ONE-DIMENSIONAL UNIFORM ARRAYS THAT WASH OUT FINITE ISLANDS

P. Gach, G. L. Kurdyumov, and L. A. Levin

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Both deterministic and probabilistic one-dimensional uniform systems of finite automata with local interaction are considered. A state of a deterministic system is called attracting if it is maintained in time and any finite deviation from it disappears over a finite time. Three simple examples are given of systems with a nonunique uniform attracting state. Results of computer simulations of probabilistic systems obtained by superimposing random noise on such systems are given. The simulation results indicate that the systems may be nonergodic in the case of low noise.

§1. Introduction

A one-dimensional uniform random array S (or, briefly, an array) is a uniform chain of interacting finite automata $\{s_i\}$ that is infinite in both directions. It operates in discrete time $t = 0, 1, \ldots$. The state of automaton s_i at time $t + 1$ (which we denote by s_i^{t+1}; $s_i^{t+1} \in X$, where X is the finite set of possible automaton states s_i) depends probabilistically on the states of a finite number of automata in the network at t. If the states of all automata s_i^u at all points in time $u \le t$ are specified, then the states at $t + 1$ are conditionally independent, and their conditional probabilities are defined by the set of integers $j_1, \ldots, j_k$ and matrix φ (of order $\underbrace{n\times n\times\ldots\times n}_{k+1}$, where n is the cardinality of set X):

$$P[s_i^{t+1}=y]=\varphi_y(s^t_{i+j_1},\ldots,s^t_{i+j_k}).$$

Matrix φ naturally generates operator S_φ on the space of all probability measures over the set of states of the array Ω (where $\Omega = X^Z$, Z is the set of all integers) (see [1]).

Measure μ is called invariant if $S_\varphi \mu = \mu$; the array is called ergodic if it has exactly one invariant measure. In this short paper we will deal primarily with deterministic arrays, i.e., ones such that $\varphi_y(x_1, \ldots, x) \in \{0, 1\}$. Such arrays are described by some function $\lambda : X^k \to X$; $\lambda(x_1, \ldots, x_k) = y \to \varphi_y(x_1, \ldots, x_k) = 1$. Assume that for $x \in \Omega$ we have $(S_\lambda x)_i = \lambda(x_{j+j_1}, \ldots, x_{i+j_k})$; then S_λ is the transition function of chain of automata s_i. We will say that array S_φ is obtained by superimposing noise of level α on array S_ψ (and write $S_\varphi = S_\psi^{(\alpha)}$) if $\varphi = (1-\alpha)\psi + (\alpha/n)e$, where n is the cardinality of X and e is a unit matrix. States x and y of deterministic array S_λ are called equivalent if there exists a t for which $S_\lambda^t x = S_\lambda^t y$. State x is called attracting if all the y for which $\{i \mid y_i \ne x_i\}$ is a finite set are equivalent to it.

Paper [1] offered a hypothesis to the effect that every one-dimensional uniform array with positive transition probabilities is ergodic. Results of computer simulations of various uniform arrays were offered in confirmation of this hypothesis. A major piece of evidence against this hypothesis is the example of a nonergodic nonuniform (either with respect to space or to time) random system with positive transition probabilities, proposed by Tsirel'son in [2]. This result was strengthened by Kurdyumov, with the result that the hypothesis in question was rejected. Thus there exists a nonergodic array for which all the transition probabilities $\varphi_y(x_1, \ldots, x_k)$ are positive. This array can be obtained by superimposing a small noise on some (extremely complex) deterministic array S_*. For S_* we set up two nonequivalent attracting states x_1 and x_2. They are also extremely complex, aperiodic, and do not maintain themselves over time. On the basis of x_1 and x_2, we can set up invariant measures for the corresponding probabilistic array.

A deterministic array will be called conservative if it has nonequivalent periodic attracting states. The notion of conservative array can be naturally generalized to the case of dimensions greater than 1. Paper [3] cites examples of multidimensional conservative arrays. The same paper establishes a relationship between

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the conservativeness of an array and the nonergodic nature of the probabilistic array obtained from it by adding a small random noise, while paper [4] offers examples in which this relationship is violated. The problem of finding simple conservative arrays arose in conjunction with the notion of finding a nonergodic one-dimensional array with positive transition probabilities which would have spatially uniform invariant measures and would be much simpler than the one proposed by Kurdyumov. The first version of a one-dimensional conservative array was proposed by Gach. Subsequently, on the basis of the same ideas, Levin proposed three simple media S_{II}, S_{IV}, S_{VI}, which will be discussed below.* Kurdyumov carried out the computer simulations of the behavior of these arrays with a small noise superimposed. In a number of cases, the simulation results give grounds for assuming that the systems are nonergodic.

