Clairvoyant embedding in one dimension

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Abstract

Let v, w be infinite 0-1 sequences, and $\tilde{m}$ a positive integer. We say that w is $\tilde{m}$-***embeddable*** in v, if there exists an increasing sequence $(n_i : i \geq 0)$ of integers with $n_0 = 0$, such that $1 \leq n_i - n_{i-1} \leq \tilde{m}$, $w(i) = v(n_i)$ for all $i \geq 1$. Let X and Y be coin-tossing sequences. We will show that there is an $\tilde{m}$ with the property that Y is $\tilde{m}$-embeddable into X with positive probability. This answers a question that was open for a while. The proof generalizes somewhat the hierarchical method of an earlier paper of the author on dependent percolation.

1 Introduction

Consider the following problem, stated in [6, 5]. Let $v = (v(1), v(2)\dots)$, $w = (w(1), w(2)\dots)$ be infinite 0-1 sequences, and $\tilde{m} > 0$. We say that w is $\tilde{m}$-***embeddable*** in v, if there exists an increasing sequence $(n_i : i \geq 1)$ of positive integers such that $w(i) = v(n_i)$, and $1 \leq n_i - n_{i-1} \leq \tilde{m}$ for all $i \geq 1$. (We set $n_0 = 0$, so $n_1 \leq \tilde{m}$ is required.) Let $X = (X(1), X(2), \dots)$ and $Y = (Y(1), Y(2), \dots)$ be sequences of independent Bernoulli variables with parameter 1/2. The question asked was whether there is any $\tilde{m}$ with the property that if Y is independent of X then it is $\tilde{m}$-embeddable into X with positive probability. The present paper answers the question positively.

Theorem 1. *There is an $\tilde{m}$ with the property that if Y is independent of X then it is $\tilde{m}$-embeddable into X with positive probability.*

It turns out that independence is not needed, see Theorem 2 below.

The proof allows the computation of an upper bound on $\tilde{m}$, but we will not do this, and not just to avoid ridicule: many steps of the proof would become less transparent when trying to do this.

Here is a useful equivalent formulation. First we define the fixed directed graph $G_{\tilde{m}} = (\mathbb{Z}_+^2, E)$. From each point (i, j) edges go to $(i+1, j+1), (i+2, j+1), \dots, (i+\tilde{m}, j+1)$. The random graph

$$\mathcal{G}_{\tilde{m}}(X, Y) = (\mathbb{Z}_+^2, \mathcal{E}) \tag{1.1}$$

is defined as follows: delete all edges going *into* points (i, j) of $G_{\tilde{m}}$ with $X(i) \neq Y(j)$. (In percolation terms, in $\mathcal{G}_{\tilde{m}}$ we would call a point "open" if it has some incoming edge.) Now Y is embeddable into X if and only if there is an infinite path in $\mathcal{G}_{\tilde{m}}$ starting at the origin. So the embedding question is equivalent to a percolation question.

Our proof generalizes slightly the method of [4], making also its technical result more explicit. Just before uploading to the arXiv, the author learned that Basu and Sly have also proved the embedding theorem, in an independent work [2]. They are citing another, simultaneous and independent, paper by Sidoravicius.

The proof of Theorem 1 relies on the independence of the processes X and Y. But the proof in [2], just like the proof of the compatible sequences result in [3], does not: it applies to any joint distribution with the coin-tossing marginals X, Y. Allan Sly showed in [7] how Theorem 1 can also be adapted to prove a version without the independence assumption:

Theorem 2. *There is an $\tilde{m}$ with the property that if Y has a joint distribution with X then it is $\tilde{m}$-embeddable into X with positive probability.*

Proof. It is easy to derive from Theorem 1, but is even more immediate from the proof as pointed out in Remark 3.3 below, that the probability of the existence of an $\tilde{m}$-embedding converges to 1 as $\tilde{m} \to \infty$. Let us choose an $\tilde{m}$ now making this probability at least $1 - \varepsilon$ for some $\varepsilon < 1/2$.

Given two coin-tossing sequences X, Y with a joint distribution, let us create a coin-tossing sequence Z independent of (X, Y). The above remark implies that there is an $\tilde{m}$ such that Y is $\tilde{m}$-embeddable into Z with probability $> 1 - \varepsilon$, and Z is $\tilde{m}$-embeddable into X with probability $> 1 - \varepsilon$. Combining the two embeddings gives an $\tilde{m}^2$-embedding of Y into X with probability $> 1 - 2\varepsilon$. □

It is unknown currently whether the theorem of [4] on clairvoyant scheduling of random walks can also be generalized to non-independent random walks.

Just like in [4], we will introduce several extra elements into the percolation picture. For consistency with what comes later, let us call open points "lower left trap-clean". Let us call any interval $(i, i+\tilde{m}]$ a "vertical wall" if $X(i+1) = X(i+2) = \cdots = X(i+\tilde{m})$. We call an interval $(a, a+1]$ "horizontal hole fitting" this wall if $Y(a+1) = X(i+1)$. The idea is that a vertical wall forms a certain obstacle for a path $n_1, n_2, \ldots$. But if the path arrives at a fitting hole $(a, a+1]$, that is it has $n_a = i$, then it can pass through, with $n_{a+1} = i + \tilde{m}$. The vertical walls are obstacles to paths, but there is hope: a wall has only probability $2^{-\tilde{m}+1}$ to start at any one place, while a hole fitting it has probability 1/2 to start at any place. Under appropriate conditions then, walls can be passed. The failure of these conditions gives rise to a similar, "higher-order" model with a new notion of walls. It turns out that in higher-order models, some more types of element (like traps) are needed. This system was built up in the paper [4], introducing a model called "mazery". We will generalize mazeries slightly (more general bounds on slopes and cleanness), to make them applicable to the embedding situation.

It is an understatement to say that the construction and proof in [4] are complex. Fortunately, much of it carries over virtually without changes, only some of the proofs (a minority) needed to be rewritten. On the other hand, giving up any attempt to find reasonable bounds on $\tilde{m}$ made it possible to simplify some parts; in particular, the proof of the Approximation Lemma (Lemma 7.1) is less tedious here than in [4].

We rely substantially on [4] for motivation of the proof structure and illustrations. Each lemma will still be stated, but we will omit the proof of those that did not change in any essential respect.

The rest of the paper is as follows. Section 2 defines mazeries. Section 3 formulates the main theorem and main lemma in terms of mazeries, from which Theorem 1 follows. Section 4 defines the scale-up operation $\mathcal{M}^k \mapsto \mathcal{M}^{k+1}$. It also proves that scale-up preserves almost all *combinatorial* properties, that is those that do not involve probability bounds. The one exception is the reachability property, formulated by Lemma 7.1 (Approximation): its proof is postponed to Section 7. Section 5 specifies the parameters in a way that guarantees that the probability conditions are also preserved by scale-up. Section 6 estimates how the probability bounds are transformed by the scale-up operation. Section 8 proves the main lemma.

2 Mazeries

This section is long, and is very similar to Section 3 in [4]: we will point out the differences.

2.1 Notation

The notation (a, b) for real numbers a, b will generally mean for us the pair, and not the open interval. Occasional exceptions would be pointed out, in case of ambiguity.

We will use

$$a \wedge b = \min(a, b), \quad a \vee b = \max(a, b).$$

To avoid too many parentheses, we use the convention

$$a \wedge b \cdot c = a \wedge (b \cdot c).$$

We will use intervals on the real line and rectangles over the Euclidean plane, even though we are really only interested in the lattice $\mathbb{Z}_+^2$. To capture all of $\mathbb{Z}_+$ this way, for our right-closed intervals $(a, b]$, we allow the left end a to range over all the values $-1, 0, 1, 2, \dots$. For an interval $I = (a, b]$, we will denote

$$X(I) = (X(a+1), \dots, X(b)).$$

The ***size*** of an interval I with endpoints a, b (whether it is open, closed or half-closed), is denoted by $|I| = b - a$. By the ***distance*** of two points $a = (a_0, a_1)$, $b = (b_0, b_1)$ of the plane, we mean

$$|b_0 - a_0| \vee |b_1 - a_1|.$$

The ***size*** of a rectangle

$$\mathrm{Rect}(a, b) = [a_0, b_0] \times [a_1, b_1]$$

in the plane is defined to be equal to the distance between a and b. For two different points $u = (u_0, u_1)$, $v = (v_0, v_1)$ in the plane, when $u_0 \le v_0$, $u_1 \le v_1$:

$$\mathrm{slope}(u, v) = \frac{v_1 - u_1}{v_0 - u_0}.$$

We introduce the following partially open rectangles

$$\begin{aligned} \mathrm{Rect}^{\rightarrow}(a, b) &= (a_0, b_0] \times [a_1, b_1], \\ \mathrm{Rect}^{\uparrow}(a, b) &= [a_0, b_0] \times (a_1, b_1]. \end{aligned}$$

The relation

$$u \rightsquigarrow v$$

says that point v is reachable from point u (the underlying graph will always be clear from the context). For two sets A, B in the plane or on the line,

$$A + B = \{\, a + b : a \in A,\ b \in B \,\}.$$

2.2 The structure

A mazery is a special type of random structure we are about to define. Eventually, an infinite series of mazeries $\mathcal{M}^1, \mathcal{M}^2, \dots$ will be defined. Each mazery $\mathcal{M}^i$ for $i > 1$ will be obtained from the preceding one by a certain scaling-up operation. Mazery $\mathcal{M}^1$ will derive directly from the original percolation problem, in Example 2.21.

The tuple

All our structures defined below refer to "percolations" over the same lattice graph $\mathcal{G} = \mathcal{G}(X, Y)$ depending on the coin-tossing sequences X, Y. It is like the graph $\mathcal{G}_{3\tilde{m}}(X, Y)$ introduced in (1.1) above, but we will not refer to $\tilde{m}$ explicitly.

A ***mazery***

$$(\mathcal{M}, \sigma, \sigma_x, \sigma_y, R, \Delta, w, q_\triangle, q_\square)$$

consists of a random process $\mathcal{M}$, and the listed nonnegative parameters. Of these, $\sigma, \sigma_x, \sigma_y$ are called ***slope lower bounds***, R is called the ***rank lower bound***, and they satisfy

$$1/2R \le \sigma_x/2 \le \sigma \le \sigma_x, \quad 2 \le \sigma_y, \tag{2.1}$$

$$\sigma_x \sigma_y < 1 - \sigma. \tag{2.2}$$

With (2.1) this implies $\sigma \le \sigma_x < \frac{1-\sigma}{2}$. We call Δ the ***scale parameter***. We also have the probability upper bounds $w, q_j > 0$ with

$$q_\triangle < 0.05, \quad q_\square < 0.55,$$

which will be detailed below, along with conditions that they must satisfy. (In [4], there was just one parameter σ and one parameter q.) Let us describe the random process

$$\mathcal{M} = (X, Y, \mathcal{G}, \mathcal{T}, \mathcal{W}, \mathcal{B}, \mathcal{C}, \mathcal{S}).$$

In what follows, when we refer to the mazery, we will just identify it with $\mathcal{M}$. We have the random objects

$$\mathcal{G}, \quad \mathcal{T}, \quad \mathcal{W} = (\mathcal{W}_x, \mathcal{W}_y), \quad \mathcal{B} = (\mathcal{B}_x, \mathcal{B}_y), \quad \mathcal{C} = (\mathcal{C}_x, \mathcal{C}_y), \quad \mathcal{S} = (\mathcal{S}_x, \mathcal{S}_y, \mathcal{S}_2).$$

all of which are functions of X, Y. The graph $\mathcal{G}$ is a random graph.

Definition 2.1 (Traps)**.** In the tuple $\mathcal{M}$ above, $\mathcal{T}$ is a random set of closed rectangles of size $\le \Delta$ called ***traps***. For trap $\mathrm{Rect}(a, b)$, we will say that it ***starts*** at its lower left corner a. ⌟

Definition 2.2 (Wall values)**.** To describe the process $\mathcal{W}$, we introduce the concept of a ***wall value*** $E = (B, r)$. Here B is the ***body*** which is a right-closed interval, and ***rank***

$$r \geq R.$$

We write $\mathrm{Body}(E) = B$, $|E| = |B|$. We will sometimes denote the body also by E. Let Wvalues denote the set of all possible wall values. ⌟

Let

$$\mathbb{Z}_+^{(2)}$$

denote the set of pairs (u, v) with $u < v$, $u, v \in \mathbb{Z}_+$. The random objects

$$\begin{aligned} \mathcal{W}_d &\subseteq \mathcal{B}_d \subseteq \mathrm{Wvalues}, \\ \mathcal{S}_d &\subseteq \mathcal{C}_d \subseteq \mathbb{Z}_+^{(2)} \times \{-1, 1\} \text{ for } d = x, y, \\ \mathcal{S}_2 &\subseteq \mathbb{Z}_+^{(2)} \times \mathbb{Z}_+^{(2)} \times \{-1, 1\} \times \{0, 1, 2\} \end{aligned}$$

are also functions of X, Y. (Note that we do not have any $\mathcal{C}_2$.)

