Clairvoyant scheduling of random walks

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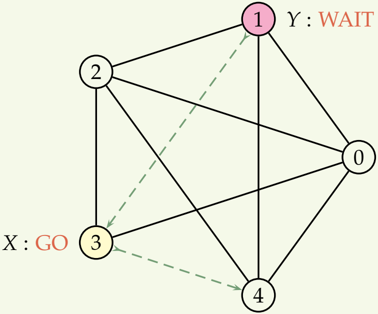
The clairvoyant demon problem



X, Y are tokens performing walks on the same graph: say, the complete graph K_m on m nodes. In each instant, either X or Y will move. A demon knows both (infinite) walks completely in advance. She decides every time, whose turn it is and wants to prevent collision. Say:

$$X = \texttt{233334002}\dots,$$

$$Y = \texttt{0012111443}\dots.$$

The repetitions are the demon's insertions.

The walks are called compatible if the demon can succeed. A graph is roomy if two independent random walks X, Y on it are compatible with positive probability.

Question

Which graphs are roomy?

Until now, no roomy graphs were known. For simplicity, let us look only at complete graphs. The triangle K_3 is definitely not roomy.

Theorem (Main)

If m is sufficiently large then the complete graph K_m is roomy.

Computer simulations suggest that already K_5 is roomy, and maybe even K_4. The bound coming from the proof of the theorem is above 10^{500}.

Origin in distributed computing

Find a leader among a finite number of processes, in a communication graph.
Proposed algorithm:

> *At start, each process has a token. Each token performs a random walk. Collision: tokens merge. The process with the remaining single token becomes the leader.*

Timing is controlled by an adversary (demon).

- Non-clairvoyant adversary: leader will be found in $O(n^2)$ expected time.
- Clairvoyant:

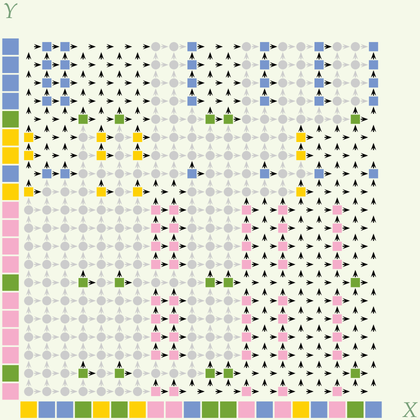
Question

Can the demon keep two distinct tokens apart forever?

Percolation

Let us transform the problem into a graphical form.

- A related problem: given two finite walks, give a polynomial algorithm to decide whether they are compatible.
- Dynamic programming leads to a 2-dim reachability picture.
- Alon: this transforms the scheduling problem into percolation (Winkler percolation...).



$0 \to$ [pink], $1 \to$ [yellow], $2 \to$ [green], $3 \to$ [blue].

Random walks on the complete graph K_4.

Related results

There are interesting variations of the original problem.

- Undirected percolation problem: demon can move backward on the schedule, as well as forward.
- Completely solved by Winkler and, independently, by Balister, Bollobás, Stacey.
- Known exactly for which Markov processes does the corresponding undirected percolation actually percolate. For random walks on K_m, one needs just $\boxed{m > 3}$.
- The undirected color percolations have exponential convergence; the directed case has power-law convergence (see next), so it needs new methods.
- Winkler introduced also a simpler "compatible sequences" problem. I have also shown it to have power-law behavior (in a work whose methods are used here).

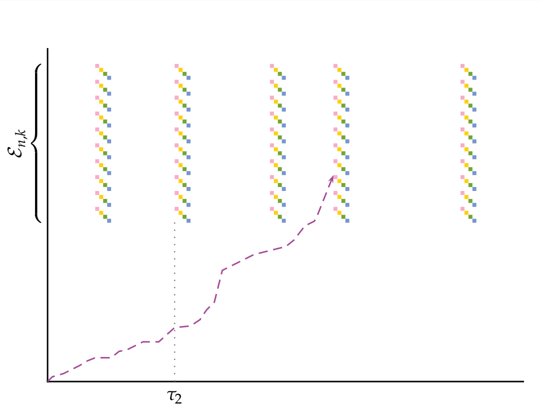
Power-law behavior

Theorem

$$\mathbf{P}[\,(0,0) \text{ is blocked at distance } n \text{ but not closer }\,] > n^{-c}$$

for some constant $c > 0$ depending on p.

In typical percolation theory, this probability decreases exponentially in n.



A long horizontal wall occurs, with only polynomially small probability.

Method: renormalization

Messy, laborious, crude, but robust. For "error-correction" situations.

