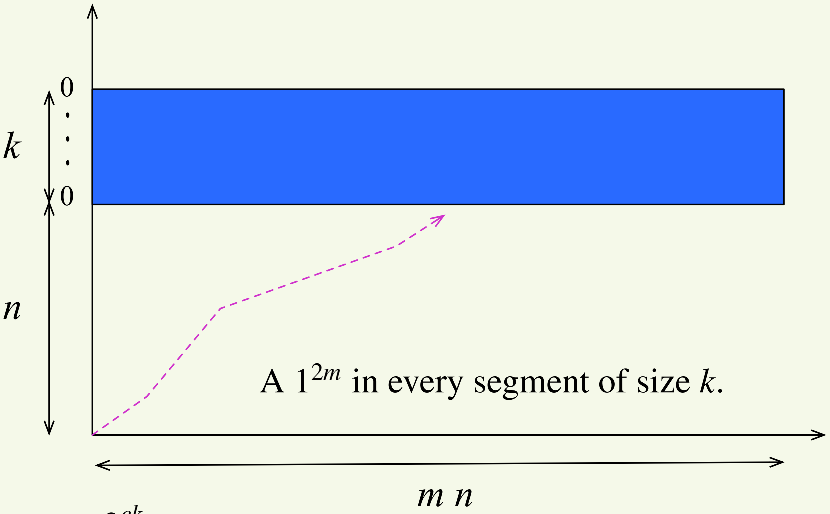
Theorem

$$\mathbb{P}\left[\,(0,0) \text{ is blocked at distance } n \text{ but not closer}\,\right] > n^{-c}$$

for some constant $c > 0$ depending on m.

In typical percolation theory, this probability decreases exponentially in n.

A situation that occurs with at least $n^{-\mathrm{const}}$ probability:



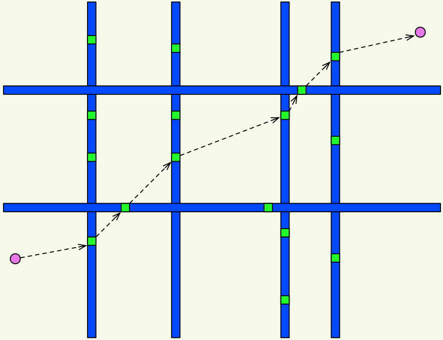
$n = 2^{ck}$

Messy, laborious, crude, but robust.
Contrary to undirected percolation, the obstacles to percolation do not not form a contour of closed point. We will classify them.

Example When 0^k occurs in the Y sequence, this forms a kind of horizontal wall of thickness k. You can only penetrate it at a place of X with at least k 0's placed closer than m to each other (a fitting vertical hole).
If the probability of a wall is p
the probability of a fitting hole is p^c, $c < 1$ constant.

We will find other obstacles: traps, and dirty points (something like closedness).

First-order approximation, using scapegoats

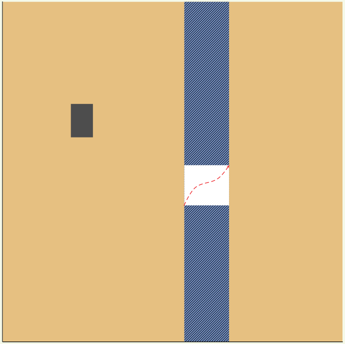


- Holes through walls normally dense
  (where not, a higher-order trap).
- Walls normally well separated from each other
  (where not, higher-order wall).
- Normally, no walls near the endpoints
  (where not, the endpoint is higher-order dirty).

An abstract random process (generating mazes. . .) that models the obstacles on top of the random graph.

Bad event

- wall (stripe),
- trap (rectangle),
- dirty point both in the plane and its two projections.



Good event To each wall, fitting holes where it can be passed.

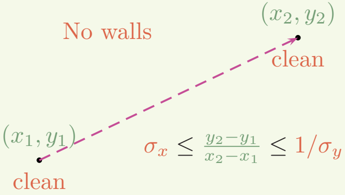
Combinatorial conditions, independences, probability bounds.
Some parameters, among them $\Delta, \sigma_x, \sigma_y$, with $1/\sigma_y > 1.5\sigma_x$.

Upper bound on the size of walls and traps Δ

Density of clean points Every trap- and wall-free square of size 3Δ contains a clean point in its middle part.

