Compatible sequences and a slow Winkler percolation

Péter Gács

Computer Science Department
Boston University

April 9, 2008

Imagine two infinite 0-1 sequences x, y running in parallel, like a conversation.

Definition

We call x, y compatible if we can delete some 0's from both, so that in the resulting sequences x', y', we never have a collision $x'(i) = y'(i) = 1$.

Example

The following two sequences are not compatible:

$$
\begin{aligned}
x &= \texttt{0001100100001111}\ldots,\\
y &= \texttt{1101010001011001}\ldots.
\end{aligned}
$$

The x, y below are. (We insert $\underline{\texttt{1}}$ in x instead of deleting the corresponding $\texttt{0}$ of y.)

$$
\begin{aligned}
x &= \texttt{0000100100001111001001001001001}\ldots,\\
y &= \texttt{0101010001011000000010101101010}\ldots,\\
x' &= \texttt{00001001}\underline{\texttt{1}}\texttt{000011110010}\underline{\texttt{1}}\texttt{01001001001}\ldots,\\
y' &= \texttt{0101010001011000000010101101}\underline{\texttt{1}}\texttt{010}\ldots.
\end{aligned}
$$

The problem

Given two independent, i.i.d. random 0-1 sequences X, Y.
$\mathbf{P}[X(i) = 1] = \mathbf{P}[Y(i) = 1] = p$.
They are never compatible with probability 1, since $\mathbf{P}[X(1) = Y(1) = 1] > 0$. It is easy to see that for $p \geqslant 1/2$ they are compatible with probability 0.

Problem

Is there a threshold for p below which they are compatible with positive probability?

An "application"

A *leisurely, erratic conversation* (say, in a retirement home), between X and Y.

- $X(i) = 1$: X is talking at turn i.
- $X(i) = 0$: he is listening.

Leisurely, since it lasts forever. Erratic, since determined by the i.i.d. sequences X, Y.
A nurse wants to help. She can only put, say, X to sleep temporarily, while Y is only listening (she inserts 1 into the sequence X). This only postpones the actions of X. She wants that every time one party talks, the other one listens.
The nurse is a fairy, she is clairvoyant, sees both infinite sequences X, Y. She can do this iff X, Y are compatible.

Phase transition

Theorem (Winkler, Kesten)

If $p > 0.44$ then X, Y are incompatible with probability 1.

Proof sketch. X, Y are compatible iff we can delete 0's from both, so that they become complementary. If $p \approx 1/2$ then we cannot delete much. So, by changing the two sequences just a little bit, X' would completely determine Y', making the joint entropy of n bits of both X' and Y' only $\approx n$; but it is really $\approx 2n$. □

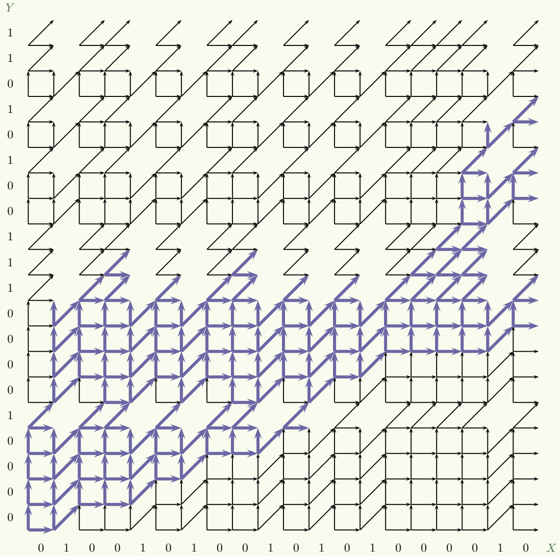
Theorem (Main)

If p is sufficiently small then with positive probability, X, Y are compatible.

So, there is some critical value p_c. Computer simulations by John Tromp suggest $p_c \approx 0.3$. My lower bound is about 10^{-300}.

An oriented percolation

The graph $G(X, Y)$.



This sort of percolation, where two infinite random sequences X, Y are given and the openness of a point or edge at position $\langle i, j \rangle$ depends on the pair $\langle X(i), Y(j) \rangle$, is called Winkler percolation.

