

THE CLAIRVOYANT DEMON HAS A HARD TASK

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ABSTRACT. Consider the integer lattice $L = \mathbb{Z}^2$. For some $m \geq 4$, let us color each column of this lattice independently and uniformly into one of m colors. We do the same for the rows, independently from the columns. A point of L will be called *blocked* if its row and column have the same color. We say that this random configuration *percolates* if there is a path in L starting at the origin, consisting of rightward and upward unit steps, and avoiding the blocked points. As a problem arising in distributed computing, it has been conjectured that for $m \geq 4$, the configuration percolates with positive probability. This has now been proved (in a later paper) for large m . We prove that the probability that there is percolation to distance n but not to infinity is not exponentially small in n . This narrows the range of methods available for proving the conjecture.

1. STATEMENT OF THE RESULT

1.1. Introduction. Let $x = (x(0), x(1), \dots)$ be an infinite sequence and $u = (u(0), u(1), \dots)$ be a binary sequence with elements in $\{0, 1\}$. Let $s_n = \sum_{i=0}^{n-1} u(i)$. We define the *delayed version* $x^{(u)}$ of x , by

$$x^{(u)}(n) = x(s_n).$$

Thus, if $i = s_n$ then $x^{(u)}(n) = x(i)$, and $x^{(u)}(n+1) = x(i)$ or $x(i+1)$ depending on whether $u(n) = 0$ or 1 . If $u(n) = 0$ then we can say that $x^{(u)}$ is *delayed* at time $n+1$. For two infinite sequences x, y we say that they *do not collide* if there is a delay sequence u such that for each n we have

$$x^{(u)}(n) \neq y^{(1-u)}(n).$$

Here, $1-u$ is the delay sequence complementary to u : thus, $y^{(1-u)}$ is delayed at time n if and only if $x^{(u)}$ is not.

For a given $m > 1$, suppose that $X = (X(0), X(1), \dots)$ is an infinite sequence of independent random variables, and $Y = (Y(0), Y(1), \dots)$ is another such sequence, also independent of X , where all variables are uniformly distributed over $\{0, \dots, m-1\}$. The following theorem has been conjectured in [2].

Proposition 1.1 (See [3]). *If m is sufficiently large then with positive probability, X does not collide with Y .*

The sequences X, Y can be viewed as two independent random walks on the complete graph K_m , and then the problem is whether a “clairvoyant demon”, i.e. a being who knows in advance both infinite sequences X and Y , can introduce delays into these walks in such a way that they never collide.

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1.2. Graph reformulation. We define a graph $G = (V, E)$ as follows. $V = \mathbb{Z}_{>0}^2$ is the set of points (i, j) where i, j are positive integers. Let us define the *distance* of two points $(i, j), (k, l)$ as $|k - i| + |l - j|$ (L_1 distance). When representing the set V of points (i, j) graphically, the right direction is the one of growing i , and the upward direction is the one of growing j . The set E of edges consists of all pairs of the form $((i, j), (i + 1, j))$ and $((i, j), (i, j + 1))$.

Given X, Y as in the theorem, let us say that a point (i, j) has color k if $X(i) = Y(j) = k$. Otherwise, it has color -1 , which we will call *white*. It is easy to see that X and Y do not collide if and only if there is an infinite directed path in G starting from $(0, 0)$ and proceeding on white points. Indeed, each path corresponds to a delay sequence u such that $u(n) = 1$ if and only if the edge is horizontal. Thus, the two sequences do not collide if and only if the graph of white points “percolates”. We will say that *there is percolation* if the probability that there is an infinite path is positive.

It has been shown independently in [1] and [4] that if the graph is undirected then there is percolation even for $m = 4$. In traditional percolation theory, when there is percolation then typically (unless the probability of blocking is at a “critical point”) the probability that there is percolation to a distance n but no percolation to infinity is exponentially small in n . This is the case also in the papers cited above, but it is not true for the directed percolation problem we are facing.