Definition 2.3 (Barriers and walls)**.** The elements of $\mathcal{W}_x$ and $\mathcal{B}_x$ are called ***walls*** and ***barriers*** of X respectively, where the sets $\mathcal{W}_x, \mathcal{B}_x$ are functions of X. (Similarly for $\mathcal{W}_y, \mathcal{B}_y$ and Y.) In particular, elements of $\mathcal{W}_x$ are called ***vertical walls***, and elements of $\mathcal{W}_y$ are called ***horizontal walls***. Similarly for barriers. When we say that a certain interval contains a wall or barrier we mean that it contains its body.

A right-closed interval is called ***external*** if it intersects no walls. A wall is called ***dominant*** if it is surrounded by external intervals each of which is either of size $\geq \Delta$ or is at the beginning of $\mathbb{Z}_+$. Note that if a wall is dominant then it contains every wall intersecting it.

For a vertical wall value $E = (B, r)$ and a value of $X(B)$ making E a barrier of rank r we will say that E is a ***potential vertical wall*** of rank r if there is an extension of $X(B)$ to a complete sequence X that makes E a vertical wall of rank r. Similarly for horizontal walls. ⌟

The last definition uses the fact following from Condition 2.18.1b that whether an interval B is a barrier of the process X depends only $X(B)$.

The set of barriers is a random subset of the set of all possible wall values, and the set of walls is a random subset of the set of barriers.

Condition 2.4. The parameter Δ is an upper bound on the size of every trap and the thickness of any barrier. ⌟

Cleanness

The set $\mathcal{C}_x$ is a function of the process X, and the set $\mathcal{C}_y$ is a function of the process Y: they are used to formalize (encode) the notions of cleanness given descriptive names below.

Definition 2.5 (One-dimensional cleanness)**.** For an interval $I = (a, b]$ or $I = [a, b]$, if $(a, b, -1) \in \mathcal{C}_x$ then we say that point b of $\mathbb{Z}_+$ is ***clean*** in I for the sequence X. If $(a, b, 1) \in \mathcal{C}_x$ then we say that point a is clean in I. From now on, whenever we talk about cleanness of an element of $\mathbb{Z}_+$, it is always understood with respect to either for the sequence X or for Y. For simplicity, let us just talk about cleanness, and so on, with respect to the sequence X. A point $x \in \mathbb{Z}_+$ is called left-clean (right-clean) if it is clean in all intervals of the form $(a, x]$, $[a, x]$ (all intervals of the form $(x, b]$, $[x, b]$). It is ***clean*** if it is both left- and right-clean. If both ends of an interval I are clean in I then we say I is ***inner clean***.

To every notion of one-dimensional cleanness there is a corresponding notion of ***strong cleanness***, defined with the help of the process $\mathcal{S}$ in place of the process $\mathcal{C}$. ⌟

Figure 8 of [4] illustrates one-dimensional cleanness.

Definition 2.6 (Trap-cleanness)**.** For points $u = (u_0, u_1)$, $v = (v_0, v_1)$, $Q = \mathrm{Rect}^{\varepsilon}(u, v)$ where $\varepsilon = \rightarrow$ or $\uparrow$ or nothing, we say that point u is ***trap-clean in*** Q (with respect to the pair of sequences (X, Y)) if $(u, v, 1, \varepsilon') \in \mathcal{S}_2$, where $\varepsilon' = 0, 1, 2$ depending on whether $\varepsilon = \rightarrow$ or $\uparrow$ or nothing. Similarly, point v is ***trap-clean in*** Q if $(u, v, -1, \varepsilon') \in \mathcal{S}_2$. It is ***upper right trap-clean***, if it is trap-clean in the lower left corner of all rectangles. It is ***trap-clean***, if it is trap-clean in all rectangles. ⌟

Definition 2.7 (Complex two-dimensional sorts of cleanness)**.** We say that point u is ***clean*** in Q when it is trap-clean in Q and its projections are clean in the corresponding projections of Q.

If u is clean in all such left-open rectangles then it is called ***upper right rightward clean***. We delete the "rightward" qualifier here if we have closed rectangles in the definition here instead of left-open ones. Cleanness with qualifier "upward" is defined similarly. Cleanness of v in Q and lower left cleanness of v are defined similarly, using $(u, v, -1, \varepsilon')$, except that the qualifier is unnecessary: all our rectangles are upper right closed.

A point is called ***clean*** if it is upper right clean and lower left clean. If both the lower left and upper right points of a rectangle Q are clean in Q then Q is called ***inner clean***. If the lower left endpoint is lower left clean and the upper right endpoint is upper right rightward clean then Q is called ***outer rightward clean***. Similarly for ***outer upward clean*** and ***outer-clean***.

We will also use a ***partial*** versions of cleanness. If point u is trap-clean in Q and its projection on the x axis is *strongly* clean in the same projection of Q then we will say that u is ***H-clean*** in Q. Clearly, if u is H-clean in Q and its projection on the y axis is clean in (the projection of) Q then it is clean in Q. We will call rectangle Q ***inner*** H-clean if both its lower left and upper right corners are H-clean in it. It is now clear what is meant for example by a point being ***upper right rightward H-clean***.

The notion ***V-clean*** is defined similarly when we interchange horizontal and vertical. ⌟

Figure 9 of [4] illustrates 2-dimensional cleanness.

Hops

Hops are intervals and rectangles for which we will be able to give some guarantees that they can be passed.

Definition 2.8 (Hops)**.** A right-closed horizontal interval I is called a ***hop*** if it is inner clean and contains no vertical wall. A closed interval $[a, b]$ is a hop if $(a, b]$ is a hop. Vertical hops are defined similarly.

We call a rectangle $I \times J$ a ***hop*** if it is inner clean and contains no trap, and no wall (in either of its projections). ⌟

Remarks 2.9. 1. An interval or rectangle that is a hop can be empty: this is the case if the interval is $(a, a]$, or the rectangle is, say, $\mathrm{Rect}^{\rightarrow}(u, u)$.

2. The slight redundancy of considering separately $R^{\uparrow}$ and $R^{\rightarrow}$ in the present paper is there just for the sake of some continuity with [4]. The present paper could just use rectangles that are both bottom-open and left-open. On the other hand, [4] started from a graph $\mathcal{G}$ with only horizontal and vertical edges. Then the bottom left point of a rectangle that is both bottom-open and left-open would be cut off completely.

⌟

Definition 2.10 (Sequences of walls)**.** Two disjoint walls are called ***neighbors*** if the interval between them is a hop. A sequence $W_i \in \mathcal{W}$ of walls $i = 1, 2, \dots, n$ along with the intervals $I_1, \dots, I_{n-1}$ between them is called a ***sequence of neighbor walls*** if for all $i > 1$, W_i is a right neighbor of W_{i-1}. We say that an interval I is ***spanned*** by the sequence of neighbor walls $W_1, W_2, \dots, W_n$ if $I = W_1 \cup I_1 \cup W_2 \cup \cdots \cup W_n$. We will also say that I is spanned by the sequence $(W_1, W_2, \dots)$ if both I and the sequence are infinite and $I = W_1 \cup I_1 \cup W_2 \cup \dots$. If there is a hop I_0 adjacent on the left to W_1 and a hop I_n adjacent on the right to W_n (or the sequence W_i is infinite) then this system is called an ***extended sequence of neighbor walls***. We say that an interval I is ***spanned*** by this extended sequence if $I = I_0 \cup W_1 \cup I_1 \cup \cdots \cup I_n$ (and correspondingly for the infinite case). ⌟

Holes

Definition 2.11 (Reachability)**.** We say that point v is ***reachable*** from point u in $\mathcal{M}$ (and write $u \rightsquigarrow v$) if it is reachable in the graph $\mathcal{G}$. ⌟

Remark 2.12. Point u itself may be closed even if v is reachable from u. ⌟

Definition 2.13 (Slope conditions)**.** We will say that points $u = (u_0, u_1)$ and $v = (v_0, v_1)$ with $u_d < v_d$, $d = 0, 1$ satisfy the ***slope conditions*** if there is a (non-integer) point $v' = (v'_0, v'_1)$ with $0 \le v_0 - v'_0, v_1 - v'_1 < 1$ such that

$$\sigma_x \le \operatorname{slope}(u, v') \le \sigma_y^{-1}.$$

⌟

The simple slope conditions would be $\sigma_x \le \operatorname{slope}(u, v) \le \sigma_y^{-1}$, but we are a little more lenient, to allow for some rounding.

Intuitively, a hole is a place at which we can pass through a wall. We will also need some guarantees of being able to reach the hole and being able to leave it.

Definition 2.14 (Holes)**.** Let $a = (a_0, a_1)$, $b = (b_0, b_1)$, be a pair of points, and let the interval $I = (a_1, b_1]$ be the body of a horizontal barrier B. For an interval $J = (a_0, b_0]$ we say that J is a vertical ***hole passing through*** B, or ***fitting*** B, if $a \rightsquigarrow b$ within the rectangle $J \times [a_1, b_1]$. For technical convenience, we also require $|J| \le \sigma^{-1}|I|$. Consider a point (u_0, u_1) with $u_i \le a_i$, $i = 0, 1$. The hole J is called ***good as seen from*** point u if a is H-clean in $\operatorname{Rect}^{\rightarrow}(u, a)$, and b is upper-right rightward H-clean (recall Definition 2.7). It is ***good*** if it is good as seen from any such point u. Note that this way the horizontal cleanness is required to be strong, but no vertical cleanness is required (since the barrier B was not required to be outer clean).

Each hole is called ***lower left clean***, upper right clean, and so on, if the corresponding rectangle is.

Horizontal holes are defined similarly. ⌟

The conditions defining the graph $\mathcal{G}$ imply that the slope of any path is between σ and 1. It follows that the width of a horizontal hole is at most Δ, and the width of a vertical hole is at most $\sigma^{-1}\Delta$.

2.3 Conditions on the random process

The conditions will depend on a constant

$$\chi = 0.015 \tag{2.3}$$

whose role will become clear soon, and on

$$\lambda = 2^{1/2}. \tag{2.4}$$

Definition 2.15. The function

$$p(r, l)$$

is defined as the supremum of probabilities (over all points t) that any vertical or horizontal barrier with rank r and size l starts at t. ⌟

Remark 2.16. In the probability bounds of the paper [4] we also conditioned on arbitrary starting values in an interval, since there the processes X, Y were Markov chains, not necessarily Bernoulli. We omit this conditioning in the interest of readability, as it is not needed for the present application. Technically speaking, in this sense the mazery defined here is not a generalization of the earlier one. ⌟

We will use some additional constants,

$$c_1 = 2,\ c_2,\ c_3, \tag{2.5}$$

some of which will be chosen later.

Definition 2.17 (Probability bounds)**.** Let

$$p(r) = c_2 r^{-c_1} \lambda^{-r}, \tag{2.6}$$

$$h(r) = c_3 \lambda^{-\chi r}. \tag{2.7}$$

⌟

Condition 2.18.

1. (Dependencies)
   a. For any rectangle $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$.
   b. For a vertical wall value E the event $\{ E \in \mathcal{B} \}$ (that is the event that it is a vertical barrier) is a function of $X(\mathrm{Body}(E))$.
   Similarly for horizontal barriers.

c. For integers $a < b$, and the events defining strong horizontal cleanness, that is $\{ (a, b, -1) \in \mathcal{S}_x \}$ and $\{ (a, b, 1) \in \mathcal{S}_x \}$, are functions of $X((a, b])$. Similarly for vertical cleanness and the sequence Y.

   When X, Y are fixed, then for a fixed a, the (strong and not strong) cleanness of a in $(a, b]$ is decreasing as a function of $b - a$, and for a fixed b, the (strong and not strong) cleanness of b in $(a, b]$ is decreasing as a function of $b - a$. These functions reach their minimum at $b - a = \Delta$: thus, for example if x is (strongly or not strongly) left clean in $(x - \Delta, x]$ then it is (strongly or not strongly) left clean.

d. For any rectangle $Q = I \times J$, the event that its lower left corner is trap-clean in Q, is a function of the pair $X(I), Y(J)$.

   Among rectangles with a fixed lower left corner, the event that this corner is trap-clean in Q is a decreasing function of Q (in the set of rectangles partially ordered by containment). In particular, the trap-cleanness of u in $\mathrm{Rect}(u, v)$ implies its trap-cleanness in $\mathrm{Rect}^{\rightarrow}(u, v)$ and in $\mathrm{Rect}^{\uparrow}(u, v)$. If u is upper right trap-clean in the left-open or bottom-open or closed square of size Δ, then it is upper right trap-clean in all rectangles Q of the same type. Similar statements hold if we replace upper right with lower left.