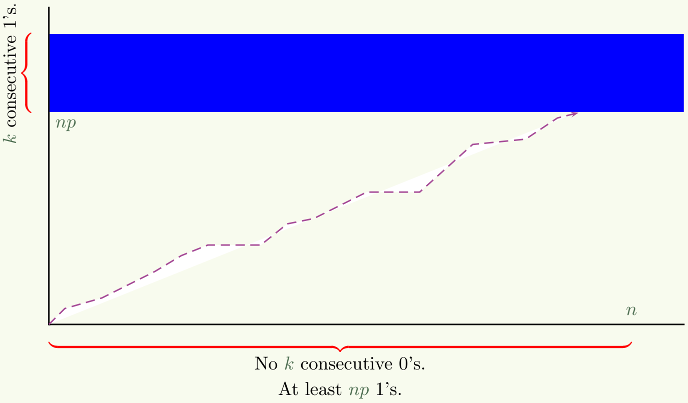
Power-law behavior

Theorem

$$\mathbf{P}\,[\,\langle 0,0\rangle \text{ is blocked at distance } n \text{ but not closer}\,] > n^{-c}$$

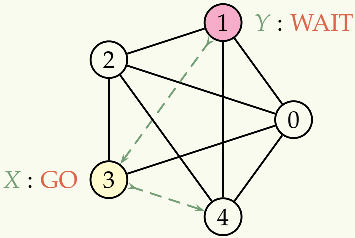
for some constant $c > 0$ depending on p.

In typical percolation theory, this probability decreases exponentially in n.



Power-law behavior. For $k = c_1 \log n$, the probability that the above three events hold simultaneously is at least $\boxed{n^{c_2 \frac{\log p}{p}}}$.

Related synchronization problems



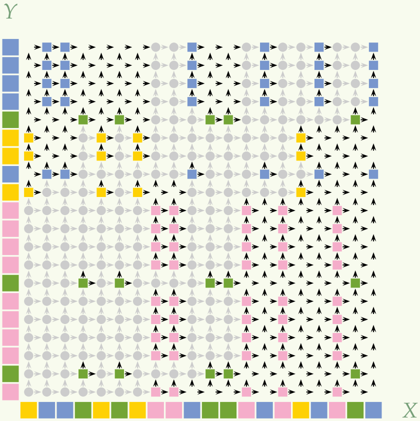
X, Y are tokens performing independent random walks on the same graph: say, the complete graph K_m on m nodes. In each instant, either X or Y will move. A demon decides every time, whose turn it is. She is clairvoyant and wants to prevent collision.

Example

$$X = \texttt{233334002}\ldots,$$
$$Y = \texttt{001211143}\ldots.$$

The repetitions are the demon's insertions.

Clairvoyant demon percolation: random walk on the complete graph K_4.



$0 \to$ ■, $1 \to$ ■, $2 \to$ ■, $3 \to$ ■.

This problem (also by Winkler) had been open for 10 years, and the current problem was designed as a similar but easier one. By now I have solved it, by a similar (somewhat more complex) method.

Related results

In the color percolation problem, we may ask: how about the undirected percolation? (Allowing the demon to move backward on the schedule, as well as forward.)

This problem has been completely solved by Winkler and, independently, by Balister, Bollobás, Stacey. It is known exactly for which Markov processes does the corresponding undirected percolation actually percolate. For random walks on K_m, there is undirected percolation for $\boxed{m > 3}$.

The undirected color percolations have exponential convergence; their methods will probably not apply to the *directed* case, which has power-law convergence (by an argument similar to the one for the "chat" percolation).

The color percolation problem is harder. In the present work we can rely on monotonicity (and the FKG inequality, and there is a natural concept of "wall" (see below).

Method: combinatorial renormalization

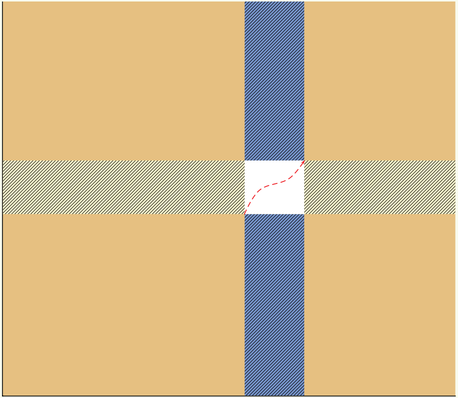
Messy, laborious, crude (forget exact constants!), but robust. For "error-correction" situations.

