Eroders

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- “Simple” ways to suppress minorities.
- Declared motivation from CS: error correction.
- “Simple” means here: monotonic cellular automata.
- This reveals a hidden motivation: natural mathematical questions, interesting answers.

- Elementary parts: cells, or sites. Set of cells: for example, $\mathbb{C} = \mathbb{Z}^3$, or $\mathbb{C} = \mathbb{Z}/m\mathbb{Z}$ (periodic boundary conditions).
- Finite set $\mathbb{S}$ of (local) states.
- (Space-) configuration: any function $\xi : \mathbb{C} \to \mathbb{S}$.

$$\mathbb{X} = \mathbb{Z} \qquad\qquad \mathbb{S} = \{0, 1, 2\}$$

$$\cdots \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 2 & 1 & 0 \\ \hline \end{array} \cdots$$

$$\begin{array}{cccccc} & & -1 & 0 & 1 & 2 \end{array}$$

$$\xi(-1) = 1, \xi(0) = 1, \xi(1) = 2, \ldots$$

History, or space-time configuration, $\eta(x, t)$.

time

0	1	0	1	1	2	0	1	0	0	0	0	2	2	1	2	1	0
1	1	1	1	0	2	0	1	0	1	0	2	0	2	1	2	1	0
2	1	0	1	1	2	1	1	0	0	0	0	2	2	1	2	2	0
	0	0	0	1	0	0	2	1	0	2	0	1	2	0	1	1	1
			-1	0	1	2											

$$\eta(1, 2) = 2, \eta(2, 2) = 1, \ldots$$

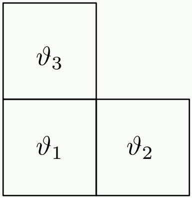
Initial configuration, or initial condition: $\eta(\cdot, 0)$.

Neighborhood array: $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_r)$. We will sometimes treat it just as a set, writing $\boldsymbol{\vartheta} \subset \mathbb{C}$. Here we will have $\mathbb{C} = \mathbb{Z}^d$ and for an arbitrary site $\boldsymbol{x}$,

$$\vartheta_i(\boldsymbol{x}) = \boldsymbol{x} + \vartheta_i,$$

Examples

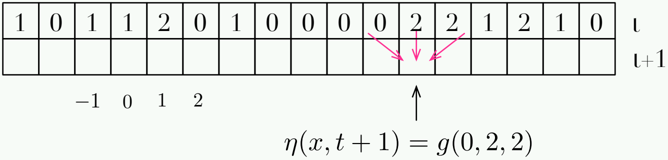
- von Neumann neighborhood: the 5 nearest neighbors (including itself) of a point, say, in the lattice $\mathbb{Z}^3$.
- Toom neighborhood: $\boldsymbol{\vartheta} = ((0, 0), (0, 1), (1, 0))$.



In discrete time, we say history η is a trajectory of local transition function $g : \mathbb{S}^r \to \mathbb{S}$ if

$$\eta(x, t+1) = g(\eta(\vartheta_1(x), t), \ldots, \eta(\vartheta_r(x), t)).$$

Example $\mathbb{C} = \mathbb{Z}$, $\boldsymbol{\vartheta} = (-1, 0, 1)$.



Here is a trajectory of Wolfram's rule 110 on $\mathbb{Z}/(17\,\mathbb{Z})$.

0	1	0	1	1	0	0	1	0	0	0	0	1	0	1	1	1	0
1	1	1	1	1	0	1	1	0	0	0	1	0	1	1	0	1	1
2	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	0
	0	0	1	1	0	0	1	0	0	0	0	1	0	0	0	1	0
			-1	0	1	2											$13=-4$

time

If your right neighbor is 1 and the neighborhood state is not 111 then become 1, otherwise 0.

So, a cellular automaton can be defined by $\mathbf{A} = \mathrm{CA}(\mathbb{S}, \mathbb{C}, \boldsymbol{\vartheta}, g)$.