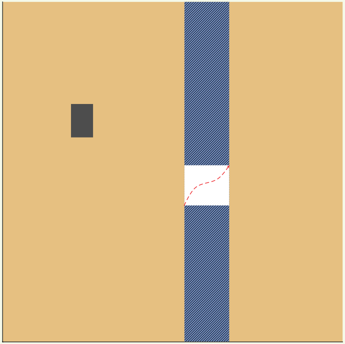
For appropriate $\Delta_1 < \Delta_2 < \cdots$, define the square $\Box_k = [0, \Delta_k]^2$. Let $\mathcal{F}_k$ be some ultimate bad event in $\Box_k$. (Here, that $(0,0)$ is blocked in $\Box_k$.) We want to prove $\boxed{\mathbf{P}(\bigcup_k \mathcal{F}_k) < 1}$.

1. Identify simple bad events and very bad events: the latter are much less probable.
2. Define a series $\mathcal{M}^1, \mathcal{M}^2, \dots$ of models similar to each other, where the very bad events of $\mathcal{M}^k$ become the simple bad events of $\mathcal{M}^{k+1}$.
3. Prove $\mathcal{F}_k \subset \bigcup_{i \leqslant k} \mathcal{F}'_i$ where $\mathcal{F}'_k$ says that some bad event of $\mathcal{M}^k$ happens in $\Box_{k+1}$.
4. Prove $\sum_k \mathbf{P}(\mathcal{F}'_k) < 1$.

Mazeries

Bad event A trap (rectangle) or a wall (stripe).

Good event To each wall, a fitting hole (see “power-law”).



Distilled leftovers

Some parts of the model $\mathcal{M}^k$ may function as the still needed effects of suppressed details of $\mathcal{M}^1, \dots, \mathcal{M}^{k-1}$. In our case, these are the notions of clean points and a condition called slope constraint.

The mazery $\mathcal{M}^k$ is a random process consisting of abstract traps and walls of various types, holes fitting the walls, and set of clean points. It obeys some conditions.
For example: there will be constant parameter $\chi < 1$ such that when a kind of wall will have a probability upper bound p, holes through it will have a probability lower bound

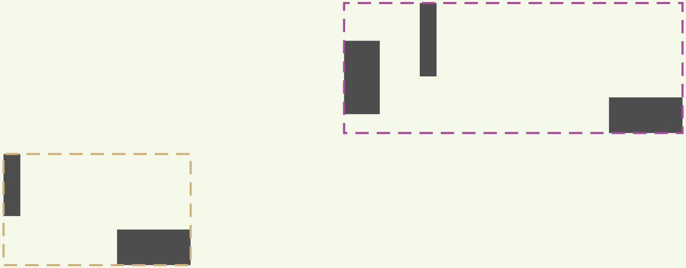
$$p^{\chi}.$$

Renormalization

The operation $\mathcal{M}^k \mapsto \mathcal{M}^{k+1}$.
Traps and walls are the bad events (those of $\mathcal{M}^k$); what are the very bad events (bad events of $\mathcal{M}^{k+1}$)?

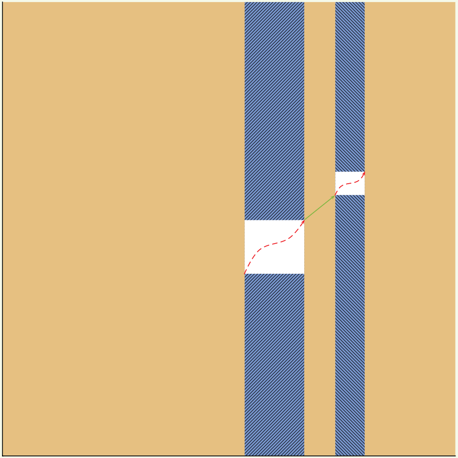
- Compound traps (3 kinds)
- Compound walls
- Emerging walls (2 kinds)

Compound traps



An uncorrelated and a horizontal correlated compound trap.
Trap of the missing-hole type: a large wall segment not penetrated by any hole.

Compund walls



Compound wall penetrable only at a fitting pair of holes.

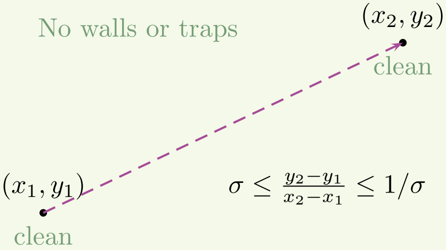
Emerging wall: where the conditional probability of a correlated compound trap or a trap of the missing-hole type is too high.

We will have the following properties, with a $\sigma < 1/2$.

Initial cleanness $\{\,0$ is not clean in $\mathcal{M}^k\,\} \subset \bigcup_{i<k} \mathcal{F}'_i$.

Cleanness density Every square of size $3\Delta_k$ that does not contain traps or walls, contains a clean point in its middle part.