§2. Simple Conservative Arrays

Let us now define three specific conservative arrays S_{II}, S_{IV}, and S_{VI} in which the number of states is 2, 4, and 6, respectively. Functions λ_{II} and λ_{IV} will be defined only partially; their values in the remaining cases will be given supplementary definitions from considerations of symmetry.

Thus, $X_{II} = \{\rightarrow, \leftarrow\}$. Function λ_{II} is permutable with the operation of reflection: x_i goes over to x_{-i} when the directions of the arrows are simultaneously reversed. If $x_i = \leftarrow$, then the direction of the arrow of $(S_{II}x)_i$ is determined by voting; it will be the predominant direction among x_i, x_{i+1}, x_{i+3}.

There are more states in S_{IV}, but the dependence is only on the nearest neighbors. We have $X_{IV} = \{\rightarrow, \leftarrow, \uparrow, \downarrow\}$, $(S_{IV}x)_i = \lambda_{IV}(x_{i-1}, x_i, x_{i+1})$. Function λ_{IV} is permutable with the reflection operation ($x_i \rightarrow x_{-i}$ when the right and left arrows are simultaneously replaced), and is given by the following expressions:

1. $\lambda_{IV}(\rightarrow, x, y) = \rightarrow$, if $x, y \neq \leftarrow$;

2. $\lambda_{IV}(x, \rightarrow, y) = \begin{cases} \downarrow \text{ for } x \in \{\leftarrow, \uparrow\}, \\ \rightarrow \text{ otherwise;} \end{cases}$

3. $\lambda_{IV}(x, y, z) = \uparrow$, if $y \in \{\downarrow, \uparrow\}$ and case 1 does not occur.

THEOREM. Arrays S_{II} and S_{IV} are conservative; the states $x_i \equiv \rightarrow$ and $x_i \equiv \leftarrow$ are attracting states for them.†

Proof. The behavior of S_{II} and S_{IV} are analogous if we note that zones of not less than three identical arrows in S_{II} correspond to zones of the same arrows in S_{IV}, while zones of alternating arrows correspond to zones of $\uparrow$ (with $\downarrow$ at the ends). Therefore, the proof as carried out for S_{IV} can be readily transferred to S_{II}. Assume that all the signs of Ω (except for a finite number) are $\leftarrow$. Assume that L(x) is the smallest and R(x) the greatest i for which $x_i = \leftarrow$. Segment (L(x), R(x)) will be called an island (see [3]). We will show that if $n = R(x^0) - L(x^0)$, then $S_{IV}^{6n}x^0$ consists only of $\leftarrow$. Let M(x) be the smallest i for which $x_i = \rightarrow$ (if it exists). By definition, we can readily establish that: 1) $R(S_{IV}x) \leq R(x)$; 2) $L(S_{IV}^2x) \geq L(x) - 1$; hence the left edge of the island cannot move to the left at a rate greater than 1/2; 3) $M(S_{IV}^2x) > M(x)$ if $M(S_{IV}^2x)$ exists, and therefore the $\rightarrow$ vanish from x not later than the 2n-th step; 4) if there are no $\rightarrow$ in x, then $R(S_{IV}x) < R(x)$. Therefore, an island which now has a side not greater than 2n will vanish over the next 4n steps. QED.

We will define yet another version of a conservative array with six states, $X_{VI} = \{+, -, \rightarrow, \leftarrow, \nrightarrow, \nleftarrow\}$. Function λ_{VI} is permutable with the operation that changes the plus and minus signs and the direction of the arrows, and also with the reflection operation (changing the direction of the arrows and carrying x_i to x_{-i}). Each of the following equations is satisfied if the value of λ_{VI} is not implied by the preceding ones:

1. $\lambda_{VI}(x, y, x) = x$;	2. $\lambda_{VI}(+, x, -) = \nleftarrow$
3. $\lambda_{VI}(x, \nrightarrow, y) = \rightarrow$;	4. $\lambda_{VI}(\leftarrow, x, \rightarrow) = +$;
5. $\lambda_{VI}(+, \rightarrow, x) = +$;	6. $\lambda_{VI}(\rightarrow, +, x) = \nrightarrow$;

For S_{VI} the states $x_i \equiv +$ and $x_i \equiv -$ are attracting states.

§3. Experiments

Results of computer simulation of some uniform arrays (with the intent of clarifying the issue of their ergodicity) are given in [5]. Below we offer simulation results for arrays $S_{VI}^{(\alpha)}$, $S_{IV}^{(\alpha)}$, and $S_{II}^{(\alpha)}$. We consider a finite segment of the array (2000-5000 automata long) that was bent into a ring (to avoid any boundary effect)

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*Some versions of one-dimensional conservative media were also proposed by M. G. Shnirman.
†Evidently S_{II} and S_{IV} do not have other attracting states.