Example (The Toom Rule) We define $\mathbf{A}_{\mathrm{Toom}}$ as follows:

$$\begin{aligned} \mathbb{S} &= \{0, 1\}, \quad \mathbb{C} = \mathbb{Z}^2, \\ \boldsymbol{\vartheta} &= ((0, 0), (0, 1), (1, 0)), \\ g(x, y, z) &= \mathrm{Maj}(x, y, z). \end{aligned}$$

The new state is the majority of the state of the cell itself, and of its northern and eastern neighbor.
We also extend $\mathrm{Maj}(x, y, z)$ to larger alphabets: when no symbol is in majority, let the result be y.

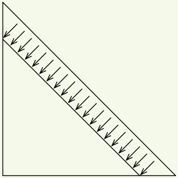
Erasing finite differences

- Consider an initial configuration ξ, cellular automaton $\mathbf{A}$ is self-restoring on ξ if whenever ξ' differs from ξ only in finitely many places, then the whole trajectory $\eta[\xi']$ differs from $\eta[\xi]$ only in finitely many places. (So the differences will be erased.)
- For a state s, let $s^{\mathbb{C}}$ be the configuration in which all cells have state s: the constant configuration.

The self-restoring property requires something from the configuration ξ as well as the automaton $\mathbf{A}$:

- ξ needs some redundancy,
- $\mathbf{A}$ needs to take advantage of the redundancy.

- $\mathbf{A}_{\mathrm{Toom}}$ is self-restoring on all constant configurations $s^{\mathbb{C}}$. Indeed, an island of non-s sites can be enclosed into a triangle, and this triangle will shrink.



- The symmetric majority (say, over the von Neumann neighborhood of 5 neighbors) is not. Does not erase even an island that is a 2×2 square!

For some cellular automaton $\mathbf{A} = \mathrm{CA}(\mathbb{S}, \mathbb{C}, \boldsymbol{\vartheta}, g)$,

- a history η has a fault at space-time point (x, t), if

$$\eta(x, t) \neq g(\eta(\vartheta_1(x), t-1), \ldots, \eta(\vartheta_r(x), t-1)).$$

- A random history with initial configuration ξ is given by a probability distribution μ over histories. It has an ε-bounded noise,

$$\mu \in \mathcal{M}(g, \varepsilon, \xi),$$

  if for all k, for all tuples $((x_1, t_1), \ldots, (x_i, t_k))$, the μ-probability that there is a fault at each point (x_i, t_i) is $\leq \varepsilon^k$.

Rule g is robustly self-restoring on initial configuration ξ, if for $\zeta = \eta[\xi]$,

$$\sup_{\mu \in \mathcal{M}(g,\varepsilon,\xi),\ \text{all } (x,t)} \mu\{\eta(x,t) \neq \zeta(x,t)\} \to 0 \text{ as } \varepsilon \to 0.$$

This means that with small enough noise, the rule does not allow non-$\eta[\xi]$ states to proliferate: it remembers everywhere, forever—probabilistically—that it started from ξ.

The following theorem is important, and highly nontrivial.

Theorem (Toom 1976) The Toom rule is robustly self-restoring on all constant configurations.

<See simulation.>

Some stories postponed (hoping time is left):

- Application of these results to fault-tolerant computation.
- A simple 1D rule (so-called GKL) that is self-restoring on the constants, but not robustly.

Monotonicity

What makes Toom's rule robust? Part of it can be monotonicity. Suppose that the state set $\mathbb{S}$ has some partial order defined on it. A transition function g is monotonic if

$$s_1 \leq t_1, \ldots, s_r \leq t_r \Rightarrow g(s_1, \ldots, s_r) \leq g(t_1, \ldots, t_r).$$

The Toom rule as well as the symmetric majority rule are monotonic.

Let 0 denote the minimal state of the ordered state set $\mathbb{S}$.

- A monotonic cellular automaton $\mathbf{A} = (\mathbb{S}, \mathbb{C}, \boldsymbol{\vartheta}, g)$ with $g(0, \dots, 0) = 0$ is an eroder if it is self-restoring on the initial configuration $0^{\mathbb{C}}$.
- It is a robust eroder if it does this robustly.

In English: an eroder is a cellular automaton that erases finite islands of non-0's.

Theorem (Toom 1976) Every two-state eroder is robust.

Characterizing two-state monotonic eroders

The above theorem comprises two parts, both nontrivial.