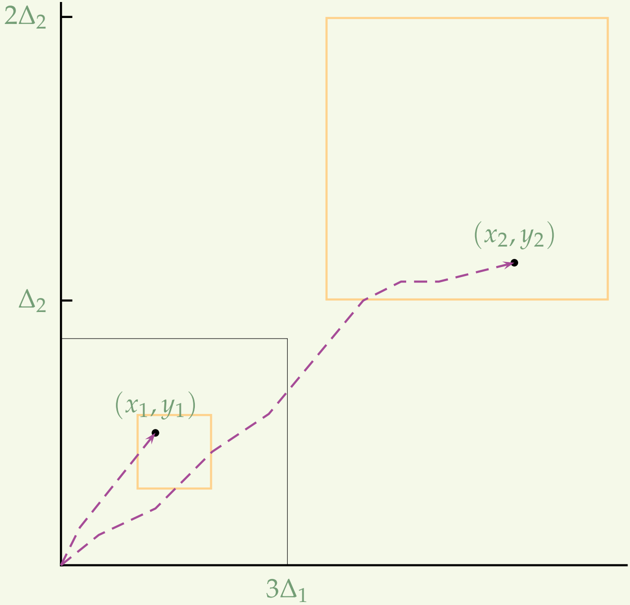
Reachability Lack of walls and traps, cleanness and the slope constraints imply reachability.



Lemma (Main)

If m is sufficiently large then the sequence $\mathcal{M}^k$ can be constructed, in such a way that it satisfies the above conditions and also $\sum_k \mathbf{P}(\mathcal{F}'_k) < 1$.

Proof of the theorem from the lemma. Assume $\bigcup_k \mathcal{F}'_k$ does not hold.
By the initial cleanness condition, 0 is clean in each $\mathcal{M}^k$.
By the cleanness densition condition, for each k, there is a point (x^k, y^k) in $[\Delta_k, 2\Delta_k]^2$ that is clean in $\mathcal{M}^k$.
For each k, it also satisfies the slope constraint $1/2 \leqslant y^k/x^k \leqslant 2$.
Hence, by the reachability condition, is reachable from $(0, 0)$. $\square$


$2\Delta_2$
Δ_2
(x_2, y_2)
(x_1, y_1)
$3\Delta_1$

Mazeries formally

Let us define all elements of the complex model now.

- All our randomness comes from $Z = (X, Y) = (Z_0, Z_1)$, where $X(t), Y(t) \in \{1, \ldots, m\}$ are independent random walks on the graph K_m for some fixed m. This defines a random lattice graph, in the northeast quadrant of $\mathbb{Z}^2$, fixed throughout.
- A mazery $\mathbb{M} = (\mathcal{M}, \Delta, \sigma, R, w, q)$ consists of a random process $\mathcal{M}$, the parameters $\Delta > 0$, $\sigma \geqslant 0$, $R > 0$, and the probability bounds $w > 0$, q. These will be detailed, along with conditions that they must satisfy.

The random process $\mathcal{M}$

We have $\mathcal{M} = (Z, \mathcal{T}, \mathcal{W}, \mathcal{B}, \mathcal{C}, \mathcal{S})$ where

- $\mathcal{T}$ is the set of rectangles called traps
- $\mathcal{W}, \mathcal{B}$ are the sets of walls and barriers,
- $\mathcal{C}, \mathcal{S}$ are functions telling which points are clean and strongly clean in various ways.

All these functions of the process Z will be explained.

Traps

The set $\mathcal{T}$ of random traps is a set of some closed rectangles of size $\leqslant \Delta$. Trap $\mathrm{Rect}(a, b)$ starts at its lower left corner a.

Remark

We will build a sequence of mazeries $\mathcal{M}^1, \mathcal{M}^2, \dots$. The first one's traps are the missing points of the percolation graph (the points (i, j) where $X_i = Y_j$). It has no walls, and all its points are strongly clean in every sense. The other random objects only come in for the higher-order mazeries introduced by renormalization.

Walls

It is the walls that seem to distinguish the Winkler percolations.

- A wall value is a pair (B, r). Here B is a right-closed interval of size $\leqslant \Delta$ called the body, and $r > R > 0$ is a real number called the rank.
  The rank lower bound R is a parameter of the mazery. (The role of ranks will be explained later.)
  Example: $(]5, 10], 108.9)$.
  Let Wvalues be the set of all possible wall values.
- We have the random sets $\mathcal{W}_0 \subseteq \mathcal{B}_0 \subseteq \text{Wvalues}$, where $\mathcal{W}_0, \mathcal{B}_0$ are the set of vertical walls and vertical barriers respectively. (Every wall is a barrier.)
  The horizontal walls and barriers are in $\mathcal{W}_1, \mathcal{B}_1$.

Why barriers?

There is some tension between the need to reason about reachability and the need to estimate probabilities.

- For any rectangle with projections $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$.
- For any interval I, the event that it is a (say, vertical) barrier depends only on $X(I)$.
- The same is not true of walls.
- It is easier to estimate the probability of barriers, but it will be easier to reason combinatorially about the penetration of walls.
- Let $X(I)$ be a potential wall of rank r if there is an extension to $X_1, X_2, \dots$ that makes it a wall of rank r.

Why ranks?

Ranks arise from the need of upper- as well as lower bounds on certain probabilities.