For appropriate $\Delta_1 < \Delta_2 < \cdots$, define the square $\square_k = [0, \Delta_k]^2$. Let $\mathcal{F}_k$ be some ultimate bad event in $\square_k$. (Here, the fact that $(0, 0)$ is blocked in $\square_k$.) We want to prove $\boxed{\mathbf{P}(\bigcup_k \mathcal{F}_k) < 1}$.

1. Identify simple bad events and very bad events: the latter are much less probable.
2. Define a series $\mathcal{M}^1, \mathcal{M}^2, \ldots$ of models similar to each other, where the very bad events of $\mathcal{M}^k$ become the simple bad events of $\mathcal{M}^{k+1}$.
3. Prove $\mathcal{F}_k \subseteq \bigcup_{i \leqslant k} \mathcal{F}'_i$ where $\mathcal{F}'_k$ says that some simple bad event of $\mathcal{M}^k$ happens in $\square_{k+1}$.
4. Prove $\sum_k \mathbf{P}(\mathcal{F}'_k) < 1$.

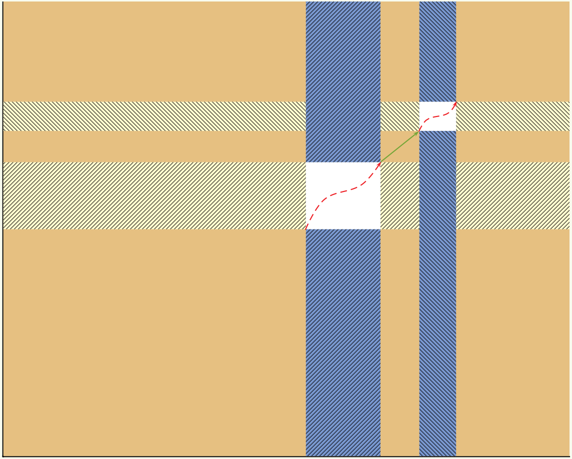
Mazeries

Bad event: a wall in $\square_{k+1}$. We also need good events: to each wall, a fitting hole (see “power-law”).



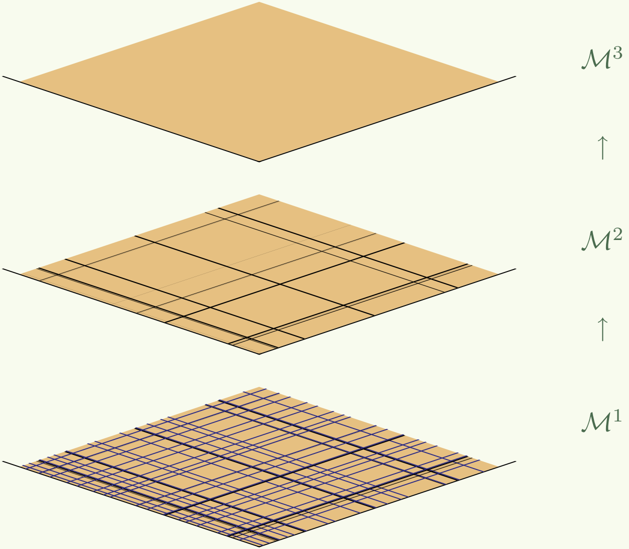
The model $\mathcal{M}^k$ is built on abstract walls of various types, and fitting holes. $\mathcal{M}^k$ itself is called a mazery (a system for making mazes).

Walls are the *simple bad events*; what are the *very bad events*?



When two (vertical) walls occur “too close” to each other (some parameter says how close), we get a compound wall of $\mathcal{M}^{k+1}$, penetrable only at a fitting (horizontal) compound hole.

The other kind of (say, horizontal) very bad event occurs if a certain type of hole is completely missing on a (some parameter says how) “large interval”. This gives rise to a (vertical) wall of $\mathcal{M}_0^{k+1}$ called an emerging wall. A fitting (horizontal) hole is an interval of comparable size in $\mathcal{M}_0^{k+1}$ without any wall at all.


$\mathcal{M}^3$
↑
$\mathcal{M}^2$
↑
$\mathcal{M}^1$

Renormalization

The operation $\mathcal{M}^k \mapsto \mathcal{M}^{k+1}$: remove isolated walls (and holes), introduce the new, higher-level walls.

Classical renormalization: Say, when the Ising model is subdivided into large blocks, and the spins of each block are summed up into super-spins.