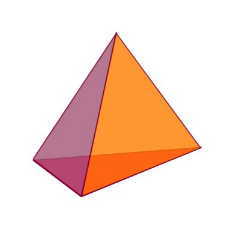
- Characterizing two-state eroders. (They turn out to be all somewhat similar to the Toom rule.)
- Showing that the rules so characterized are also robust eroders. (The argument is similar to the one used for the Toom rule.)

What does "characterization" mean? It gives an algorithm to decide about a given monotonic cellular automaton whether it is an eroder.

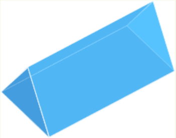
Pyramids

In what follows, I am referring only to the 2D case: $\mathbb{C} = \mathbb{Z}^2$, but everything here holds also for arbitrary dimensions.
Consider the space-time of our 2D cellular automaton as a 3D upper half-space. A generalized triangular pyramid is an object with finite height, one of the following kinds:

triangular-based pyramid



triangular prism (2-way infinite), base is one of the faces



Let $\mathbf{A} = \mathrm{CA}(\{0, 1\}, \mathbb{Z}^2, \boldsymbol{\vartheta}, g)$ be a monotonic cellular automaton. The trajectory defined by initial configuration ξ is denoted by $\eta[\xi]$.

Theorem (Toom 1976) $\mathbf{A}$ is an eroder if and only if there is a generalized pyramid P with the following property for any initial configuration ξ.

> If for some $c > 0$ the base of cP contains all non-zero sites of $I(\xi)$ then cP contains all non-zero sites of $\eta[\xi]$.

- Such an eroder erodes an island clearly in time linear in the diameter.
- How to decide the existence of P? If time permits I will show an equivalent characterization by Toom that does it.

Recall that if the initial configuration is $\xi = \eta(\cdot, 0)$ then the state of the origin at time 1 in a trajectory η is

$$\eta(\mathbf{0}, 1) = g(\xi(\vartheta_1), \ldots, \xi(\vartheta_r)).$$

A set $E \subset \boldsymbol{\vartheta}$ is called an annihilator for $\mathbf{A}$ if

$$(\forall\, \vartheta \in E\ \xi(\vartheta) = 0) \Rightarrow g(\xi(\vartheta_1), \ldots, \xi(\vartheta_r)) = 0.$$

That is, setting to 0 all inputs in E forces g to 0.
Example: For the Toom rule, the pairs $\{(0, 0), (0, 1)\}$, $\{(0, 0), (1, 0)\}$, $\{(0, 1), (1, 0)\}$ are annihilators.

For the characterization, we need to extend the lattice $\mathbb{Z}^2$ into the plane $\mathbb{R}^2$. Let $\mathrm{Conv}(E)$ denote the convex hull of the set E in $\mathbb{R}^2$.

Theorem (Toom 1976) A monotonic cellular automaton $\mathbf{A} = \mathrm{CA}(\{0, 1\}, \mathbb{Z}^2, \boldsymbol{\vartheta}, g)$ is an eroder if and only if

$$\bigcap_{\text{annihilators } E} \mathrm{Conv}(E) = \emptyset. \tag{1}$$

- The Toom rule is an eroder, since

$$\mathrm{Conv}(\{(0, 0), (0, 1)\}) \cap \mathrm{Conv}(\{(0, 0), (1, 0)\}) \cap \mathrm{Conv}(\{(0, 1), (1, 0)\}) \\ = \emptyset.$$

- The rule $(\xi(0, 0) \vee \xi(1, 1)) \wedge (\xi(0, 1) \vee \xi(1, 0))$ is not an eroder. Though its annihilators $\{(0, 0), (1, 1)\}$ and $\{(0, 1), (1, 0)\}$ don't intersect, their convex hulls intersect in the point $(1/2, 1/2)$.

More than two states

Let the set of states be $\mathbb{S} = \{0, 1, \ldots, m\}$, with its natural ordering. Questions for monotonic rules with $m > 1$:

Are all eroders linear-time? Not in 2D.

Are all eroders robust? Not even in 1D, example in [Toom 1976].

Are linear-time eroders robust? Not even in 1D, same example.

Can robust eroders be characterized? Yes in 1D [G-Törmä, in preparation]. Result seems extendable to all dimensions.