Whether a certain wall value $E = (B, r)$ is a vertical barrier depends only on $X(B)$. Whether it is a vertical wall depends also only on X—however, it may depend on the values of X outside B. Similarly, whether a certain horizontal interval is inner clean depends only the sequence X but may depend on the elements outside it, but whether it is strongly inner clean depends only on X inside the interval.

Similar remarks apply to horizontal wall values and vertical cleanness with the process Y.

2. (Combinatorial requirements)

   a. A maximal external interval (see Definition 2.3) of size $\geq \Delta$ or one starting at -1 is inner clean.

   b. An interval I that is surrounded by maximal external intervals of size $\geq \Delta$ is spanned by a sequence of (vertical) neighbor walls (see Definition 2.10). This is true even in the case when I starts at 0 and even if it is infinite. To accommodate these cases, we require the following, which is somewhat harder to parse: Suppose that interval I is adjacent on the left to a maximal external interval that either starts at -1 or has size $\geq \Delta$. Suppose also that it is either adjacent on the right to a similar interval or is infinite. Then it is spanned by a (finite or infinite) sequence of neighbor walls. In particular, the whole line is spanned by an extended sequence of neighbor walls.

   c. If a (not necessarily integer aligned) right-closed interval of size $\geq 3\Delta$ contains no wall, then its middle third contains a clean point.

d. Suppose that a rectangle $I \times J$ with (not necessarily integer aligned) right-closed I, J with $|I|, |J| \geq 3\Delta$ contains no horizontal wall and no trap, and a is a right clean point in the middle third of I. There is an integer b in the middle third of J such that the point (a, b) is upper right clean. A similar statement holds if we replace upper right with lower left (and right with left). Also, if a is clean then we can find a point b in the middle third of J such that (a, b) is clean.

   There is also a similar set of statements if we vary a instead of b.

e. (Reachability) If points u, v satisfying the slope conditions are the starting and endpoint of a rectangle that is a hop, then $u \rightsquigarrow v$. The rectangle in question is allowed to be bottom-open or left-open, but not both.

   (In the present paper, we could even allow the rectangle to be both bottom open and left open, since the graph $\mathcal{G}$ has no horizontal and vertical edges anyway. But we will not use this.)

3. (Probability bounds)

a. Given a string $x = (x(0), x(1), \dots)$, a point (a, b), let $\mathcal{F}$ be the event that a trap starts at (a, b). We have

$$\mathbb{P}(\mathcal{F} \mid X = x) \leq w.$$

   The same is required if we exchange X and Y.

b. We have $p(r) \geq \sum_l p(r, l)$.

c. We require that for all $a < b$ and all $u = (u_0, u_1)$, $v = (v_0, v_1)$

$$\mathbb{P}\{\, a \text{ (resp. } b) \text{ is not strongly clean in } (a, b] \,\} \leq q_{\triangle}, \tag{2.8}$$

   Further, for $Q = \mathrm{Rect}^{\rightarrow}(u, v)$ or $\mathrm{Rect}^{\uparrow}(u, v)$ or $\mathrm{Rect}(u, v)$, for all sequences y

$$\mathbb{P}\{\, u \text{ is not trap-clean in } Q \mid Y = y \,\} \leq q_{\triangle},$$
$$\mathbb{P}\{\, v \text{ is not trap-clean in } Q \mid Y = y \,\} \leq q_{\square}$$

   and similarly with X and Y reversed.

d. Let $u \leq v < w$, and a be given with $v - u \leq \sigma^{-2}\Delta$, and define

$$b = a + \lceil \sigma_y (v - u) \rceil,$$
$$c = b \vee (a + \lfloor \sigma_x^{-1} (v - u) \rfloor).$$

   Assume that $Y = y$ is fixed in such a way that B is a horizontal wall of rank r with body $(v, w]$. For a $d \in [b, c]$ let $Q(d) = \mathrm{Rect}^{\rightarrow}((a, u), (d, v))$. Let

$$F(u, v;\ a, d)$$

be the event (a function of X) that $Q(d)$ contains no traps or vertical barriers, and is inner H-clean. Let

$$E = E(u,v,w;\ a)$$

be the event that at some point $d \in [b,c]$ a vertical hole fitting B starts, and event $F(u,v;\ a,d)$ holds. Then

$$\mathbb{P}(E \mid Y = y) \geq (v-u+1)^{\chi} h(r).$$

The same is required if we exchange horizontal and vertical, X with Y, further σ_y with σ_x, and define $Q(d) = \mathrm{Rect}^{\uparrow}((u,a),(v,d))$.

Figure 10 of [4] illustrates the last condition.

⌟

The following lemma shows how the above condition will serve for passing from point (a,u) past the wall.

Lemma 2.19. *In Condition 2.18.3d, the points (a,u), (d,v) always satisfy the slope conditions.*

Proof. Consider the case of horizontal walls. We have $b-a \geq \sigma_y(v-u)$ by definition. If $c > b$ then also $d-a \leq \sigma_x^{-1}(v-u)$ by definition for any $d \in [b,c]$. Assume therefore $c = b$, then $d = b = c$. We claim that the points (a,u) and (b,v) satisfy the slope conditions. Indeed, set $b' = a + \sigma_y(v-u)$, then $b-1 < b' \leq b$, and

$$1/\sigma_y = \mathrm{slope}((a,u),(b',v)) > \sigma_x$$

since $\sigma_x\sigma_y < 1$. The case for vertical walls is similar. □

Remarks 2.20.

1. Conditions 2.18.2c and 2.18.2d imply the following. Suppose that a right-upper closed square Q of size 3Δ contains no wall or trap. Then its middle third contains a clean point.
2. Note the following asymmetry: the probability bound on the upper right corner of a rectangle not being trap-clean in it is $q_{\square}$ which is bounded only by 0.55, while the bound of the lower left corner not being trap-clean in it is $q_{\triangle}$, which is bounded by 0.05.
3. With respect to condition 2.18.2e note that not all individual edges satisfy the slope condition; indeed, some arguments will make use of this fact.
4. The most important special case of Condition 2.18.3d is $v = u$, then it says that for any horizontal wall B of rank r, at any point a, the probability that there is a vertical hole passing through B at point a is at least $h(r)$.

⌟

2.4 Base mazery

Let us define a mazery $\mathcal{M}^1$ corresponding to the embedding problem.

Example 2.21 (Embedding mazery)**.** Let

$$\begin{aligned}\sigma = \sigma_{x1} &= 1/2\tilde{m}, \quad \sigma_{y1} = \tilde{m}, \quad R_1 = 2\tilde{m},\\ \Delta_1 &= \lambda^{\delta R_1},\\ q_{\triangle 1} &= 0, \quad q_{\square 1} = 0.5, \quad w_1 = 0,\end{aligned}$$

$\delta = 0.15$ (the choice will be justified in Section 5).

Let $\mathcal{G}(X, Y) = \mathcal{G}_{3\tilde{m}}(X, Y)$ be the graph defined in the introduction. Let $\mathcal{T} = \emptyset$, that is there are no traps.

An interval $(i, i + l]$ is a vertical barrier and wall if and only if $\tilde{m} \le l < 2\tilde{m}$, and $X(i + 1) = X(i + 2) = \cdots = X(i + l)$. Similarly, it is a horizontal barrier and wall if and only if $Y(i + 1) = Y(i + 2) = \cdots = Y(i + l)$. We define the common rank of these barriers to be R_1.

Every point is strongly clean in all one-dimensional senses. All points are upper right trap clean. A point (i, j) is lower left trap-clean if $X(i) = Y(j)$. On the other, hand if $X(i) \neq Y(j)$ then it is not trap-clean in any nonempty rectangles whose upper right corner it is.

Note that even though the size of the largest walls or traps is bounded by $\tilde{m}$, the bound Δ_1 is defined to be exponential in $\tilde{m}$. This will fit into the scheme of later definitions. ⌟

Lemma 2.22. *The definition given in Example 2.21 satisfies the mazery conditions, for sufficiently large $R_1(= 2\tilde{m})$.*

Proof. We will write $R = R_1$ throughout the proof.

1. Almost all combinatorial and dependency conditions are satisfied trivially; here are the exceptions. Condition 2.18.2b says that an interval I surrounded by maximal external intervals of size $\ge \Delta$ is spanned by a sequence of (vertical) neighbor walls. Since I is a surrounded by maximal external intervals, there is a wall of size $\tilde{m}$ at the beginning of I and one of size $\tilde{m}$ at the end of I. If $|I| < 2\tilde{m}$ then I is itself a wall. Otherwise, we start with the wall J_1 of size $\tilde{m}$ at the beginning, and build a sequence of disjoint walls $J_1, J_2, \ldots$ of size $\tilde{m}$ recursively with each J_i at a distance $\ge \tilde{m}$ from the right end of I. The next wall is chosen always to be the closest possible satisfying these conditions. Finally, we add the wall of size $\tilde{m}$ at the end of I. Since every point is by definition strongly clean in all one-dimensional senses, the sequence we built is a spanning sequence of neighbor walls.

In Condition 2.18.2d, only lower left cleanness is not automatic. Suppose that a rectangle $I \times J$ with right-closed I, J with $|I|, |J| \geq 3\Delta_1$ contains no horizontal wall and no trap, and a is a point in the middle third of I. We must show that there is an integer b in the middle third of J such that the point (a, b) is lower left clean. This condition would now only be violated if $Y(b) \neq X(a)$ for all b in the middle third. But since $\Delta_1 > \tilde{m}$, this would create a horizontal wall, which was excluded. The same argument applies if we vary a instead of b.

2. Let us verify the reachability condition. Let $u < v$, $v = (v_0, v_1)$ be points with the property that there is a $v' = (v'_0, v'_1)$ with $0 \leq v_d - v'_d < 1$ for $d = 0, 1$, and $\mathrm{slope}(u, v') \geq \sigma_{x1} = 1/2\tilde{m}$, $1/\mathrm{slope}(u, v') \geq \sigma_{y1} = \tilde{m}$. If they are the starting and endpoint of a (bottom-open or left-open) rectangle that is a hop, then the condition requires $u \rightsquigarrow v$. The hop property implies $X(v_0) = Y(v_1)$: indeed, otherwise the rectangle would not be inner clean.

   Without loss of generality, let $u = (0, 0)$, $v = (a, b)$, $v' = (a', b')$. Now the slope requirements mean $\tilde{m} \leq a'/b' \leq 2\tilde{m}$, hence $\tilde{m} < a/(b-1)$, $(a-1)/b < 2\tilde{m}$, and so

$$\tilde{m}(b-1) < a \leq 2\tilde{m}b.$$

   It is then easy to see that we can choose a sequence

$$0 \leq s_1 < s_2 < \cdots < s_{b-1} < a$$

   with the properties

$$\begin{aligned} s_1 &\leq 2\tilde{m}, \\ s_i + \tilde{m} \leq s_{i+1} &\leq s_i + 2\tilde{m}, \\ s_{b-1} + \tilde{m} < a &\leq s_{b-1} + 2\tilde{m}. \end{aligned}$$

   Indeed, if the s_i are all made minimal then $s_{b-1} + \tilde{m} = \tilde{m}(b-2) + \tilde{m} < a$. On the other hand, if all these values are maximal then $s_{b-1} + 2\tilde{m} = 2\tilde{m}(b-1) + 2\tilde{m} \geq a$. Choosing the values in between we can satisfy both inequalities.

   The hop requirement implies that there is no vertical wall in $(0, a]$, that is there are no $\tilde{m}$ consecutive numbers in this interval with identical values of $X(i)$. It also implies $X(a) = Y(b)$. Let us choose a_i from the interval $(s_i, s_i + \tilde{m}]$ such that $X(a_j) = Y(j)$. By construction we have $0 < a_1 \leq 2\tilde{m}$, $0 < a_{i+1} - a_i < 3\tilde{m}$, $0 < a - a_{b-1} \leq 2\tilde{m}$. Thus the points (a_j, j) form a path in the graph $\mathcal{G} = \mathcal{G}_{3\tilde{m}}(X, Y)$ from $(0, 0)$ to v.