- In defining mazery $\mathcal{M}^{k+1}$ from mazery $\mathcal{M}^k$, we will drop low rank walls of $\mathcal{M}^k$, (those with $\leqslant R_{k+1}$). These walls will have high probability of holes through them, so reachability will be conserved.
- To control the proliferation of walls, a pair of close walls of $\mathcal{M}^k$ will give rise to a compound wall of $\mathcal{M}^{k+1}$ only if at least one of the components has low rank.

One-dimensional cleanness

Cleanness and strong cleanness are described by the random functions $\mathcal{C}, \mathcal{S}$.

- For an interval $I =]a, b]$ or $I = [a, b]$, the point a or b may be called clean in I for the sequence X (clean in the horizontal interval I). It can also be called clean for Y (clean in the vertical interval I).
- A point c is called left-clean for X if it is clean for X in all intervals of the form $]a, c]$ and $[a, c]$.
- To every notion of one-dimensional cleanness there is a corresponding notion of strong cleanness.
- Intuitively a is clean for X in $I =]a, b]$ for mazery $\mathcal{M}^k$ if there are no vertical walls of $\mathcal{M}^i$ very near it in I, for any $i < k$.

Trap-cleanness

Trap-cleanness is described by the random function $\mathcal{T}$.

- For points $u = (u_0, u_1)$, $v = (v_0, v_1)$, different kinds of rectangles: $\mathrm{Rect}^{\uparrow}(u, v)$ is bottom-open, $\mathrm{Rect}^{\rightarrow}(u, v)$ is left-open, $\mathrm{Rect}(u, v)$ is closed.
- Let $Q = \mathrm{Rect}^{\varepsilon}(u, v)$ where $\varepsilon = \rightarrow$ or $\uparrow$ or nothing. Point u or v can be trap-clean in Q.

Complex sorts of cleanness

In what follows we introduce some definitions that will be needed in formulating the conditions.
First, we need combinations of one- and two-dimensional cleanness notions.

- An interval $]a, b]$ is inner clean if both a and b are clean in it.
- Point u is clean in rectangle Q when it is trap-clean in Q and its projections are clean in the corresponding projections of Q.
- If u is clean in all such left-open rectangles then it is called upper right rightward-clean.
- Point u is H-clean in Q if it is trap-clean in Q and its projection on the x axis is strongly clean in the same projection of Q. We define V-clean similarly.

Hops

Hops are intervals and rectangles with some guarantees that they can be passed.

- A right-closed or closed interval is called a hop if it is inner clean and contains no wall. It is a jump if it is strongly inner clean and contains no barrier.
- A rectangle is hop if it is inner-clean (both the lower left and the upper right corners are clean in it) and contains no trap or wall.

Good sequences of walls

Let us define the sequences of walls with some passability.

- Two disjoint walls are called neighbors if the interval between them is a hop.
- An interval I is spanned by the sequence of neighbor walls $W_1, W_2, \dots W_n$ and intervals $I_1, \dots, I_{n-1}$ between them if $I = W_1 \cup I_1 \cup W_2 \cup \cdots \cup W_n$. We allow the sequence to be infinite.
- If there are hops adjacent on the left of W_1 and to the right of W_n then this (possibly infinite) system is called an extended sequence of neighbor walls.

Holes

A (vertical) hole is a rectangle in a (horizontal) barrier where we can pass through. It is called good if it is lower-left and upper-right H-clean.

Conditions on the random process

Dependencies

Most conditions are fairly natural. The first set requires the expected localities and mononotonicities.

- For any rectangle $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$.
- For a vertical wall value E the event that it is a vertical barrier is a function of $X(\mathrm{Body}(E))$.
- For any endpoint of a horizontal interval I, the event that it is strongly clean in I is a function of $X(I)$.
- When X is fixed, then for a fixed a, the (strong and not strong) cleanness of a in $]a, b]$ is decreasing as a function of $b - a$. This function reaches its minimum at $b - a = \Delta$.

- For any rectangle $Q = I \times J$, the event that its lower left corner is trap-clean in Q, is a function of the pair $X(I), Y(J)$.
- Among rectangles Q with a fixed lower left corner, the event that this corner is trap-clean in Q is a decreasing function of rectangles Q (partially ordered by containment). This function reaches its minimum for squares of size Δ.

Density of clean

We want many clean points.

- If a (not necessarily integer aligned) right-closed interval of size $\geqslant 3\Delta$ contains no wall, then its middle third contains a clean point.
- Suppose that a rectangle $I \times J$ with (not necessarily integer aligned) right-closed I, J with $|I|, |J| \geqslant 3\Delta$ contains no horizontal wall and no trap, and a is a right clean point in the middle third of I. There is an integer b in the middle third of J such that the point (a, b) is upper right clean.

Combinatorial requirements

These requirements are somewhat subtle. They are needed for the passing of walls. Their proof in the renormalization will take some work.