Are robust eroders linear-time? Yes, same paper.

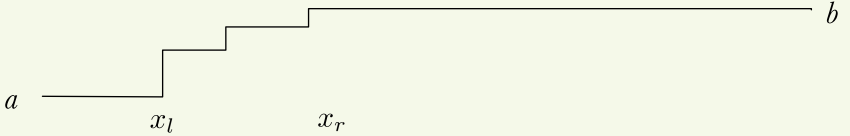
Can eroders be characterized? Not known in 2D.
Yes in 1D [Gal'perin 1976], extended to a partially ordered state set in [G-Hilaire, in preparation].

Velocities

Consider a 1D cellular automaton $\mathbf{A} = \mathrm{CA}(\mathbb{S}, \mathbb{Z}, \boldsymbol{\vartheta}, g)$ where $\mathbb{S} = \{0, 1, \ldots, m\}$, and for each $s \in \mathbb{S}$, $g(s, \ldots, s) = s$.

- Let $a < b$. An increasing ab-ladder is a configuration ξ such that $i < j$ implies $\xi(i) \le \xi(j)$, the leftmost values are a and the rightmost values are b. Decreasing ba-ladders are defined similarly.
  The automaton takes ab-ladders into ab-ladders.
- For an ab-ladder ξ let

$$x_l(\xi) = \max\{\, i : \xi(i) = a \,\}, \quad x_r(\xi) = \min\{\, i : \xi(i) = b \,\}.$$



- If for a ladder ξ, $\xi(i) = a$ for $i < 0$, $\xi(i) = b$ for $i \geq 0$ then we call it an ab-jump J_{ab}.
- For real numbers $\alpha < \beta$ and v, the set $\{ (x, t) : \alpha \leq x - vt < \beta \}$ is called a a space-time stripe, of velocity (or slope) v.

Theorem (Gal'perin 1976) For every pair $a < b$ consider the trajectory $\eta_{ab} = \eta[J_{ab}]$ whose initial configuration is an ab-jump. There is space-time stripe computable from $\mathbf{A}$ containing the points $x_l(\eta_{ab}(\cdot, t))$ for all t. There are also corresponding stripes for x_r, and also for $a > b$.

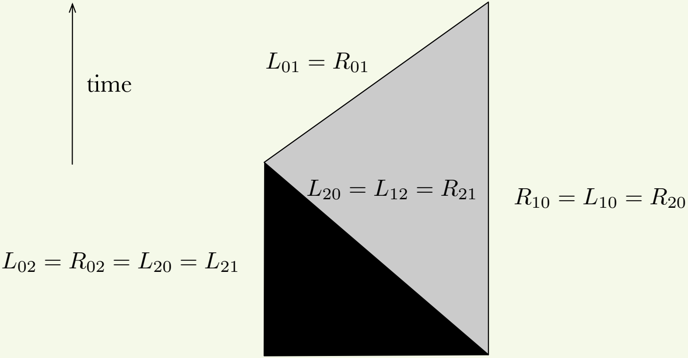
The slope of this stripe is called the left velocity L_{ab}.
The corresponding right velocity is R_{ab}.

The above theorem implies (with some work):

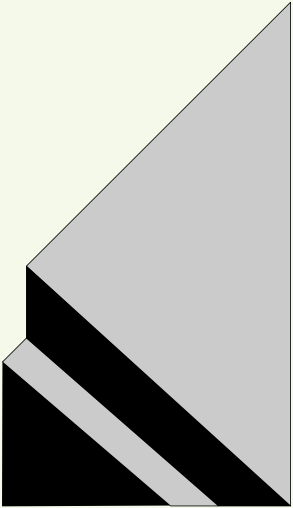
Theorem (Gal'perin 1976) Eroder criterion:

$$\forall\, b > 0\ R_{0b} > L_{b0}.$$

The criterion decreases the maximum of any island in linear time. Applying it repeatedly erodes the island.



This picture shows more possibilities:



Some 1D eroders are not robust.