3. Since there are no traps, the trap probability upper bound is satisfied trivially.

4. Consider the probability bounds for barriers. Since the rank is the same for both horizontal and vertical barriers, it is sufficient to consider vertical ones. Clearly $p(r, l) = 0$ unless $r = R$, $l \geq \tilde{m}$, in which case it is 2^{-l}; hence $\sum_l p(R, l) \leq 2^{-\tilde{m}+1}$. For $p(R) \geq \sum_l p(R, l)$, we need:
$$2^{-\tilde{m}+1} \leq c_2 R^{-c_1} \lambda^{-R},$$
which holds for R sufficiently large, since $\lambda = 2^{1/2}$ by (2.4).
5. Consider the bounds 2.18.3c on the probability that some point in not clean in some way. Only the lower left trap-cleanness is now in question, so only the bound
$$\mathbb{P}\{\, v \text{ is not trap-clean in } Q \mid Y = y \,\} \leq q_{\square}$$
must be checked. The event happens here only if $X(v_0) \neq y(v_1)$: its probability, $\frac{1}{2}$, is now equal to $q_{\square}$ by definition, so the inequality holds. The argument is the same when horizontal and vertical are exchanged.
6. Consider Condition 2.18.3d for a vertical wall, with the parameters a, u, v, w. With our parameters, it gives $b = a + \lceil (v - u)/2\tilde{m} \rceil$, and events $F(u, v;\ a, d)$ and $E = E(u, v, w;\ a)$. By definition of cleanness now, the lower left corner of any rectangle is automatically V-clean in it. The requirement is
$$\mathbb{P}(E \mid X = x) \geq (v - u + 1)^{\chi} h(r).$$
Since B is a wall we have $X(v + 1) = \cdots = X(w)$.
Let A denote the event that interval $(a, b]$ contains a horizontal barrier. Then the probability of A is bounded by $1/8$ if $R = 2\tilde{m}$ is sufficiently large. Indeed,
$$b - a \leq \sigma^{-2} \Delta = 4\tilde{m}^2 \lambda^{2\delta \tilde{m}},$$
while the probability of a barrier at a point is $\leq 2^{-\tilde{m}}$. Via the union bound, we bound the probability by the product of these two numbers.
Let E' be the event that $Y(b + 1) = X(w)$, further $b > a \Rightarrow Y(b) = X(v)$. It has probability at least $\frac{1}{4}$. This event implies that (v, b) is trap-clean in $Q(b)$, so $Q(b)$ becomes inner V-clean. It also implies a horizontal hole $(b, b + 1]$ fitting the wall B, as we can simply go from (v, b) to $(w, b + 1)$ on an edge of the graph $\mathcal{G}$.
Lemma 2.19 implies that $Q(b)$ satisfies the slope conditions, so $E' \setminus A$ implies also event $F(u, v;\ a, b)$, so also event $E(u, v, w;\ a)$. So $\frac{1}{4} - \frac{1}{8}$ lowerbounds the probability of event $E' \setminus A \subseteq E$. It is sufficient to lowerbound therefore $\frac{1}{8}$ by
$$(v - u + 1)^{\chi} h(R) = (v - u + 1)^{\chi} \lambda^{-\chi R}.$$

Using the value $\Delta = \lambda^{\delta R}$ and the bound $v - u \le \sigma^{-2}\Delta$, it is sufficient to have

$$c_3(2\sigma^{-2}\Delta)^{\chi}\lambda^{-\chi R_1} = c_3(8R^2)^{\chi}\lambda^{-\chi R(1-\delta)} \le 1/8,$$

which is true with R sufficiently large.

7. Consider now the probability lower bound on passing a horizontal wall of size l, where $\tilde{m} \le l < 2\tilde{m}$, that is Condition 2.18.3d. This condition, for our parameters, defines $b = a + \tilde{m}(v - u)$. It assumes that $Y = y$ is fixed in such a way that B is a horizontal wall of rank r with body $(v, w]$. The requirement is $\mathbb{P}(E \mid Y = y) \ge (v - u + 1)^{\chi} h(r)$. Now, since B is a wall we have $Y(v + 1) = \cdots = Y(w)$.

   Let A_1 denote the event that there is a vertical wall in $(a, b]$. Let E' be the event that $b > a \Rightarrow X(b) = Y(v)$. It implies that $Q(b)$ is inner H-clean. The event $E' \setminus A_1$ implies $F(u, v;\ a, b)$.

   Let A_2 denote the event that there is an interval $I \subseteq (b, b + l\tilde{m}]$ of size $\tilde{m}$ with $X(i) \ne Y(w)$ for all $i \in I$. Let E'' denote the event $X(b + l\tilde{m}) = Y(w)$. Then $E'' \setminus A_2$ implies that $(b, b + l\tilde{m}]$ is a vertical hole fitting the wall B. Indeed, just as in the proof of the reachability condition, already the fact that there is no interval $I \subseteq (b, b + l\tilde{m}]$ of size $\tilde{m}$ with $X(i) \ne Y(w)$ for all i, and that the pair of points (b, v), $(b + l\tilde{m}, w)$ satisfies the slope conditions, implies that the second point is reachable from the first one.

   So the event $E' \cap E'' \setminus (A_1 \cup A_2)$ implies $E(u, v, w;\ a)$. Let us upperbound the probability that this does not occur. Since events E', E'' are independent, the probability that $E' \cap E''$ does not occur is at most $\frac{3}{4}$. The probability of $A_1 \cup A_2$ can be bounded by $\frac{1}{8}$, just as in the case of passing a horizontal wall. Thus we found $\mathbb{P}(E) \ge 1 - \frac{7}{8} = \frac{1}{8}$. It is sufficient to lowerbounded this by $(v - u + 1)^{\chi} c_3 \lambda^{-\chi R}$. So we will be done if

$$(2\sigma^{-2}\Delta)^{\chi} c_3 \lambda^{-\chi R} = (8R^2)^{\chi} c_3 \lambda^{-\chi R(1-\delta)} \le 1/8,$$

   which holds if R is sufficiently large.

□

3 Application to the theorem

Theorem 1 follows from Lemma 2.22 and the following theorem:

Theorem 3. *In every mazery with a sufficiently large rank lower bound there is an infinite path starting from the origin, with positive probability.*

The proof will use the following definitions.

Definition 3.1. In a mazery $\mathcal{M}$, let $\mathcal{Q}$ be the event that the origin $(0, 0)$ is not upper right clean, and $\mathcal{F}(n)$ the event that the square $[0, n]^2$ contains some wall or trap. ⌟

Lemma 3.2 (Main). *Let $\mathcal{M}^1$ be a mazery. If its rank lower bound is sufficiently large then a sequence of mazeries $\mathcal{M}^k$, $k > 1$ can be constructed on a common probability space, sharing the graph $\mathcal{G}$ of $\mathcal{M}^1$ and the parameter σ, and satisfying*

$$\begin{aligned}\sigma_{j,k+1} &\geq \sigma_{j,k} \text{ for } j = x, y,\\ \Delta_k/\Delta_{k+1} &< \sigma^2/2,\\ 1/4 &> \sum_{k=1}^{\infty} \mathbb{P}\big(\mathcal{F}_k(\Delta_{k+1}) \cup (\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k)\big).\end{aligned} \tag{3.1}$$

Most of the paper will be taken up with the proof of this lemma. Now we will use it to prove the theorem.

Proof of Theorem 3. Let $u = (0, 0)$ denote the origin. The mazery conditions imply $\mathbb{P}(\mathcal{Q}_1) \leq 0.15$. Let us construct the series of mazeries $\mathcal{M}^k$ satisfying the conditions of Lemma 3.2. These conditions imply that the probability that one of the events $\mathcal{F}_k(\Delta_{k+1})$, $\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k$ hold is less than 0.25. Hence the probability that $\bigcup_{k=1}^{\infty} \mathcal{Q}_k \cup \mathcal{F}_k(\Delta_{k+1})$ holds is at most 0.4. With probability at least 0.6 none of these events holds. Assume now that this is the case. We will show that there is an infinite number of points v of the graph reachable from the origin. The usual compactness argument implies then an infinite path starting at the origin.

Under the assumption, in all mazeries $\mathcal{M}^k$ the origin u is upper right clean, and the square $[0, \Delta_{k+1}]^2$ contains no walls or traps. Let $\sigma_x = \sigma_{x,k}$, $\sigma_y = \sigma_{y,k}$. Consider the point $(a, b) = (\Delta_{k+1}, \sigma_x \Delta_{k+1})$. Then the square $(a - 3\Delta_k, b) + [0, 3\Delta_k]^2$ is inside the square $[0, \Delta_{k+1}]^2$, and contains no walls or traps. The mazery conditions imply that then its middle, the square $(a - 2\Delta_k, b + \Delta_k) + [0, \Delta_k]^2$ contains a clean point $v = (v_0, v_1)$. By its construction, the rectangle $\mathrm{Rect}(u, v)$ is a hop. Let us show that it also satisfies the slope lower bounds of mazery $\mathcal{M}^k$, and therefore by the reachability condition, $u \rightsquigarrow v$. Indeed, by its construction, v is above the line of slope σ_x starting from u. On the other hand, using the bound on Δ_k/Δ_{k+1} of Lemma 3.2 and $\sigma \leq 1/2$:

$$\begin{aligned}\mathrm{slope}(u, v) = \frac{v_1}{v_0} &\leq \frac{\sigma_x \Delta_{k+1} + 2\Delta_k}{\Delta_{k+1} - 2\Delta_k} \leq \frac{\sigma_x + \sigma^2}{1 - \sigma^2}\\ &\leq \sigma_x \frac{1 + \sigma}{1 - \sigma^2} = \frac{\sigma_x}{1 - \sigma} \leq 1/\sigma_y\end{aligned}$$

by (2.2). □

Remark 3.3. It follows from the proof that if we use the base mazery of Example 2.21 then the probability of the existence of an infinite path in Theorem 3 converges to 1 as $\tilde{m} \to \infty$. Indeed in this case $\mathbb{P}(Q_1) = 0$, and the sum in (3.1) converges to 0 as $\tilde{m} \to \infty$. ⌟

4 The scaled-up structure

In this section, we will define the scaling-up operation $\mathcal{M} \mapsto \mathcal{M}^*$ producing $\mathcal{M}^{k+1}$ from $\mathcal{M}^k$; however, we postpone to Section 5 the definition of several parameters and probability bounds for $\mathcal{M}^*$.

4.1 The scale-up construction

Some of the following parameters will be given values only later, but they are introduced by name here.

Definition 4.1. The positive parameters Δ, Γ, Φ will be different for each level of the construction, and satisfy

$$\begin{aligned} \Delta/\Gamma = (\Gamma/\Phi)^{1/2} &\ll \sigma^4, \\ \Phi &\ll \Delta^*. \end{aligned} \tag{4.1}$$

More precisely the $\ll$ is understood here as $\lim_{R\to\infty} \Phi/\Delta^* = 0$. ⌟

Here is the approximate meaning of these parameters: Walls closer than Φ to each other, and intervals larger than Γ without holes raise alarm, and a trap closer than Γ makes a point unclean. (The precise equality of the quotients above is not crucial for the proof, but is convenient.)

Definition 4.2. Let $\sigma_i^* = \sigma_i + \Lambda\sigma^{-3}\Delta/\Gamma$ for $i = x, y$, where Λ is a constant to be defined later (in the proof of Lemma 7.10). ⌟

For the new value of R we require

$$R^* \le 2R - \log_\lambda \Phi. \tag{4.2}$$

Definition 4.3 (Light and heavy)**.** Barriers and walls of rank lower than R^* are called ***light***, the other ones are called ***heavy***. ⌟

Heavy walls of $\mathcal{M}$ will also be walls of $\mathcal{M}^*$ (with some exceptions given below). We will define walls only for either X or Y, but it is understood that they are also defined when the roles of X and Y are reversed.

The rest of the scale-up construction will be given in the following steps.

Step 1 (Cleanness)**.** For an interval I, its right endpoint x will be called clean in I for $\mathcal{M}^*$ if

- It is clean in I for $\mathcal{M}$.
- The interval I contains no wall of $\mathcal{M}$ whose right end is closer to x than $\Phi/3$.

We will say that a point is strongly clean in I for $\mathcal{M}^*$ if it is strongly clean in I for $\mathcal{M}$ and I contains no barrier of $\mathcal{M}$ whose right end is closer to it than $\Phi/3$. Cleanness and strong cleanness of the left endpoint is defined similarly.

Let a point u be a starting point or endpoint of a rectangle Q. It will be called trap-clean in Q for $\mathcal{M}^*$ if

- It is trap-clean in Q for $\mathcal{M}$.
- Any trap contained in Q is at a distance $\geq \Gamma$ from u.

⌟

Step 2 (Uncorrelated traps)**.** A rectangle Q is called an ***uncorrelated compound trap*** if it contains two traps with disjoint projections, with a distance of their starting points at most Φ, and if it is minimal among the rectangles containing these traps. ⌟

Clearly, the size of an uncorrelated trap is bounded by $\Delta + \Phi$.