Reachability

A clean point is reachable from another clean point if there is no trap or wall between, and the slope between them is bounded below and above:



Upper bounds on the probability of walls, traps, dirt.

Lower bound on the probability of holes.

We will prove

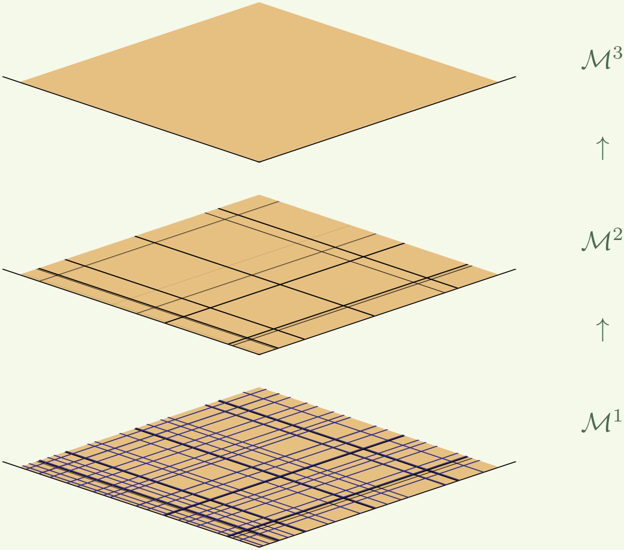
Lemma If m is sufficiently large then a sequence of mazeries $\mathcal{M}^k$, $k > 1$ can be constructed on a common probability space, sharing the original random graph, and satisfying

$$\sum_{k=1}^{\infty} \mathbb{P}\,(\text{ trap or wall of } \mathcal{M}^k \text{ in } [0, \Delta_{k+1}]^2\,) \leqslant 1/8,$$

$$\sum_{k=1}^{\infty} \mathbb{P}\,(\,(0,0) \text{ is clean in } \mathcal{M}^k,\ \text{dirty in } \mathcal{M}^{k+1}\,) < 1/8,$$

$$8\Delta_k/\Delta_{k+1} < \sigma_{x,k}, \sigma_{y,k}.$$

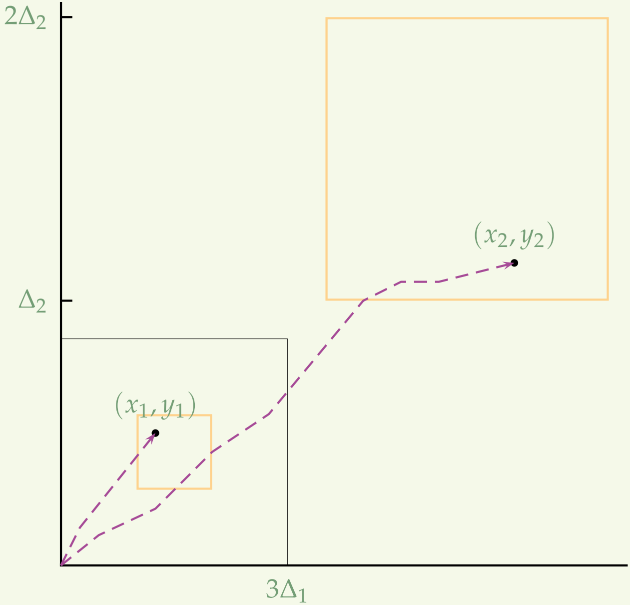
Walls in higher-order mazeries are much farther apart.



Proof of the embedding theorem

Using the lemma show that with positive probability, arbitrarily far points are reachable from the origin.

- We can assume that for all k, the origin is clean, and the square $[0, \Delta_{k+1}]^2$ is trap- and wall-free.
- The density condition gives a clean point (x_k, y_k) with $x_k \geqslant \Delta_{k+1}/2$ that satisfies the slope bounds in $\mathcal{M}^k$ with respect to $(0, 0)$.
- The reachability condition of $\mathcal{M}^k$ implies that (x_k, y_k) is reachable from $(0, 0)$.