We call an interval external if it does not intersect any walls.

We call a wall dominant if it contains every wall intersecting with it.

- A maximal external interval of size $\geqslant \Delta$ or one starting at the beginning is inner clean.
- Suppose that interval I is adjacent on the left to a maximal external interval that has size $\geqslant \Delta$ (or starts at the beginning). Suppose also that it is adjacent on the right to a similar interval (or is infinite and contains no such interval). Then it is spanned by a sequence of neighbor walls. (In particular, the whole line is spanned by an extended sequence of neighbor walls.)

Probability conditions

Trap probability bound

In these probability bounds, we have frequently a condition like $Y(b-1)=k$ in the conditional probability, since the processes X, Y are Markov processes, and this is equivalent to conditioning on the whole past $(Y(0), \dots, Y(b-1))$.

The bound on traps: Given a string $x = (x(0), x(1), \dots)$ and an interval $I \ni a$,

$$\mathbf{P}\left[\begin{array}{l}\text{a trap starts at } (a,b)\\ \text{with projection in } I\end{array}\;\middle|\; X(I) = x(I),\ Y(b-1) = k\right] \leqslant w.$$

Parameters

Let

$$\lambda = 2^{1/2}.$$

and c_1, c_2 some constants to be chosen later. The probability bound on a wall of rank r will be

$$p(r) = c_2 r^{-c_1} \lambda^{-r}.$$

The constant

$$\chi = 0.015$$

is part of the definition. The lower bound for the probability of holes for rank r, will be, with an appropriate constant c_3:

$$h(r) = c_3 \lambda^{-\chi r}.$$

Barrier probability bound

Let

$$p(r,l)$$

be the supremum of probabilities (over all points t) that any barrier with rank r and size l starts at t, conditional over all possible $X(t)=k$.
The function $p(r)$ is a parameter in the definition of mazeries (we will define it explicitly). We require

$$p(r) \geqslant \sum_{l} p(r,l).$$

Cleanness bounds

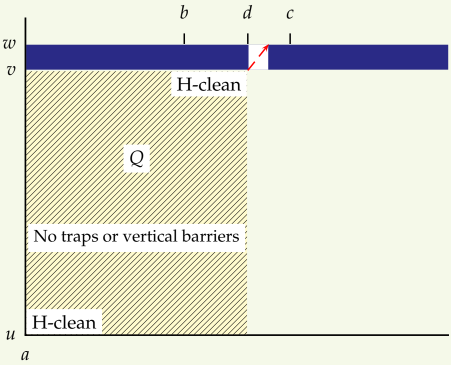
We require $q < 0.1$, and following inequalities for $k \in \{1, \dots, m\}$, for all $a < b$, for all sequences y such that u_1 (resp. v_1) is clean in $]u_1, v_1]$:

$$q/2 \geqslant \mathbf{P}\left[\, a \text{ is not strongly clean in }]a, b] \mid X(a) = k \,\right],$$
$$q/2 \geqslant \mathbf{P}\left[\, u \text{ is not trap-clean in } \mathrm{Rect}^{\rightarrow}(u, v) \mid X(u_0) = k, Y = y \,\right].$$

As always, there are several similar requirements obtained by interchanging X and Y, a for b, and so on.

Hole probability lower bound

We need a better lower bound than $h(r)$ for the case when we approach the wall from a certain distance, as in passing compound walls.



Given a, u, v, w.

$$b = a + \lceil (v - u)/2 \rceil,$$
$$c = a + (v - u) + 1.$$

Then

$$\text{prob} > (c - b)^{\chi} h(r).$$

Given a and $u \leqslant v < w$ with $v - u \leqslant 12\Delta$, define

$$b = a + \lceil (v-u)/2 \rceil, \qquad c = a + (v-u) + 1.$$

Let $Y = y$ be such that B is a horizontal potential wall of rank r with body $]v, w]$.
For a $d \in [b, c-1]$ let $Q = Q(d) = \mathrm{Rect}^{\rightarrow}((a,u),(d,v))$. Let $E = E(u, v, w;\ a)$ be the event (a function of X) that there is a d with

1. A vertical hole fitting B starts at d.
2. Q contains no traps, or vertical barriers.
3. Points (a,u) and (d,v) are H-clean in Q,

Then

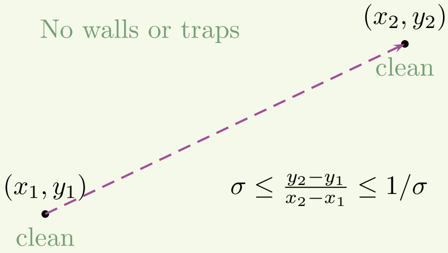
$$\mathbf{P}\left[E \mid X(a) = k,\ Y = y\right] \geqslant (c-b)^{\chi} h(r).$$

Reachability

We require $0 \leqslant \sigma < \frac{1}{2}$.