Step 3 (Correlated trap)**.** Let

$$L_1 = 29\sigma^{-1}\Delta, \quad L_2 = 9\sigma^{-1}\Gamma. \tag{4.3}$$

(Choice motivated by the proof of Lemmas 4.11 and 7.1.) Let I be a closed interval with length L_i, $i = 1, 2$, and $b \in \mathbb{Z}_+$, with $J = [b, b + 5\Delta]$. We say that event

$$\mathcal{L}_i(X, Y, I, b)$$

holds if $I \times J$ contains at least four traps with disjoint x projections. Let $x(I), y(J)$ be given. We will say that $I \times J$ is a ***horizontal correlated trap*** of kind i if $\mathcal{L}_i(X, Y, I, b)$ holds and

$$\mathbb{P}(\mathcal{L}_i(X, Y, I, b) \mid X(I) = x(I)) \leq w^2.$$

Vertical correlated traps are defined analogously. Figure 11 of [4] illustrates correlated traps. ⌟

Remark 4.4. In the present paper, traps of type 1 are used only in Part 2 of the proof of Lemma 7.1. ⌟

Step 4 (Traps of the missing-hole kind)**.** Let I be a closed interval of size Γ, let b be a site with $J = [b, b + 3\Delta]$. We say that event

$$\mathcal{L}_3(X, Y, I, b)$$

holds if there is a $b' > b + \Delta$ such that $(b + \Delta, b']$ is the body of a light horizontal potential wall W, and no good vertical hole (in the sense of Definition 2.14) $(a_1, a_2]$ with $(a_1 - \Delta, a_2 + \Delta] \subseteq I$ passes through W.

Let $x(I), y(J)$ be fixed. We say that $I \times J$ is a ***horizontal trap of the missing-hole kind*** if event $\mathcal{L}_3(X, Y, I, b)$ holds and

$$\mathbb{P}\big(\, \mathcal{L}_3(X, Y, I, b) \mid X(I) = x(I) \,\big) \le w^2.$$

Figure 12 of [4] illustrates traps of the missing-hole kind. ⌟

Note that the last probability is independent of the value of b.

The value L_2 bounds the size of all new traps, and it is $\ll \Phi$ due to (4.1).

Step 5 (Emerging walls)**.** We define some objects as barriers, and then designate some of the barriers (but not all) as walls.

A vertical emerging barrier is, essentially, a horizontal interval over which the conditional probability of a bad event $\mathcal{L}_j$ is not small (thus preventing a new trap). But in order to find enough barriers, the ends are allowed to be slightly extended. Let x be a particular value of the sequence X over an interval $I = (u, v]$. For any $u' \in (u, u + 2\Delta]$, $v' \in (v - 2\Delta, v]$, let us define the interval $I' = [u', v']$. We say that interval I is the body of a vertical ***barrier*** of the ***emerging kind***, of type $j \in \{1, 2, 3\}$ if the following inequality holds:

$$\sup_{I'} \mathbb{P}\big(\, \mathcal{L}_j(x, Y, I', 1) \mid X(I') = x(I') \,\big) > w^2.$$

To be more explicit, for example interval I is an emerging barrier of type 2 for the process X if it has a closed subinterval I' of size L_2 within 2Δ of its two ends, such that conditionally over the value of $X(I')$, with probability $> w^2$, the rectangle $I \times [b, b + 5\Delta]$ contains four traps with disjoint x projections. More simply, the value $X(I')$ makes not too improbable (in terms of a randomly chosen Y) for a sequence of closely placed traps to exist reaching horizontally across $I' \times [b, b + 5\Delta]$.

Let

$$L_3 = \Gamma.$$

Then emerging barriers of type j have length in $L_j + [0, 4\Delta]$. Figure 13 of [4] illustrates emerging barriers.

We will designate some of the emerging barriers as walls. We will say that I is a ***pre-wall*** of the emerging kind if also the following properties hold:

(a) Either I is an external hop of $\mathcal{M}$ or it is the union of a dominant light wall and one or two external hops of $\mathcal{M}$, of size $\geq \Delta$, surrounding it.

(b) Each end of I is adjacent to either an external hop of size $\geq \Delta$ or a wall of $\mathcal{M}$.

Figure 14 of [4] illustrates pre-walls.

Now, for $j = 1, 2, 3$, list all emerging pre-walls of type j in a sequence $(B_{j1}, B_{j2}, \dots)$. First process the sequence $(B_{11}, B_{12}, \dots)$. Designate B_{1n} a wall if and only if it is disjoint of all emerging pre-walls designated as walls earlier. Then process the sequence $(B_{31}, B_{32}, \dots)$. Designate B_{3n} a wall if and only if it is disjoint of all emerging pre-walls designated as walls earlier. Finally process the sequence $(B_{21}, B_{22}, \dots)$ similarly.

To emerging barriers and walls, we assign rank

$$\hat{R} > R^* \tag{4.4}$$

to be determined later.

⌟

Step 6 (Compound walls)**.** A ***compound barrier*** occurs in $\mathcal{M}^*$ for X wherever barriers W_1, W_2 occur (in this order) for X at a distance $d \leq \Phi$, and W_1 is light. (The distance is measured between the right end of W_1 and the left end of W_2.) We will call this barrier a wall if W_1, W_2 are neighbor walls (that is, they are walls separated by a hop). We denote the new compound wall or barrier by

$$W_1 + W_2.$$

Its body is the smallest right-closed interval containing the bodies of W_j. For r_j the rank of W_j, we will say that the compound wall or barrier in question has ***type***

$$\langle r_1, r_2, i \rangle, \text{ where } i = \begin{cases} d & \text{if } d \in \{0, 1\}, \\ \lfloor \log_\lambda d \rfloor & \text{otherwise.} \end{cases}$$

Its rank is defined as

$$r = r_1 + r_2 - i. \tag{4.5}$$

Thus, a shorter distance gives higher rank. This definition gives

$$r_1 + r_2 - \log_\lambda \Phi \leq r \leq r_1 + r_2.$$

Inequality (4.2) will make sure that the rank of the compound walls is lower-bounded by R^*.

Now we repeat the whole compounding step, introducing compound walls and barriers in which now W_2 is required to be light. The barrier W_1 can be any barrier introduced until now, also a compound barrier introduced in the first compounding step. ⌟

The walls that will occur as a result of the compounding operation are of the type $L + W$, $W + L$, or $(L + W) + L$, where L is a light wall of $\mathcal{M}$ and W is any wall of $\mathcal{M}$ or an emerging wall of $\mathcal{M}^*$. Figure 15 of [4] illustrates the different kinds of compound barriers. Thus, the maximum size of a compound wall is

$$\Delta + \Phi + (L_2 + 4\Delta) + \Phi + \Delta < \Delta^*,$$

for sufficiently large R_1, where we used (4.1).

Step 7 (Finish)**.** The graph $\mathcal{G}$ does not change in the scale-up: $\mathcal{G}^* = \mathcal{G}$. Remove all traps of $\mathcal{M}$.

Remove all light walls and barriers. If the removed light wall was dominant, remove also all other walls of $\mathcal{M}$ (even if not light) contained in it. ⌟

4.2 Combinatorial properties

The following lemmas are taken straight from [4], and their proofs are unchanged in every essential respect.

Lemma 4.5. *The new mazery $\mathcal{M}^*$ satisfies Condition 2.18.1.*

This lemma corresponds to Lemma 4.1 of [4].

Lemma 4.6. *The mazery $\mathcal{M}^*$ satisfies conditions 2.18.2a and 2.18.2b.*

This lemma corresponds to Lemma 4.2 of [4].

Lemma 4.7. *Suppose that interval I contains no walls of $\mathcal{M}^*$, and no wall of $\mathcal{M}$ closer to its ends than $\Phi/3$ (these conditions are satisfied if it is a hop of $\mathcal{M}^*$). Then it either contains no walls of $\mathcal{M}$ or all walls of $\mathcal{M}$ in it are covered by a sequence $W_1, \dots, W_n$ of dominant light neighbor walls of $\mathcal{M}$ separated from each other by external hops of $\mathcal{M}$ of size $> \Phi$.*

If I is a hop of $\mathcal{M}^*$ then either it is also a hop of $\mathcal{M}$ or the above end intervals are hops of $\mathcal{M}$.

This lemma corresponds to Lemma 4.3 of [4].

Lemma 4.8. *Let us be given intervals $I' \subset I$, and also $x(I)$, with the following properties for some $j \in \{1, 2, 3\}$.*

(a) *All walls of $\mathcal{M}$ in I are covered by a sequence $W_1, \dots, W_n$ of dominant light neighbor walls of $\mathcal{M}$ such that the W_i are at a distance $> \Phi$ from each other and at a distance $\geq \Phi/3$ from the ends of I.*

(b) *I' is an emerging barrier of type j.*

(c) *I' is at a distance $\geq L_j + 7\Delta$ from the ends of I.*

Then I contains an emerging wall.

This lemma corresponds to Lemma 4.4 of [4].

Lemma 4.9. *Let the rectangle Q with X projection I contain no traps or vertical walls of $\mathcal{M}^*$, and no vertical wall of $\mathcal{M}$ closer than $\Phi/3$ to its sides. Let $I' = [a, a + \Gamma]$, $J = [b, b + 3\Delta]$ with $I' \times J \subseteq Q$ be such that I' is at a distance $\geq \Gamma + 7\Delta$ from the ends of I. Suppose that a light horizontal wall W starts at position $b + \Delta$. Then $[a + \Delta, a + \Gamma - \Delta]$ contains a vertical hole passing through W that is good in the sense of Definition 2.14. The same holds if we interchange horizontal and vertical.*

This lemma corresponds to Lemma 4.5 of [4].

Lemma 4.10. *Let rectangle Q with X projection I contain no traps or vertical walls of $\mathcal{M}^*$, and no vertical walls of $\mathcal{M}$ closer than $\Phi/3$ to its sides. Let L_j, $j = 1, 2$ be as introduced in the definition of correlated traps and emerging walls in Steps 3 and 5 of the scale-up construction. Let $I' = [a, a + L_j]$, $J = [b, b + 5\Delta]$ with $I' \times J \subseteq Q$ be such that I' is at a distance $\geq L_j + 7\Delta$ from the ends of I. Then I' contains a subinterval I'' of size $L_j/4 - 2\Delta$ such that the rectangle $I'' \times J$ contains no trap of $\mathcal{M}$. The same holds if we interchange horizontal and vertical.*

This lemma corresponds to Lemma 4.6 of [4]. Note that

$$L_2/4 - 2\Delta > 2.2\sigma^{-1}\Gamma.$$

Lemma 4.11. *The new mazery $\mathcal{M}^*$ defined by the above construction satisfies Conditions 2.18.2c and 2.18.2d.*

This lemma corresponds to Lemma 4.7 of [4].

5 The scale-up functions

Mazery $\mathcal{M}^1$ is defined in Example 2.21. The following definition introduces some of the parameters needed for scale-up. The choices will be justified by the lemmas of Section 6.

Definition 5.1. At scale-up by one level, to obtain the new rank lower bound, we multiply R by a constant:

$$R = R_k = R_1\tau^{k-1}, \quad R_{k+1} = R^* = R\tau, \quad 1 < \tau < 2. \tag{5.1}$$

The rank of emerging walls, introduced in (4.4), is defined using a new parameter τ':

$$\hat{R} = \tau' R.$$

⌟

We require

$$\tau < \tau' < \tau^2. \tag{5.2}$$

We need some bounds on the possible rank values.

Definition 5.2. Let $\overline{\tau} = 2\tau/(\tau - 1)$. ⌟

Lemma 5.3 (Rank upper bound)**.** *In a mazery, all ranks are upper-bounded by $\overline{\tau}R$.*

This lemma and its corollary correspond to Lemma 6.1 and Corollary 6.2 of [4].

Corollary 5.4. *Every rank exists in $\mathcal{M}^k$ for at most $\lceil \log_\tau \frac{2\tau}{\tau-1} \rceil$ values of k.*

It is convenient to express several other parameters of $\mathcal{M}$ and the scale-up in terms of a single one, T:

Definition 5.5 (Exponential relations)**.** Let $T = \lambda^R$,

$$\varDelta = T^\delta, \quad \varGamma = T^\gamma, \quad \varPhi = T^\varphi, \quad w = T^{-\omega}.$$

We require

$$0 < \delta < \gamma < \varphi < 1. \tag{5.3}$$

⌟

Note that the requirement (4.2) is satisfied as long as

$$\tau \le 2 - \varphi. \tag{5.4}$$

Our definitions give $\varDelta^* = \varDelta^\tau$. Let us see what is needed for this to indeed upper-bound the size of any new walls in $\mathcal{M}^*$. Emerging walls can have size as large as $L_2 + 4\varDelta$, and at the time of their creation, they are the largest existing ones. We get the largest new walls when the compound operation combines these with light walls on both sides, leaving the largest gap possible, so the largest new wall size is

$$L_2 + 2\varPhi + 6\varDelta < 3\varPhi,$$

where we used $\varDelta \ll \varGamma \ll \varPhi$ from (4.1), and that R_1 is large enough. In the latter case, we always get $3\varPhi \le \varDelta^*$ if

$$\varphi < \tau\delta. \tag{5.5}$$

As a reformulation of (4.1), we require

$$2(\gamma - \delta) = \varphi - \gamma. \tag{5.6}$$

We also need

$$2\gamma - \tau\delta + 1 < \omega, \tag{5.7}$$
$$4(\gamma + \delta) < \omega(4 - \tau), \tag{5.8}$$
$$4\gamma + 6\delta + \tau' < 2\omega, \tag{5.9}$$
$$\tau(\delta + 1) < \tau'. \tag{5.10}$$

(Lemma 6.21 uses (5.7), Lemma 6.6 uses (5.8), Lemma 6.8 uses (5.9), and Lemma 6.16 uses (5.10).)