TABLE 1. Arrays $S_{VI}^{(\alpha)}$

No. of trial	Ring length	Initial state	Value of α
1	5000	$Q_+=1$	0,01
2	2000	$Q_+=1$	0,03
3	2000	$Q_+=1/3;\ Q_-=2/3$	0,03
4	5000	$Q_+=1$	$\alpha=0{,}05$ for $t\leq 1000$
5			$\alpha=0{,}01$ for $t>1000$

and that operated for 1000-2000 time cycles. We employed pseudorandom numbers generated by one of the standard algorithms. The proportions Q_x of automata in particular states x at the end of certain definite time intervals were printed out. Arrays $S_{VI}^{(\alpha)}$ were studied in greatest detail. Figure 1 shows the time dependence of Q_+ and Q_- in four different trials. Different trials corresponded to different values of the ring length, noise (α), and initial state (see Table 1). Since S_{VI} is symmetrical with respect to replacement of + and −, when $S_{VI}^{(\alpha)}$ is ergodic the limits Q_+ and Q_- should be equal. They should be observed to converge in simulation. In trials 1 and 2, for an initial state $x_i \equiv +$, the value of Q_+ (beginning roughly with t = 200) fluctuates only randomly around a certain value (much greater than 0.5). At all moments that are multiples of 100, $Q_+ > 0.85$. In trial 3, the initial state was taken to be a random sequence of plus and minus signs with a probability of a minus sign of 2 / 3. In this case, Q_- increases sharply by t = 300, and then is not observed to be less than 0.8. We should note, however, that in this trial Q_- is always less than Q_+ in trial 2. In trial 4 the initial state was $x_i \equiv +$; the value of α was 0.05 up to t = 1000, changing thereupon to 0.01. On the interval $0 \leq t \leq 1000$, Q_+ displayed a clear tendency to decrease; this can be regarded as an argument favoring the ergodicity of $S_{VI}^{(0.05)}$. But the rapid increase in Q_+ after t = 1000 is an additional argument in favor of $S_{VI}^{(0.01)}$ being nonergodic. We also simulated an array $S_{VI}^{(0.15)}$ beginning with $x_i \equiv +$, but even for t = 100, Q_+ and Q_- were virtually equal and subsequently their plots crossed repeatedly. The results of this trial give every reason to assume that $S_{VI}^{(0.15)}$ is ergodic. Thus the simulation results indicate that $S_{VI}^{(\alpha)}$ is ergodic for $\alpha \geq 0.05$ but nonergodic for $\alpha \leq 0.03$.

It seems very natural to assume that the ergodicity of arrays of type $S_{VI}^{(\alpha)}$ may result from random creation of large islands. Correspondingly, in one of the trials with $S_{VI}^{(0.03)}$ we took an initial state $x_i = -$ for $10 \leq i \leq 100$ and $x_i = +$ for the remaining i. The island ceased to be "monolithic," its boundaries shifting at random, but up to t = 1000 its dimensions did not change markedly.

Much greater noise stability is displayed by arrays S_{II} and S_{IV}. In simulating $S_{IV}^{(0.05)}$ from the initial $x_i \equiv -$ for $t \leq 2000$ we did not observe Q_- values lower than 0.78; their systematic decrease ceased even for t = 100. 100. In simulating $S_{II}^{(0.2)}$ an island of length 100 was completely eliminated over the first 420 cycles, while Q_- was not observed to be lower than 0.66.

In concluding, we should note that, despite the simulation results, the conclusion that $S_{II}^{(\alpha)}$, $S_{IV}^{(\alpha)}$, and $S_{VI}^{(\alpha)}$ are nonergodic for small α is not completely persuasive. The authors wish to allow for the possibility that all these arrays are ergodic; this seems to be particularly plausible when the noise is asymmetrical (e.g., the

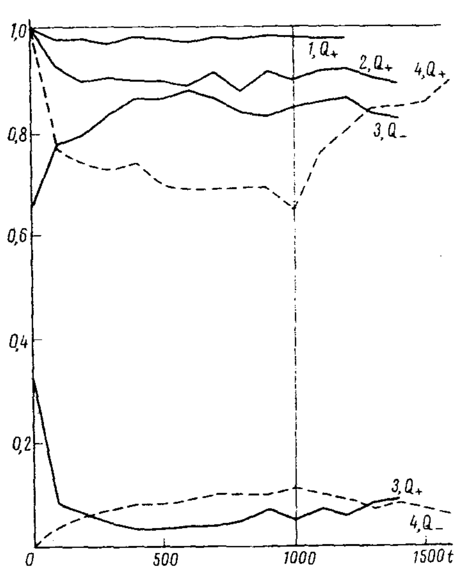

Fig. 1. Proportion of automata Q_+ (Q_-) in state + (−), as a function of time t.

probability of a random transition to state → for S_{II} is greater than the corresponding probability for state ←). Nonetheless, even if convergence to a single invariant measure occurs, we are dealing with extremely slow convergence. This "quasinonergodicity" may be of independent interest.

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