Let u, v be points with $\mathrm{minslope}(u, v) \geqslant \sigma$. If they are the starting and endpoint of a rectangle that is a hop, then $u \rightsquigarrow v$.

(The rectangle in question is allowed to be bottom-open or left-open, but not both.)



Base case

Example (Base case)

The original clairvoyant demon problem is a special case.

- Traps are points (i, j) with $X(i) = Y(j)$.
- There are no barriers.
- Every point is clean, its projections are also strongly clean.
- We have $\Delta = 1$ and $\sigma = 0$.
- The trap upper bound is satisfied if $m - 1 \geqslant 1/w$.

Renormalization

After defining the mazery $\mathcal{M}^*$, eventually we will have to prove the required properties. To be able to prove the reacheability condition for $\mathcal{M}^*$, we will introduce some new walls and traps in $\mathcal{M}^*$ whenever some larger-scale obstacles prevent reachability.

Parameters

Distances and slope

Some comments on the parameters of $\mathcal{M}^*$ and others used in the renormalization itself.

- Parameters $f \gg g \gg \Delta$, to be determined later, with $\Delta/g \leqslant g/f$.
  Here f is (among others) the minimum tolerated distance of walls, and g (among others) the minimum tolerated distance without hole on a wall.
- The choice of Δ^* will make sure $3f \leqslant \Delta^*$, since $3f$ will be an upper bound on the size of our compound walls.
- $\sigma^* := \sigma + 500g/f$. We will have $g/f < (0.5 - \sigma)/1000$, guaranteeing that σ_k never goes beyond 0.5.

Light and heavy

The barrier probability bounds will depend exponentially on ranks.

- The new rank lower bound R^* can be almost twice as large as the previous one: it will satisfy $R^* \leqslant 2R - \log_\lambda f$. Walls of rank lower than R^* are called light, the other ones are called heavy. Heavy walls of $\mathcal{M}$ will also be walls of $\mathcal{M}^*$.

Cleanness

The definition of cleanness is straightforward.

1 dim For an interval I, its right endpoint b will be called clean in I for $\mathcal{M}^*$ if

1. It is clean in I for $\mathcal{M}$.
2. I contains no wall of $\mathcal{M}$ whose right end is closer to b than $f/3$.

b is strongly clean in I for $\mathcal{M}^*$ if it is strongly clean in I for $\mathcal{M}$ and I contains no barrier of $\mathcal{M}$ whose right end is closer to x than $f/3$.

2 dim A starting point or endpoint u of a rectangle Q is trap-clean in Q for $\mathcal{M}^*$ if

1. It is trap-clean in Q for $\mathcal{M}$.
2. Any trap contained in Q is at a distance $\geqslant g$ from u.

New traps

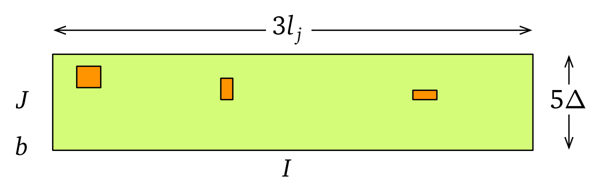
Uncorrelated A rectangle Q is called an uncorrelated compound trap if it contains two traps with disjoint projections, with a distance of their starting points at most f, and if it is minimal among all rectangles containing these traps.

Correlated and missing-hole This kind of horizontal trap $I \times J$ occurs, where $I = [b, c]$ if

- A certain bad event $\mathcal{L}_j(x, y, I, b)$, of three possible types $j = 1, 2, 3$ occurs.
- We have for all k

$$\mathbf{P}\left[\mathcal{L}_j(x, Y, I, b) \mid X(I) = x(I),\ Y(b-1) = k\right] \leqslant w^2.$$

Correlated traps



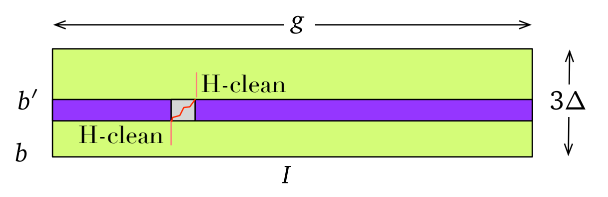
Let (for two versions of bad event leading to correlated traps)

$$g' = 2.2g, \quad l_1 = 7\Delta, \quad l_2 = g'.$$

Let I be a closed interval with length $|I| = 3l_j$, and $J = [b, b + 5\Delta]$. Fixing $x(I), y(J)$, we say that $\mathcal{L}_j(x, y, I, b)$ holds if every subinterval of I size l_j contains the projection of a trap from $I \times J$.