Using the exponent χ introduced in (2.3), we require

$$\tau\chi < \gamma - \delta, \tag{5.11}$$
$$\overline{\tau}\chi < 1 - \tau\delta, \tag{5.12}$$
$$\overline{\tau}\chi < \omega - 2\tau\delta. \tag{5.13}$$

(Lemmas 6.6 and 6.8 use (5.11), Lemmas 6.18 and 6.19 use (5.12), and Lemma 6.18 uses (5.13).)

The condition these inequalities impose on χ is just to be sufficiently small (and, of course, that the bounds involved are positive). On ω the condition is just to be sufficiently large.

Lemma 5.6. *The exponents $\delta, \gamma, \varphi, \tau, \tau', \chi$ can be chosen to satisfy the inequalities* (5.1), (5.2), (5.3)-(5.13).

Proof. It can be checked that the choices $\delta = 0.15$, $\gamma = 0.18$, $\varphi = 0.24$, $\tau = 1.75$, $\tau' = 2.5$, $\omega = 4.5$, $\overline{\tau} = 4.66\ldots$ satisfy all the inequalities in question. □

Definition 5.7. Let us fix now the exponents $\delta, \varphi, \gamma, \tau, \tau', \chi$ as chosen in the lemma. In order to satisfy all our requirements also for small k, we will fix c_2 sufficiently small, then c_3 sufficiently large, and finally R_1 sufficiently large. ⌟

We need to specify some additional parameters.

Definition 5.8. Let $q_i^* = q_i + \Delta^* T^{-1}$ for $i = \triangle, \square$. ⌟

In estimates that follow, in order to avoid cumbersome calculations, we will liberally use the notation $\ll$, $\gg$, $o()$, $O()$. The meaning is always in terms of $R_1 \to \infty$.

6 Probability bounds

In this section, we derive the bounds on probabilities in $\mathcal{M}^k$, sometimes relying on the corresponding bounds for $\mathcal{M}^i$, $i < k$.

6.1 New traps

Lemma 6.1 (Uncorrelated Traps). *Given a string $x = (x(0), x(1), \dots)$, a point (a_1, b_1), let $\mathcal{F}$ be the event that an uncorrelated compound trap of $\mathcal{M}^*$ starts at (a_1, b_1). Then*

$$\mathbb{P}\big(\mathcal{F} \mid X = x\big) \le 2\Phi^2 w^2.$$

This lemma corresponds to Lemma 5.4 of [4].

Lemma 6.2 (Correlated Traps). *Let a site (a, b) be given. For $j = 1, 2$, let $\mathcal{F}_j$ be the event that a horizontal correlated trap of type j starts at (a, b).*

(a) *Let us fix a string $x = (x(0), x(1), \dots)$. We have*

$$\mathbb{P}\big(\mathcal{F}_j \mid X = x\big) \le w^2.$$

(b) *Let us fix a string $y = (y(0), y(1), \dots)$. We have*

$$\mathbb{P}\big(\mathcal{F}_j \mid Y = y\big) \le (5\Delta L_j w)^4.$$

This lemma corresponds to Lemma 5.5 of [4].

Before considering missing-hole traps, recall the definitions needed for the hole lower bound condition, Condition 2.18.3d, in particular the definition of the numbers a, u, v, w, b, c, and event E.

Since we will hold the sequence y of values of the sequence Y of random variables fixed in this subsection, we take the liberty and omit the condition $Y = y$ from the probabilities: it is always assumed to be there.

Recall the definitions of events F and E in Condition 2.18.3d. For integers a and $u \le v$ and a horizontal wall $(v, w]$ we defined b, c by appropriate formulas, and for a $d \in [b, c]$ the event $F(u, v;\ a, d)$ (a function of X) saying that $\mathrm{Rect}^{\rightarrow}((a, u), (d, v))$ contains no traps or vertical barriers, and is inner H-clean. We elaborate now on the definition of event $E(u, v, w;\ a)$ as follows. For $t > d$ let $\tilde{E}(u, v, w;\ a, d, t)$ be the event that $(d, t]$ is a hole fitting wall $(v, w]$, and event $F(u, v;\ a, d)$ holds. Then event $E(u, v, w;\ a)$ holds if there are d, t such that event $\tilde{E}(u, v, w;\ a, d, t)$ holds. Let $\hat{E}(u, v, w;\ a)$ hold if there are d, t such that event $\tilde{E}(u, v, w;\ a, d, t)$ holds and the point (t, w) is upper right rightward H-clean (that is the hole $(d, t]$ is good as seen from (a, u), in the sense of Definition 2.14).

Lemma 6.3. *We have*

$$\mathbb{P}(\hat{E}) \geq (1 - 2q_{\triangle})\,\mathbb{P}(E) \geq 0.9\,\mathbb{P}(E).$$

This lemma corresponds to Lemma 5.1 of [4].

Lemma 6.4. *Let $v < w$, and let us fix the value y of the sequence of random variables Y in such a way that there is a horizontal wall B of rank r, with body $(v, w]$. For an arbitrary integer b, let $G = G(v, w; b)$ be the event that a good hole through B starts at position b (this event still depends on the sequence $X = (X(1), X(2), \dots)$ of random variables). Then*

$$\mathbb{P}(G) \geq (1 - q_{\triangle} - q_{\square})(1 - 2q_{\triangle})h(r) \geq 0.3h(r).$$

This lemma corresponds to Lemma 5.2 of [4].

Recall the definition of traps of the missing-hole kind in Step 4 of the scale-up algorithm in Section 4.

Lemma 6.5 (Missing-hole traps)**.** *For $a, b \in \mathbb{Z}_{+}$, let $\mathcal{F}$ be the event that a horizontal trap of the missing-hole kind starts at (a, b).*

(a) *Let us fix a string $x = (x(0), x(1), \dots)$. We have*

$$\mathbb{P}\big(\mathcal{F} \mid X = x\big) \leq w^2.$$

(b) *Let us fix a string $y = (y(0), y(1), \dots)$. Let $n = \left\lfloor \frac{\Gamma}{(\sigma^{-1}+2)\Delta} \right\rfloor$. We have*

$$\mathbb{P}\big(\mathcal{F} \mid Y = y\big) \leq e^{-0.3nh(R^*)}.$$

This lemma corresponds to Lemma 5.6 of [4]. There, we had $(1-q)^2$ in place of 0.3 which stands here for $(1 - q_{\triangle} - q_{\square})(1 - 2q_{\triangle})$, and $n = \lfloor \Gamma / 3\Delta \rfloor$. The latter change is needed here since we use $\sigma^{-1}\Delta$ instead of Δ to upperbound the width of holes. The proof is otherwise identical.

Lemma 6.6. *For any value of the constant c_3, if R_1 is sufficiently large then the following holds: if $\mathcal{M} = \mathcal{M}^k$ is a mazery then $\mathcal{M}^*$ satisfies the trap upper bound 2.18.3a.*

This lemma corresponds to Lemma 7.1 of [4].

6.2 Upper bounds on walls

Recall the definition of $p(r)$ in (2.6), used to upperbound the probability of walls. Recall the definition of emerging walls in Step 5 of the scale-up algorithm in Section 4.

Lemma 6.7. *For any point u, let $\mathcal{F}(t)$ be the event that a barrier $(u,v]$ of X of the emerging kind, of length t, starts at u. Denoting $n = \left\lfloor \frac{\Gamma}{(\sigma^{-1}+2)\Delta} \right\rfloor$ we have:*

$$\sum_t \mathbb{P}(\mathcal{F}(t)) \leq 4\Delta^2 w^2 \big(2 \cdot (5\Delta L_2)^4 + w^{-4} e^{-0.3nh(R^*)} \big).$$

This lemma corresponds to Lemma 5.7 of [4]. There we had $(1-q)^2$ in place of 0.3, and $n = \lfloor \Gamma/3\Delta \rfloor$. There was also a factor of m due to Markov conditioning (with a meaning different from the present $\tilde{m}$) that is not needed here. The proof is otherwise identical.

Lemma 6.8. *For every possible value of c_2, c_3, if R_1 is sufficiently large then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. Fixing any point a, the sum of the probabilities over l that a barrier of the emerging kind of size l starts at a is at most $p(\hat{R})/2 = p(\tau' R)/2$.*

This lemma corresponds to Lemma 7.2 of [4].

Let us use the definition of compound walls given in Step 6 of the scale-up algorithm of Section 4.

Lemma 6.9. *Consider ranks r_1, r_2 at any stage of the scale-up construction. Assume that Condition 2.18.3b already holds for rank values r_1, r_2. For a given point x_1 the sum, over all l, of the probabilities for the occurrence of a compound barrier of type $\langle r_1, r_2, i \rangle$ and width l at x_1 is bounded above by*

$$\lambda^i p(r_1) p(r_2).$$

This lemma corresponds to Lemma 5.8 of [4].

Lemma 6.10. *For a given value of c_2, if we choose the constant R_1 sufficiently large then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. After one operation of forming compound barriers, fixing any point a, for any rank r, the sum, over all widths l, of the probability that a compound barrier of rank r and width l starts at a is at most $p(r)R^{-c_1/2}$.*

This lemma corresponds to Lemma 7.3 of [4].

Lemma 6.11. *For every choice of c_2, c_3 if we choose R_1 sufficiently large then the following holds. Suppose that each structure $\mathcal{M}^i$ for $i \le k$ is a mazery. Then Condition 2.18.3b holds for $\mathcal{M}^{k+1}$.*

This lemma corresponds to Lemma 7.4 of [4].

Lemma 6.12. *For small enough c_2, the probability of a barrier of $\mathcal{M}$ starting at a given point b is bounded by T^{-1}.*

This lemma corresponds to Lemma 7.5 of [4] (where the intermediate notation $\overline{p}$ was also used for the upper bound).

6.3 Lower bounds on holes

Before proving the hole lower bound condition for $\mathcal{M}^*$, let us do some preparation.

Definition 6.13. Recall the definition of event E in Condition 2.18.3d, and that it refers to a horizontal wall with body $(v, w]$ seen from a point (a, u). Take the situation described above, possibly without the bound on $v - u$.

Let event $F^*(u, v;\ a, d)$ (a function of the sequence X) be defined just as the event $F(u, v;\ a, d)$ in Condition 2.18.3d, except that the part requiring inner H-cleanness and freeness from traps and barriers of the rectangle $Q(d)$ must now be understood in the sense of both $\mathcal{M}$ and $\mathcal{M}^*$. Let

$$E^* = E^*(u, v, w;\ a)$$

be the event that there is a $d \in [b, c]$ where a vertical hole fitting wall $B = (v, w]$ starts, and event $F^*(u, v;\ a, d)$ holds. ⌟

Note that in what follows we will use the facts several times that

$$w_{k+1} < w_k,\ T_{k+1} > T_k,$$

in other words that the bound w_k on the conditional probability of having a trap at some point in $\mathcal{M}^k$ serves also as a bound on the conditional probability of having one in $\mathcal{M}^{k+1}$, and similarly with the bound T_k^{-1} for walls.

Lemma 6.14. *Suppose that the requirement $v - u \le \sigma^{-2}\Delta$ in the definition of the event E^* is replaced with $v - u \le \sigma^{-2}\Delta^*$, while the rest of the requirements are the same. Then we have*

$$\mathbb{P}(E^* \mid Y = y) \ge 0.25 \wedge (v - u + 1)^{\chi} h(r) - U,$$

where $U = T^{-\overline{\tau}\chi-\varepsilon}$ *for some constant* $\varepsilon > 0$*. If* $v - u > \sigma^{-2}\Delta$ *then we also have the somewhat stronger inequality*

$$\mathbb{P}(E^* \mid Y = y) \geq 0.25 \wedge 2(v - u + 1)^{\chi}h(r) - U.$$

The same statement holds if we replace horizontal with vertical.