Example (Toom 1976) Let $\mathbb{S} = \{0, 1, 2\}$, $\boldsymbol{\vartheta} = (-1, 0, 1)$. The transition function $g(x, y, z)$ is defined as the maximal monotonic rule obeying the following:

$$g(x, y, z) = \begin{cases} 1 & \text{if } y = 2, z \leq 1, \text{ (1's sweeep 2's from right)}, \\ 0 & \text{if } y = z = 0 \text{ or } x = 0, y, z \leq 1 \text{ (0's sweep 1's from left)}. \end{cases}$$

This rule is an eroder, since

$$0 = R_{02} > L_{20} = -1, \quad 1 = R_{01} > L_{10} = 0.$$

An island of 2's is first converted to 1's from the right and then the result converted to 0's from the left.

But this rule is not robust, here is a proof sketch.

- Assume that the noise always creates 2's, with probability ε.
- Consider a big island I of 2's, and imagine each 1 traveling left into I as a message. It keeps traveling left even if the noise turns it into 2. Each 1 message has only $\leq (1-\varepsilon)^{|I|}$ chance of reaching the left end.
- Simultaneously, the noise keeps extending I with constant speed ε.

There is a strengthening of the eroder conditions that characterizes robust eroders.

Theorem (G-Törmä) Robust eroder criterion:

$$\forall\, b > 0\ \exists\, a < b\ L_{ab} > R_{ba}.$$

For $m = 2$ the criterion says

$$L_{02} > R_{20} \vee (L_{12} > R_{21} \wedge L_{01} > R_{10}).$$

They do not hold in the above example, since
$L_{02} = R_{20} = 0, L_{12} = R_{21} = -1.$

- The eroder criterion $R_{0b} > L_{b0}$ relies on the borders with 0's to reduce the maximum of an island of b's to $b - 1$.
- The robust eroder criterion $L_{ab} > R_{ba}$ reduces the maximum of such an island to a, without relying on the borders with 0. (The proof of sufficiency uses this.)
- Negation of the criterion says

$$\exists\, b > 0\ \forall\, a < b\ L_{ab} \le R_{ba}.$$

(The proof of necessity uses this in a generalization of the reasoning of the above example.)

In 2D, there is a 3-state eroder that erodes in exponential time. To interpret the following example, consider an island that is a square of 2's.

Example In the notation below, let $\underline{s}$ be the value of the site at position (0, 0). Let g be the maximal monotonic rule obeying the following transitions.

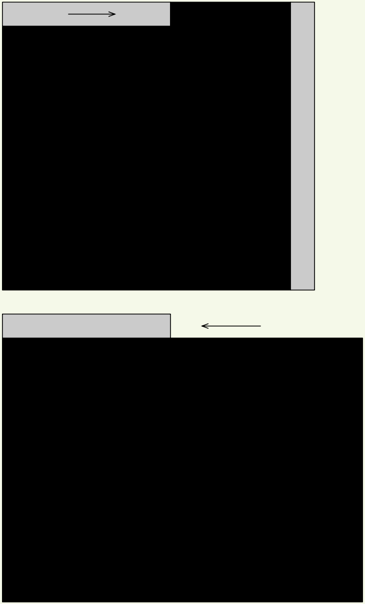
$$\begin{bmatrix} & 0 \\ \leq 1 & \underline{2} \end{bmatrix} \to 1 \text{ (1's sweep right into the top row of the island)}$$

$$\begin{bmatrix} 0 & \\ \underline{1} & 0 \end{bmatrix} \to 0, \text{ (0's sweep left into 1's on top)}$$

$$\begin{bmatrix} 2 & \underline{0} \end{bmatrix} \to 1, \begin{bmatrix} 2 & \underline{1} \end{bmatrix} \to 2, \text{ (2's extend right with half speed)}$$

$$\begin{bmatrix} \leq 1 & \underline{1} & \geq 1 \end{bmatrix} \to 1, \text{ (1's don't change otherwise)}$$

$$\begin{bmatrix} \leq 1 & \underline{0} \end{bmatrix} \to 0, \text{ (0's don't change otherwise)}$$



- In this example, an $n \times n$ rectangle of 2's is eliminated row-by-row from the top. This would already take time n^2.
- But while it is happening, the rows extend to the right with half speed, so by the time the top row turns to 1's, the rectangle became twice larger. So the eroding time is exponential.