Missing-hole traps

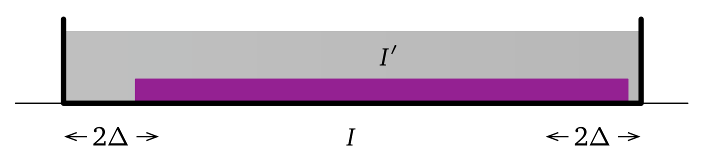
Let I be a closed interval of size g, $J = [b, b + 3\Delta]$. Fixing $x(I), y(J)$, we say that $\mathcal{L}_3(x, y, I, b)$ holds if there is a $b' > b + \Delta$ such that $]b + \Delta, b']$ is a light horizontal potential wall, and no good hole (recall the meaning) passes through it, at distance $\geqslant \Delta$ from its ends.



Example of a good hole. All such holes are missing now.

Emerging barriers

A vertical emerging barrier is, essentially, a horizontal interval over which the conditional probability of a bad event $\mathcal{L}_j$ is not small (thus preventing a new trap). But in order to find enough barriers, the ends are allowed to be sligthly extended.

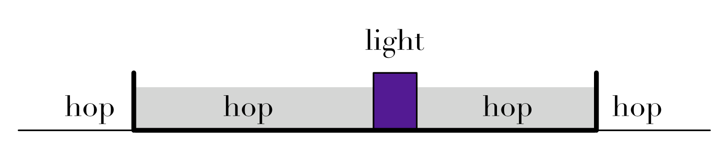


Fix the sequence X over $I =]a, b]$ as $x(I)$. Consider intervals $I' = [a', b']$ for any $a' \in]a, u + 2\Delta]$, $b' \in]b - 2\Delta, b]$. Interval I is the body of a vertical barrier of the emerging kind, of type $j \in \{1, 2, 3\}$ if

$$\exists(I', k)\ \mathbf{P}\left[\mathcal{L}_j(x, Y, I', 1) \mid X(I') = x(I'),\ Y(0) = k\right] \geqslant w^2.$$

Pre-walls

Not all barriers can be walls. First, we restrict ourselves to barriers that can be traversed in a predictable way.



Interval I is a pre-wall if following properties hold:

- (a) Either I is an external hop of $\mathcal{M}$ or it is the union of a dominant light wall and one or two external hops of $\mathcal{M}$, of size $\geqslant \Delta$, surrounding it.
- (b) Each end of I is adjacent to either an external hop of size $\geqslant \Delta$ or a wall of $\mathcal{M}$.

Emerging walls

The pre-walls that are allowed to become walls will be disjoint.

- For $j = 1, 2, 3$, list all emerging pre-walls of type j in a sequence $(B_{j1}, B_{j2}, \ldots)$.
- Process pre-walls $B_{11}, B_{12}, \ldots$ one-by-one. Select B_{1n} as a wall if and only if it is disjoint of all earlier selections.
- Next process the sequence $(B_{31}, B_{32}, \ldots)$, and then the sequence $(B_{21}, B_{22}, \ldots)$ similarly (watch the order), designating B_{in} a wall if and only if it is disjoint of all earlier selections.
- To all emerging barriers and walls, we assign one and the same rank $\hat{R} > R^*$ (to be fixed later).

Compound walls

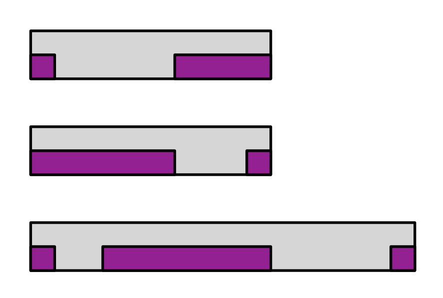
The distance of barriers is measured on an exponential scale:

$$d_i = \begin{cases} i & \text{if } i = 0, 1, \\ \lceil \lambda^i \rceil & \text{if } i \geqslant 2. \end{cases}$$

- A horizontal compound barrier $W_1 + W_2$ occurs wherever barriers W_1, W_2 occur (in this order) at a distance $d \in [d_i, d_{i+1}[$, $d \leqslant f$, and W_1 is light. Its rank is defined as

  $$r_1 + r_2 - i.$$

  Call this barrier a wall if W_1, W_2 are neighbor walls.
- Repeat the compounding step, requiring now W_2 to be light. W_1 can be any barrier introduced until now, also a compound barrier introduced in the first compounding step.



Three (overlapping) types of compound barrier obtained: light-any, any-light, light-any-light. Here, “any” can also be a recently defined emerging barrier.

Clean-up

Let us finish the construction of $\mathcal{M}^*$:

- Remove all traps of $\mathcal{M}$.
- Remove all light walls and barriers. If the removed light wall was dominant, remove also all other walls of $\mathcal{M}$ (even if not light) contained in it.

Combinatorial conditions

Dependencies

Let us start proving the mazery conditions for $\mathcal{M}^*$.

- The dependency conditions (for example, that whether an interval is a barrier of a certain rank) are easy to verify, straight from the form of the definition of $\mathcal{M}^*$.
- Recall the combinatorial requirements:
  - A maximal external interval of size $\geqslant \Delta$ or one starting at the beginning is inner clean.
  - Suppose that interval I is adjacent on the left to a maximal external interval that has size $\geqslant \Delta$ (or starts at the beginning). Suppose also that it is adjacent on the right to a similar interval (or is infinite and contains no such interval).
    Then it is spanned by a sequence of neighbor walls.