Proof. This lemma corresponds to Lemma 5.3 of [4] (incorporating the estimate of the expression called U there), but there are some parameter refinements due to the refined form of Condition 2.18.3d. For ease of reading, we will omit the condition $Y = y$ from the probabilities. We will make the proof such that it works also if we interchange horizontal and vertical, even though $\sigma_x \neq \sigma_y$.
Consider first the simpler case, showing that $v - u \leq \sigma^{-2}\Delta$ implies $\mathbb{P}(E^*) \geq (v - u + 1)^{\chi}h(r)$. Condition 2.18.3d implies this already for $\mathbb{P}(E)$, so it is sufficient to show $E \subseteq E^*$ in this case. As remarked after its definition, the event E^* differs from E only in requiring that rectangle Q contain no traps or vertical barriers of $\mathcal{M}^*$, not only of $\mathcal{M}$, and that points (a, u) and (d, v) are H-clean in Q for $\mathcal{M}^*$ also, not only for $\mathcal{M}$. A trap of $\mathcal{M}^*$ in Q cannot be an uncorrelated or correlated trap, since its components traps, being traps of $\mathcal{M}$, are already excluded. It cannot be a trap of the missing-hole kind either, since that trap, of length Γ on one side, is too big for Q when $v - u \leq \sigma^{-2}\Delta$, and $c - a$ is also of the same order. The same argument applies to vertical barriers of $\mathcal{M}^*$. The components of the compound barriers that belong to $\mathcal{M}$ are excluded, and the emerging barriers are too big, of the size of correlated or missing-hole traps.
These considerations take care also of the issue of H-cleanness for $\mathcal{M}^*$, since the latter also boils down to the absence of traps and barriers.
Take now the case $v - u > \sigma^{-2}\Delta$. Let

$$\begin{aligned}
u' &= v - \lfloor \Delta \rfloor, \quad \Delta' = \lfloor 4\sigma^{-1}\Delta \rfloor,\\
n &= \lceil (c - b)/\Delta' \rceil,\\
a_i &= b + i\Delta', \quad E'_i = E(u', v, w;\ a_i) \quad \text{for } i = 0, \ldots, n - 1,\\
E' &= \bigcup_i E'_i.
\end{aligned}$$

From (2.1) and (2.2) follows $\sigma_x^{-1} - \sigma_y \geq \sigma/\sigma_x$. Recall

$$\begin{aligned}
b &= a + \lceil \sigma_y(v - u) \rceil,\\
c &= b \vee (a + \lfloor \sigma_x^{-1}(v - u) \rfloor).
\end{aligned}$$

Hence

$$\begin{aligned} c - b &\geq (\sigma_x^{-1} - \sigma_y)(v - u) - 2 \geq (v - u)\sigma/\sigma_x - 2, \\ n &\geq ((v - u)\sigma/\sigma_x - 2)/\Delta' \geq (v - u + 1)\sigma/5\Delta \geq \sigma^{-1}/5, \end{aligned} \tag{6.1}$$

where the factor $1/5$ instead of $1/4$ allows omitting the -2 and adding the $+1$, and ignoring the integer part in Δ'. Let C be the event that point (a, u) is upper right rightward H-clean in $\mathcal{M}$. Then by Conditions 2.18.3c

$$\mathbb{P}(\neg C) \leq 2q_\triangle \leq 0.1. \tag{6.2}$$

Let D be the event that the rectangle $(a, c] \times [u, v]$ contains no trap or vertical barrier of $\mathcal{M}$ or $\mathcal{M}^*$. (Then $C \cap D$ implies that (a, u) is also upper right rightward H-clean in the rectangle $(a, c] \times [u, v]$ in $\mathcal{M}^*$.) By Lemmas 6.6, 6.12:

$$\mathbb{P}(\neg D) \leq 2(c - a)T^{-1} + 2(c - a)(v - u + 1)w.$$

Now

$$\begin{aligned} 2(c - a) &\leq 2\sigma^{-2}\sigma_x^{-1}\Delta^* \leq 2\sigma^{-3}\Delta^*, \\ v - u + 1 &\leq \sigma^{-2}\Delta^* + 1 \leq 2\sigma^{-2}\Delta^*, \end{aligned}$$

hence

$$\begin{aligned} \mathbb{P}(\neg D) &\leq 2\sigma^{-3}\Delta^* T^{-1} + 6\sigma^{-5}(\Delta^*)^2 w \\ &= 2\sigma^{-3}T^{\tau\delta - 1} + 6\sigma^{-5}T^{-\omega + 2\tau\delta} \\ &< (2\sigma^{-3} + 6\sigma^{-5})T^{-\overline{\tau}\chi - 2\varepsilon}, \end{aligned}$$

where $\varepsilon > 0$ is a constant and we used (5.12-5.13). Now the statement follows since $T^{-\varepsilon} = \lambda^{-R\varepsilon}$ decreases to 0 faster as a function of $\tilde{m}$ than the expression in parentheses in front.

1. Let us show $C \cap D \cap E' \subseteq E^*(u, v, w;\ a)$.

   Indeed, suppose that $C \cap D \cap E'_i$ holds with some hole starting at d. Then there is a rectangle $Q'_i = \mathrm{Rect}^{\rightarrow}((a_i, u'), (d, v))$ containing no traps or vertical barriers of $\mathcal{M}$, such that (d, v) is H-clean in Q'_i. It follows from D that the rectangle

   $$Q^*_i = \mathrm{Rect}^{\rightarrow}((a, u), (d, v)) \supseteq Q'_i$$

   contains no traps or vertical barriers of $\mathcal{M}$ or $\mathcal{M}^*$. Since event C occurs, the point (a, u) is H-clean for $\mathcal{M}$ in Q^*_i. The event E'_i and the inequalities $d - a_i, v - u' \geq \Delta$ imply that (d, v) is H-clean in Q^*_i, and a hole passing through the potential wall starts at d in X. The event D implies that there is no trap or vertical barrier of $\mathcal{M}$ in Q^*_i. Hence Q^*_i is also inner H-clean in $\mathcal{M}^*$, and so E^* holds.

We have $\mathbb{P}(E^*) \geq \mathbb{P}(C)\,\mathbb{P}(E' \mid C) - \mathbb{P}(\neg D)$.

2. The events E'_i are independent of each other and of the event C.

   Proof. By assumption, $v - u > \sigma^{-2}\Delta$, so $b - a \geq \sigma_y(v - u) \geq \sigma_y\sigma^{-2}\Delta \geq \Delta$, hence the event C depends only on the part of the process X before point b. This shows that the events E'_i are independent of C. The hole starts within $\sigma_x^{-1}(v-u') \leq \sigma_x^{-1}\Delta$ after a_i. The width of the hole through the wall B is at most $\sigma^{-1}\Delta$. After the hole, the property that the wall be upper right rightward H-clean depends on at most Δ more values of X on the right. So the event E_i depends at most on $(2\sigma^{-1}+1)\Delta < \Delta'$ values of the sequence X on the right of a_i.

3. It remains to estimate $\mathbb{P}(E' \mid C) = \mathbb{P}(E')$.

   The following inequality can be checked by direct calculation. Let $\alpha = 1 - 1/e = 0.632\ldots$, then for $x > 0$ we have

$$1 - e^{-x} \geq \alpha \wedge \alpha x. \tag{6.3}$$

   Condition 2.18.3d is applicable to E'_i, so we have

$$\mathbb{P}(E'_i) \geq \Delta^{\chi} h(r) =: s,$$

   hence $\mathbb{P}(\neg E'_i) \leq 1 - s \leq e^{-s}$. Due to the independence of the sequence X, this implies

$$\mathbb{P}(E') = 1 - \mathbb{P}\big(\textstyle\bigcap_i \neg E'_i\big) \geq 1 - e^{-ns} \geq \alpha \wedge \alpha n s, \tag{6.4}$$

   where we used (6.3). Using (6.1) twice (for lowerbounding n and $n\Delta$):

$$\begin{aligned} n\Delta^{\chi} &= n^{1-\chi}(n\Delta)^{\chi} \\ &\geq (\sigma^{-1}/5)^{1-\chi} 5^{-\chi}\sigma^{\chi}(v - u + 1)^{\chi} = 5^{-1}\sigma^{2\chi-1}(v - u + 1)^{\chi}. \end{aligned}$$

   Substituting into (6.4):

$$\begin{aligned} \mathbb{P}(E') &\geq \alpha \wedge \alpha \cdot 5^{-1}\sigma^{2\chi-1}(v - u + 1)^{\chi} h(r), \\ \mathbb{P}(C)\,\mathbb{P}(E') &\geq 0.9 \cdot \big(\alpha \wedge \alpha \cdot 5^{-1}\sigma^{2\chi-1}(v - u + 1)^{\chi} h(r)\big) \\ &\geq 0.5 \wedge 2(v - u + 1)^{\chi} h(r) \end{aligned}$$

   where we used (6.2).

□

The lower bound on the probability of holes through an emerging wall is slightly more complex than the corresponding lemma in [4]. Recall F^* from Definition 6.13.

Lemma 6.15. *Using the notation of Condition 2.18.3d for $\mathcal{M}^*$, a,u,v,b, assume that $Y = y$ is fixed and $v > u$. Then $\mathbb{P}(F^*(u,v;\ a,b)) \geq 0.25$.*

This lemma corresponds to Lemma 7.8 of [4].

Proof. Consider the case of a horizontal wall, the argument also works for the case of a vertical wall. The probability that it is not inner H-clean is at most $q_{\square} + 3q_{\triangle}$ (adding up the probability bounds for the inner horizontal non-cleanness and the inner trap non-cleanness of the two endpoints). The probability of finding a vertical barrier or trap (of $\mathcal{M}$ or $\mathcal{M}^*$) is bounded by U as in Lemma 6.14, so the total bound is at most

$$q_{\square} + 3q_{\triangle} + U.$$

Here, U can be made less than 0.05 if R_1 is sufficiently large, so the total is at most 0.75. □

Lemma 6.16. *For emerging walls, the fitting holes satisfy Condition 2.18.3d if R_1 is sufficiently large.*

This lemma corresponds to Lemma 7.9 of [4], with Figure 22 there illustrating the proof.

Consider now a hole through a compound wall. In the lemma below, we use w_1, w_2: please note that these are integer coordinates, and have nothing to do with the trap probability upper bound w: we will never have these two uses of w in a place where they can be confused.

Lemma 6.17. *Let $u \le v_1 < w_2$, and a be given with $v_1 - u \le \sigma^{-2}\Delta^*$. Assume that $Y = y$ is fixed in such a way that W is a compound horizontal wall with body $(v_1, w_2]$, and type $\langle r_1, r_2, i\rangle$, with rank r as given in (4.5). Assume also that the component walls W_1, W_2 already satisfy the hole lower bound, Condition 2.18.3d. Let*

$$E_2 = E_2(u, v_1, w_2;\ a) = E^*(u, v_1, w_2;\ a)$$

where E^* was introduced in Definition 6.13. Assume

$$(\sigma^{-2}\Delta^* + 1)^{\chi} h(r_j) \le 0.25, \text{ for } j = 1, 2. \tag{6.5}$$

Then

$$\mathbb{P}\left(E_2 \mid Y = y\right) \ge (v_1 - u + 1)^{\chi} \lambda^{i\chi} h(r_1) h(r_2)(1 - V)$$

with $V = 2U/h(r_1 \vee r_2)$, where U comes from Lemma 6.14.

The statement also holds if we exchange horizontal and vertical.

The lemma corresponds to Lemma 5.9 of [4], with Figure 21 there illustrating the proof. Some parts of the proof are simpler, due to using $v_1 - u + 1$ in place of $c - b$.

Proof. Let D be the distance between the component walls W_1, W_2 of the wall W, where the body of W_j is $(v_j, w_j]$. Consider first passing through W_1. For each integer $x \in [b, c + \sigma^{-1}\Delta]$, let A_x be the event that $E^*(u, v_1, w_1;\, a)$ holds with the vertical projection of the hole ending at x, and that x is the smallest possible number with this property. Let $B_x = E^*(w_1, v_2, w_2;\, x)$.

1. We have $E_2 \supseteq \bigcup_x (A_x \cap B_x)$.

   Proof.

   If for some x we have A_x, then there is a hole $\mathrm{Rect}((t_1, v_1), (x, w_1))$ through the first wall with the property that rectangle $\mathrm{Rect}((a, u), (t_1, v_1))$ contains no traps or barriers of $\mathcal{M}$ and is inner clean in $\mathcal{M}$. Given that by assumption this rectangle contains no traps or barriers of $\mathcal{M}^*$, event $E^*(u, v_1, w_1;\, a)$ holds. If also B_x holds, then there is a rectangle $\mathrm{Rect}((x, w_1), (t_2, v_2))$ satisfying the requirements of $E^*(w_1, v_2, w_2;\, x)$, and also a hole $\mathrm{Rect}((t_2, v_2), (x', w_2))$ through the second wall.