Question

- How much worse can it be?
- Is the eroder property is decidable at all, even just for 3 states, in 3D?

Partial ordering

We may restrict 2D cellular automata to a single horizontal stripe of constant width. The columns, viewed as states, are partially ordered. This raises the following question:

Can we characterize 1D eroders on a partially ordered set?

The answer is yes, by the next generalization of Gal'perin's result (which is not mechanical).

Theorem (G-Hilare)

- For every pair $a < b$ consider the trajectory $\eta_{ab} = \eta[J_{ab}]$ whose initial configuration is an ab-jump. There is space-time stripe computable from $\mathbf{A}$ containing the points $x_l(\eta_{ab}(\cdot, t))$ for all t. There are also corresponding stripes for x_r, and also for $a > b$.
- This allows a decision procedure for the eroder property (the criterion seems not as neat as for the totally ordered case).

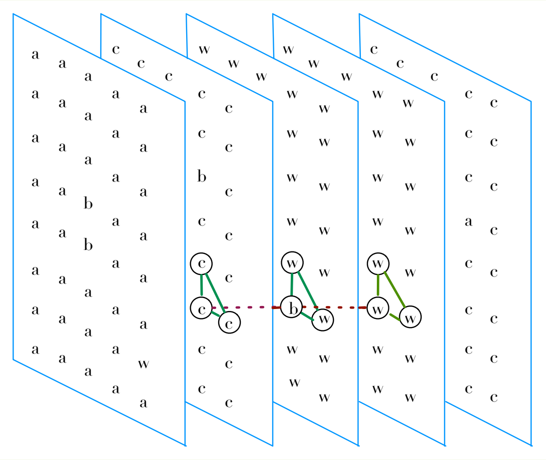
The simplest known fault-tolerant computation model is the following 3D cellular automaton.

Definition (Toom-layering) Let $\mathbf{U}$ be an arbitrary 1D cellular automaton with transition rule $g_{\mathbf{U}}$. Its Toom-layering is a 3D automaton $\overline{\mathbf{U}}$ defined as follows.

- Slice the space into planes by the value of the first coordinate. Let ξ be an arbitrary initial configuration of $\mathbf{U}$. The layered initial configuration $\overline{\xi}$ of $\overline{\mathbf{U}}$, constant on each plane $\{x\} \times \mathbb{Z}^2$, is defined by

$$\overline{\xi}(x, y, z) = \xi(x).$$

- The transition $g_{\overline{\mathbf{U}}}$ applies Toom's rule within each plane, and $g_{\mathbf{U}}$ across the planes.



Theorem (G-Reif 1988) The automaton $\overline{\mathbf{U}}$ is robustly self-restoring on the Toom-layering $\overline{\xi}$ of every initial configuration ξ of $\mathbf{U}$.

- The 3D cellular automaton $\overline{\mathbf{U}}$ simulates in a fault-tolerant way the (arbitrary) 1D cellular automaton $\mathbf{U}$. If $\mathbf{U}$ simulates a universal Turing machine, then $\overline{\mathbf{U}}$ carries out a universal computation reliably.
- In a finite version of the result, the redundancy brought by the two extra dimensions is only $\log^2$.
- Today this result is mentioned only to motivate the interest in cellular automata related to Toom's rule.

Here is an example of a rule that is self-restoring on the constants, but not robustly.

Definition (GKL rule) Let

$$\mathbb{S} = \{-1, 1\}, \quad \boldsymbol{\vartheta} = (-3, -2, -1, 0, 1, 2, 3),$$
$$g(s_{-3}, \ldots, s_3) = \mathrm{Maj}(s_0, s_{s_0}, s_{3s_0}).$$

This rule is not monotonic.

Theorem (G-Kurdyumov-Levin 1976) The GKL rule is self-restoring on constant configurations.

Theorem (G-Park, 1996) In some kind of noise, the GKL rule is not only not self-restoring, but forgets everything about its initial configuration ("is ergodic").

Remarks on fault-tolerance

Self-restoration is generally too much to require for fault-tolerance. We expect some results to be correct (possibly with high probability), but can allow many paths to reaching them and may ignore some scratch information—all depending on the noise.