Theorem 1. *If there is percolation from the origin to infinity with positive probability then the probability of percolating from the origin to distance n but not to infinity is at least $Cn^{-\alpha}$ for some constants $C, \alpha > 0$ depending only on m .*

2. THE PROOF

Let $b_m = (0, 1, 2, \dots, m - 1)$ be called the *basic color sequence* of length m : it is simply the list of all different colors. Let b'_m be the reverse of b_m , i.e. $b'_m(i) = b_m(m - i - 1)$. Let $\mathcal{E}_{n,k}$ be the event that for all $i \in [0, k - 1], j \in [0, m - 1]$ we have

$$Y(n + im + j - 1) = b'_m(j),$$

i.e. starting with the index $n - 1$, the sequence Y has k consecutive repetitions of b'_m . We say that i is an index of the occurrence of b_m in the sequence X if $X(i + j) = b_m(j)$ for $j \in [0, m - 1]$. For $i > 0$, let τ_i be the i -th index of occurrence of b_m in X . Let $\mathcal{F}_{n,k}$ be the event that for all $i \in [1, n]$ we have

$$\tau_{i+1} - m - \tau_i \leq k - 1, \tag{1}$$

and also $\tau_1 \leq k - 1$.

Lemma 2.1. *If for some integer $n > 0$ both $\mathcal{E}_{n,k}$ and $\mathcal{F}_{n,k}$ hold then there is no white infinite directed path.*

Proof. Let us assume that there is a white infinite path. Since there are n consecutive copies of b_m in X , the i -th one starting at index τ_i , for each $p \in [1, n]$ there must be a vertical step in the path with an x projection in $[\tau_p, \tau_p + m - 1]$. Therefore the path ascends to the segment $[0, \tau_n + m - 1] \times \{n - 1\}$, before its x projection reaches t_n .

For each $1 \leq p \leq n, 0 \leq q < k$ there is a diagonally descending sequence of m colored points

$$\{(\tau_p + j, n + (q + 1)m - j - 2) : j \in [0, m - 1]\}.$$

For a fixed p there are k such diagonal barriers stacked above each other, forming an impenetrable column of height km . The path would have to ascend between two of these

columns, say between column i and $i + 1$. The distance of two consecutive columns from each other is

$$\tau_{i+1} - m - \tau_i \leq k - 1.$$

Since there are k consecutive copies of b'_m in Y starting at index $n - 1$, for each $q \in [0, k - 1]$ there must be a horizontal step in the path with a height in $n + qm - 1 + [0, m - 1]$. Since the distance of the two columns is at most $k - 1$, it is not possible for the path to pass between the two columns. (The same holds for the space before the first column.) \square

Lemma 2.2. *There is a constant α such that for all $s > 0$, there is a k with*

$$\text{Prob}(\mathcal{E}_{n,k}) \geq m^{-m} n^{-\alpha(s+1)}, \quad \text{Prob}(\mathcal{F}_{n,k}) \geq 1 - n^{-s}.$$

Proof. With $p_1 = m^{-m}$, we have $\text{Prob}(\mathcal{E}_{n,k}) = p_1^k$. Let us estimate the probability of $\mathcal{F}_{n,k}$. The probability that $\tau_1 > k - 1$ is upperbounded by the probability that a copy of b_m does not begin at i in X for any i in $\{0, m, \dots, \lfloor (k-1)/m \rfloor m\}$, which is

$$(1 - p_1)^{\lfloor (k-1)/m \rfloor + 1} \leq (1 - p_1)^{k/m} < e^{-p_1 k/m}.$$

The same estimate holds for the probability of (1) assuming that the similar conditions for smaller i have already been satisfied. Hence

$$\text{Prob}(\mathcal{F}_{n,k}) > 1 - ne^{-p_1 k/m}.$$

Let us choose

$$k = \lceil (s + 1)(m \log n) / p_1 \rceil \tag{2}$$

for some $s > 0$, then $\text{Prob}(\mathcal{F}_{n,k}) > 1 - n^{-s}$, while

$$\text{Prob}(\mathcal{E}_{n,k}) \geq p_1 n^{\frac{m \log p_1}{p_1}(s+1)}.$$

\square

Proof of Theorem 1. Assume that there is an infinite path with some positive probability P_0 . Choose s such that $n^{-s} < 0.5P_0$ and choose k as a function of s as in (2). Then the probability that $\mathcal{F}_{n,k}$ holds and there is an infinite path is at least $0.5P_0$. Let \mathcal{G}_n be the event that there is a path leaving the rectangle $[0, \tau_n] \times [0, n - 1]$. Then $\text{Prob}(\mathcal{F}_{n,k} \wedge \mathcal{G}_n) \geq 0.5P_0$. Since $\mathcal{F}_{n,k} \wedge \mathcal{G}_n$ is independent of $\mathcal{E}_{n,k}$, we have

$$\text{Prob}(\mathcal{E}_{n,k} \wedge \mathcal{F}_{n,k} \wedge \mathcal{G}_n) \geq 0.5P_0 m^{-m} n^{-\alpha(s+1)}.$$

\square

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