Combinatorial renormalization is more complex: the system of concepts delivering self-similarity is different in each situation.

Distilled leftovers

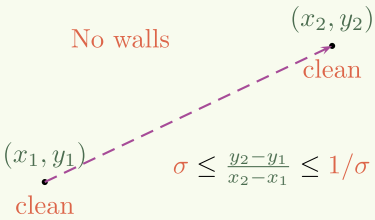
Some parts of the model $\mathcal{M}^k$ function as the still needed effects of suppressed details of $\mathcal{M}^1, \ldots, \mathcal{M}^{k-1}$. These are the notions of clean points and a slope constraint.

We will have the following properties, with a

$$\sigma < 1/2.$$

Condition (Reachability)

Lack of walls, cleanness and the slope constraints imply reachability.

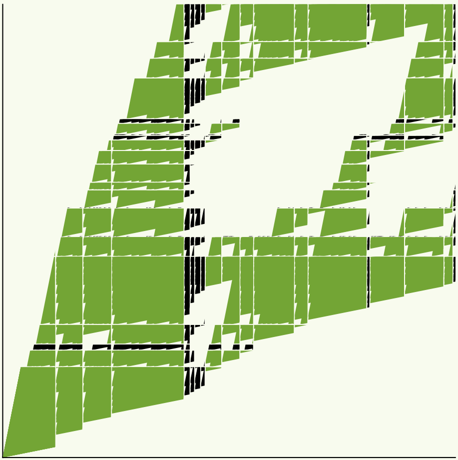


Condition (Enough clean points)

Every interval of size $3\Delta_k$ that does not contain walls, contains a clean point in its middle part.

Condition (Inherited cleanness)

The event that 0 is not clean is in $\mathcal{M}^k$ is in $\bigcup_{i<k} \mathcal{F}'_i$.



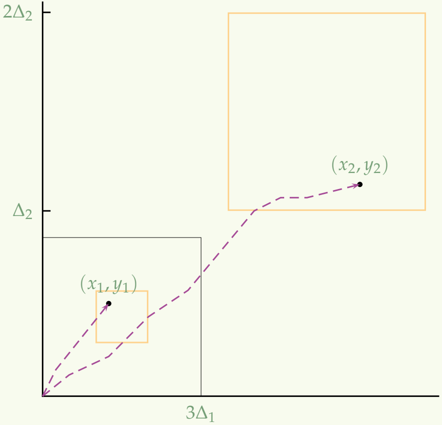
Walls and holes are shown with size 0. There are two wall types. Holes appear 10 times more frequently than walls. The minimum slope is 0.2. Cleanness not shown. Only the clean dark points are really reachable. Black: compound walls?

Main lemma

Lemma (Main)

If p is sufficiently small then the sequence $\mathcal{M}^k$ can be constructed, in such a way that it satisfies Conditions 1,2,3 and $\sum_k \mathbf{P}(\mathcal{F}'_k) < 1$.

Proof of the theorem from the lemma: Assume $\bigcup_k \mathcal{F}'_k$ does not hold.
By the Inherited Cleanness condition, 0 is clean in each $\mathcal{M}^k$.
By the condition on Enough Clean Points, for each k, there is a point $\langle x^k, y^k \rangle$ in $[\Delta_k, 2\Delta_k]^2$ that is clean in $\mathcal{M}^k$.
For each k, it also satisfies the slope constraint $1/2 \leqslant y^k/x^k \leqslant 2$. Hence, by the Reachability Condition, is reachable from $\langle 0, 0 \rangle$.


$2\Delta_2$
Δ_2
$3\Delta_1$
(x_1, y_1)
(x_2, y_2)

Remarks on the proof

The challenging parts of the proof are the following.

- To give the combinatorial definitions of walls, cleanness, and so on in a way that provides the independence and monotonicity properties needed for the probability estimates.
- There will be a constant $0 < \gamma < 1$ (independent of the level k) with the property that if a wall has probability upper bound p then the corresponding hole has probability lower bound p^{γ}. The proof of the probability lower bound on holes is a little delicate.
- Wall types have to be defined carefully, to avoid a proliferation of them (that would cause problem with probability bounds).

Open problems

- Simplify!
- Improve the lower bound on the threshold p_c! (Mine is only 10^{-300}.)
- Three sequences?