$2\Delta_2$
(x_2, y_2)
Δ_2
(x_1, y_1)
$3\Delta_1$

We outline the operation $\mathcal{M}^k \mapsto \mathcal{M}^{k+1}$.
The obstacles of $\mathcal{M}^{k+1}$ are scapegoats for the violation of reachability at the scale Δ_{k+1}. These are

New dirt is caused by traps or walls of $\mathcal{M}^k$ nearby a point.

Emerging traps due to lack of holes on a too long stretch of a wall of $\mathcal{M}^k$.

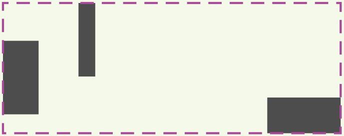
Compound traps: pairs of traps that are too close (uncorrelated and correlated).

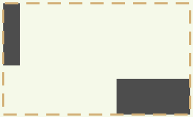
Emerging walls (2 kinds) caused by high conditional probability of some new traps.

Compound walls: too close pairs of certain walls.

Emerging trap of the missing-hole type: a large wall segment not penetrated by any hole.

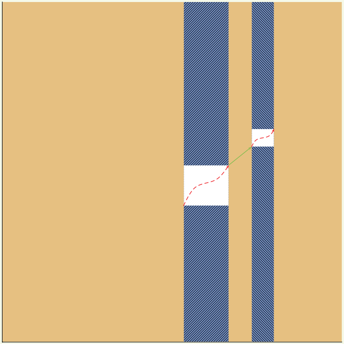
Compound trap uncorrelated and horizontal correlated:





Emerging wall where the conditional probability of a missing-hole trap or a correlated compound trap is not small.

Compound wall penetrable only at a fitting pair of holes.



The actual mazery concept comes with a number of finer distinctions.

Examples

1. We distinguish barriers and walls.
   - Barriers have good independence properties (are determined by the X or Y sequence contained in them).
   - Walls have good combinatorial properties (can be cleanly separated from each other).

   All walls are barriers, so we will be able to benefit from the useful properties of both.
2. Each wall has a positive rank. Higher rank implies lower probability. At $\mathcal{M}^k \mapsto \mathcal{M}^{k+1}$ we delete only the walls of low rank, and use only low-rank walls for compounding.

The following combinatorial conditions on a mazery always allow separating the walls:

- A maximal wall-free interval is inner clean.
- The area between two maximal wall-free intervals of size $\geqslant \Delta$ is spanned by a sequence of walls with inner-clean wall-free intervals between them.

Exact definition of compound wall achieves two things:

- upperbound its probability,
- lowerbound the probability of a hole through it.

Solution: A horizontal compound barrier $W_1 + W_2$ occurs wherever barriers W_1, W_2 occur (in this order) at some small distance d, and W_1 has small rank. Its rank is defined as

$$r_1 + r_2 - \lceil \log d \rceil.$$

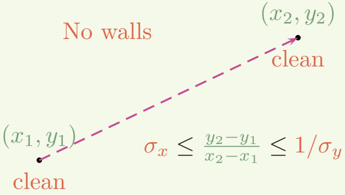
Call this barrier a wall if W_1, W_2 are walls separated by an inner-clean wall-free interval.

Some hard parts

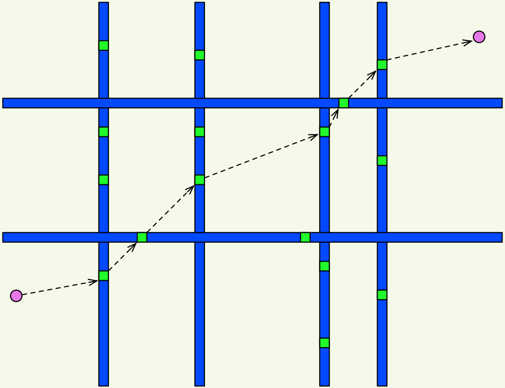
- The lower bound condition on holes, and its proof on a compound holes.
- Proving the reachability condition in $\mathcal{M}^{k+1}$.

Recall the reachability condition:

A clean point is reachable from another clean point if there is no trap or wall between, and the slope between them is bounded below and above:



To prove the same condition in $\mathcal{M}^{k+1}$, we can use the same condition in $\mathcal{M}^k$, plus:



- Enough holes through walls.
- No walls or traps near endpoints.
- Walls well separated from each other.

- The remaining traps of $\mathcal{M}^k$ are controlled by absence of compound traps (messy).