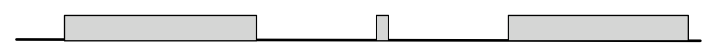
It will take hard work to prove these.

Let $(U_1, U_2, \dots)$ be a (finite or infinite) sequence of disjoint walls of $\mathcal{M}$ and $\mathcal{M}^*$, and let $I_0, I_1, \dots$ be the (possibly empty) intervals separating them (interval I_0 is the interval preceding U_1). This sequence is pure if

a. I_j are hops of $\mathcal{M}$.
b. I_0 is an external interval of $\mathcal{M}$ starting at the beginning, while every I_j for $j > 0$ is external if its size is $\geqslant 3\Delta$.

An element is isolated if it is farther than f from its neighbors.

Initial pure sequence



- The grey areas are between maximal external intervals of size $\geqslant \Delta$.
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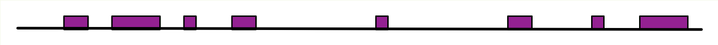
Adding emerging walls



- Start from a pure sequence.
- On-by-one, consider emerging walls. Such a wall can only intersect an isolated light wall of the sequence, and then cover it.
- Add it to the sequence, replacing what it covers.

It can be shown that the new sequence is pure again.

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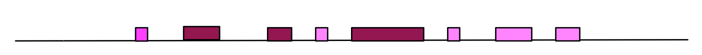
Forming compound walls

Incorporate all non-isolated light walls of the sequence into some compound wall.



At the end (before a gap of size $\geqslant f$) we may have to join 3.

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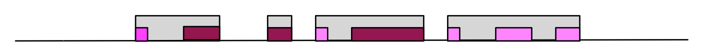

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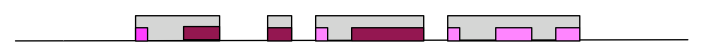

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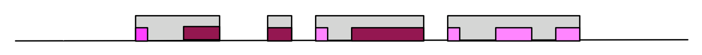

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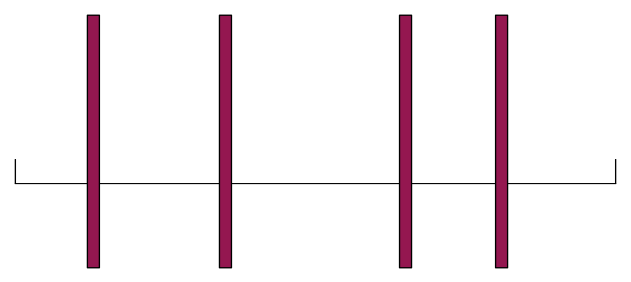
New hops

Lemma

Suppose that interval I is a hop of $\mathcal{M}^*$. Then it is either also a hop of $\mathcal{M}$ or it contains a sequence $W_1, \ldots, W_n$ of dominant light neighbor walls $\mathcal{M}$ separated from each other by external hops of $\mathcal{M}$ of size $\geqslant f$, and from the ends by hops of $\mathcal{M}$ of size $\geqslant f/3$.



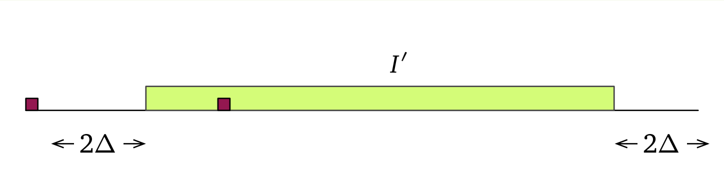
Finding emerging walls

Lemma

Let us be given intervals $I' \subset I$, and also $x(I)$, with the following properties for some $j \in \{1, 2, 3\}$.

- a. *I is spanned by an extended sequence $W_1, \ldots, W_n$ of dominant light neighbor walls of $\mathcal{M}$ such that the W_i are at a distance $> f$ from each other and at a distance $> f/3$ from the ends of I.*
- b. *I' is an emerging barrier of type j.*
- c. *I' is at a distance $\geqslant L_j + 7\Delta$ from the ends of I.*

Then I contains an emerging wall.



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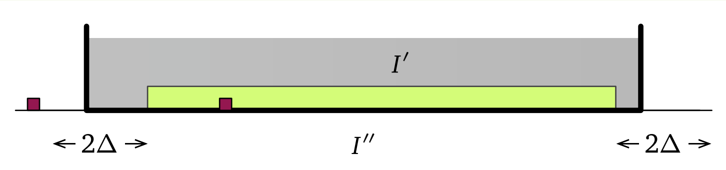

Cleanness density

The proof of the cleanness density conditions for $\mathcal{M}^*$ is straightforward, one just goes through the motions.