   Let us show that $(t_1, v_1) \rightsquigarrow (x', w_2)$, and thus the interval $(t_1, x']$ is a hole that passes through the compound wall W.

   The reachabilies $(t_1, v_1) \rightsquigarrow (x, w_1)$ and $(t_2, v_2) \rightsquigarrow (x', w_2)$ follow by the definition of holes; the reachability $(x, w_1) \rightsquigarrow (t_2, v_2)$ remains to be proven.

   Since the event B_x holds, by Lemma 2.19 (x, w_1), (t_2, v_2) satisfy the slope conditions. Let us show that then actually $\mathrm{Rect}((x, w_1), (t_2, v_2))$ is a hop of $\mathcal{M}$: then its endpoint is reachable from its starting point according to the reachability condition of $\mathcal{M}$.

   To see that the rectangle is a hop: the inner H-cleanness of $(x, t_2]$ in the process X follows from B_x; the latter also implies that there are no vertical walls in $(x, t_2]$. The inner cleanness of $(w_1, v_2]$ in the process Y is implied by the fact that $(v_1, w_2]$ is a compound wall. The fact that W is a compound wall also implies that the interval $(w_1, v_2]$ contains no horizontal walls. These facts imply the inner cleanness of the rectangle $\mathrm{Rect}((w_1, x), (v_1, t_2))$.

It remains to lower-bound $\mathbb{P}\big(\bigcup_x (A_x \cap B_x)\big)$. For each x, the events A_x, B_x belong to disjoint intervals, and the events A_x are disjoint of each other.

2. Let us lower-bound $\sum_x \mathbb{P}(A_x)$.

   We have, using the notation of Lemma 6.14: $\sum_x \mathbb{P}(A_x) = \mathbb{P}(E^*(u, v_1, w_1;\, a))$. Lemma 6.14 is applicable and we get $\mathbb{P}(E^*(u, v_1, w_1;\, a)) \geq F_1 - U$ with $F_1 = 0.25 \wedge (v_1 - u + 1)^{\chi} h(r_1)$, and U coming from Lemma 6.14. Now $(v_1 - u + 1)^{\chi} h(r_1) \leq (\sigma^{-2}\Delta^* + 1)^{\chi} h(r_1)$ which by assumption (6.5) is ≤ 0.25. So the operation $0.25\wedge$ can be deleted from F_1:

   $$F_1 = (v_1 - u + 1)^{\chi} h(r_1).$$

3. Let us now lower-bound $\mathbb{P}(B_x)$.

   We have $B_x = E^*(w_1, v_2, w_2;\, x)$. The conditions of Lemma 6.14 are satisfied for $u = w_1$, $v = v_2$, $w = w_2$, $a = x$. It follows that $\mathbb{P}(B_x) \geq F_2 - U$ with $F_2 = 0.25 \wedge (D+1)^{\chi} h(r_2)$, which can again be simplified using assumption (6.5) and $D \leq \Phi$:
   $$F_2 = (D+1)^{\chi} h(r_2).$$

4. Let us combine these estimates, using $G = F_1 \wedge F_2 > h(r_1 \vee r_2)$.

   We have
   $$\begin{aligned}\mathbb{P}(E_2) &\geq \sum_x \mathbb{P}(A_x)\,\mathbb{P}(B_x) \geq (F_1 - U)(F_2 - U)\\ &\geq F_1 F_2 (1 - U(1/F_1 + 1/F_2)) \geq F_1 F_2 (1 - 2U/G)\\ &= (v_1 - u + 1)^{\chi} (D+1)^{\chi} h(r_1) h(r_2)(1 - 2U/G)\\ &\geq (v_1 - u + 1)^{\chi} (D+1)^{\chi} h(r_1) h(r_2)(1 - 2U/h(r_1 \vee r_2)).\end{aligned}$$

5. We conclude by showing $(D+1) \geq \lambda^i$.

   If $D = 0$ or 1 then $i = D$, so this is true. If $D > 1$ then $i \leq \log_\lambda D$, so even $D \geq \lambda^i$.

□

The lemma below is essentially the substitution of the scale-up parameters into the above one.

Lemma 6.18. *After choosing c_3, R_1 sufficiently large in this order, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery: then every compound wall satisfies the hole lower bound, Condition 2.18.3d, provided its components satisfy it.*

This lemma corresponds to Lemma 7.10 of [4].

For the hole lower bound condition for $\mathcal{M}^*$, there is one more case to consider.

Lemma 6.19. *After choosing c_3, R_1 sufficiently large in this order, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery: then every wall of $\mathcal{M}^{k+1}$ that is also a heavy wall of $\mathcal{M}^k$ satisfies the hole lower bound, Condition 2.18.3d.*

This lemma corresponds to Lemma 7.11 of [4].

6.4 Auxiliary bounds

The next lemma shows that the choice made in Definition 5.8 satisfies the requirements.

Lemma 6.20. *If R_1 is sufficiently large then inequality (3.1) holds, moreover*

$$\sum_k \big(2\Delta_{k+1} T_k^{-1} + \Delta_{k+1}^2 w_k \big) < 1/4.$$

Proof. The event $\mathcal{F}_k(\Delta_{k+1})$ says that some wall or trap of level k appears in $[0, \Delta_{k+1}]^2$. The event $\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k$ implies that a trap of level k appears $[0, \Delta_{k+1}]^2$. The probability that a wall of level k appears in $[0, \Delta_{k+1}]^2$ is clearly bounded by $2\Delta_{k+1} T_k^{-1}$. The probability that a trap of level k appears there is bounded by $\Delta_{k+1}^2 w_k$. Hence $\mathbb{P}\big(\mathcal{F}_k(\Delta_{k+1}) \cup \mathcal{Q}_{k+1} \setminus \mathcal{Q}_k \big)$ is bounded by $2\Delta_{k+1} T_k^{-1} + \Delta_{k+1}^2 w_k$.

The rest of the statement and its proof correspond to Lemma 7.6 of [4]. □

Note that for R_1 large enough, the relations

$$\Delta^* T^{-1} < 0.5(0.05 - q_{\triangle}), \quad \Delta^* T^{-1} < 0.5(0.55 - q_{\square}), \tag{6.6}$$

$$\Lambda\sigma^{-3}\Delta/\Gamma < 0.5(1.1/2R_1 - \sigma_x), \quad \Lambda\sigma^{-3}\Delta/\Gamma < 0.5(1.1R_1 - \sigma_y) \tag{6.7}$$

hold for $\mathcal{M} = \mathcal{M}^1$ as defined in Example 2.21. This is clear for (6.6). For (6.7), we only need the two inequalities $1/40R_1 > \Lambda\sigma^{-3}\Delta/\Gamma = 8\Lambda R_1^3 T^{-(\gamma-\delta)}$, $R_1/20 > 8\Lambda R_1^3 T^{-(\gamma-\delta)}$, both of which are satisfied if R_1 is large enough.

Lemma 6.21. *Suppose that the structure $\mathcal{M} = \mathcal{M}^k$ is a mazery and it satisfies (6.6) and (6.7). Then $\mathcal{M}^* = \mathcal{M}^{k+1}$ also satisfies these inequalities if R_1 is chosen sufficiently large (independently of k), and also satisfies Condition 2.18.3c.*

This lemma corresponds to Lemma 7.7 of [4], and its proof is essentially also: the changed initial values and bounds of σ_x, σ_y and $q_{\triangle}, q_{\square}$ do not change the arguments due to the negative exponential dependence of their increments on R_1. Recall the definition of σ_i^* in Definition 4.2, and the definition of q_i^* in Definition 5.8.

Proof. Let us show first that $\mathcal{M}^*$ also satisfies the inequalities if R_1 is chosen sufficiently large.

For sufficiently large R_1, we have $\Delta^{**}(T^*)^{-1} < 0.5\Delta^* T^{-1}$. Indeed, this says $T^{(\tau\delta-1)(\tau-1)} < 0.5$. Hence using (6.6) and the definition of $q_{\triangle}^*$ in Definition 5.8:

$$\begin{aligned}
\Delta^{**}(T^*)^{-1} &\le 0.5\Delta^* T^{-1} \le 0.5(0.05 - q_{\triangle}) - 0.5\Delta^* T^{-1} \\
&= 0.5(0.05 - q_{\triangle}^*).
\end{aligned}$$

This is the first inequality of (6.6) for $\mathcal{M}^*$. The second one is proved the same way. To verify Condition 2.18.3c for $\mathcal{M}^*$, recall Definition 5.8 of q_i^*. For inequality (2.8), for an upper bound on the conditional probability that a point a of the line is strongly clean in $\mathcal{M}$ but not in $\mathcal{M}^*$ let us use

$$(2\Phi/3 + \Delta)T^{-1},$$

which upper-bounds the probability that a vertical barrier of $\mathcal{M}$ starts in $(a - \Phi/3 - \Delta, a + \Phi/3]$. This can be upper-bounded by $\Phi T^{-1} < \Delta^* T^{-1}$ by (4.1) for sufficiently large R_1. Hence an upper bound on the conditional probability of not strong cleanness in $\mathcal{M}^*$ is $q_\triangle + \Delta^* T^{-1} = q^*_\triangle$ as required, due to Definition 5.8.

For the other inequalities in Condition 2.18.3c, consider a rectangle $Q = \mathrm{Rect}^{\rightarrow}(u, v)$ and fix $Y = y$. The conditional probability that a point u is trap-clean in Q for $\mathcal{M}$ but not for $\mathcal{M}^*$ is upper-bounded by the probability of the appearance of a trap of $\mathcal{M}$ within a distance Γ of point u in Q. There are at most Γ^2 positions for the trap, so a bound is

$$\Gamma^2 w = T^{2\gamma - \omega} < T^{\tau\delta - 1},$$

where the last inequality follows from (5.7). We conclude the same way for the first inequality. The argument for the other inequalities in Condition 2.18.3c is identical.

For the first inequality of (6.7), the scale-up definition Definition 4.2 says $\sigma^*_x - \sigma_x = \Lambda\sigma^{-3}\Delta/\Gamma$. The inequality $\Delta^*/\Gamma^* < 0.5\Delta/\Gamma$ is guaranteed if R_1 is large. From here, we can conclude the proof as for q_i; similarly for σ_y. □

7 The approximation lemma

The crucial combinatorial step in proving the main lemma is the following.

Lemma 7.1 (Approximation)**.** *The reachability condition, Condition 2.18.2e, holds for $\mathcal{M}^*$ if R_1 is sufficiently large.*

The present section is taken up by the proof of this lemma.

Recall that we are considering a bottom-open or left-open or closed rectangle Q with starting point $u = (u_0, u_1)$ and endpoint $v = (v_0, v_1)$ with $u_d < v_d$, $d = 0, 1$ with the property that there is a (non-integer) point $v' = (v'_0, v'_1)$ with $0 \le v_0 - v'_0, v_1 - v'_1 < 1$ such that

$$\sigma^*_x \le \mathrm{slope}(u, v') \le (\sigma^*_y)^{-1}. \tag{7.1}$$

We require Q to be a hop of $\mathcal{M}^*$. Thus, the points u, v are clean for $\mathcal{M}^*$ in Q, and Q contains no traps or walls of $\mathcal{M}^*$. We have to show $u \rightsquigarrow v$. Assume

$$Q = I_0 \times I_1 = \mathrm{Rect}^{\varepsilon}(u, v)$$

where $\varepsilon = \rightarrow, \uparrow$ or nothing.

7.1 Walls and trap covers

Let us determine the properties of the set of walls in Q.

Lemma 7.2. *Under conditions of Lemma 7.1, with the notation given in the discussion after the lemma, the following holds.*

(a) *For $d = 0, 1$, for some $n_d \geq 0$, there is a sequence $W_{d,1}, \dots, W_{d,n_d}$ of dominant light neighbor walls of $\mathcal{M}$ separated from each other by external hops of $\mathcal{M}$ of size $> \Phi$, and from the ends of I_d (if $n_d > 0$) by hops of $\mathcal{M}$ of size $\geq \Phi/3$.*

(b) *For every (horizontal) wall $W_{0,i}$ of $\mathcal{M}$ occurring in I_1, for every subinterval J of I_0 of size Γ such that J is at a distance $\geq \Gamma + 7\Delta$ from the ends of I_0, there is an outer rightward clean hole fitting $W_{0,i}$, with endpoints at a distance of at least Δ from the endpoints of J. The same holds if we interchange vertical and horizontal.*

Proof. This is a direct consequence of Lemmas 4.7 and 4.9. The vertical cleanness needed in the outer rightward cleanness of the hole through $W_{0,i}$ follows from part (a). □

From now on, in this proof, whenever we mention a ***wall*** we mean one of the walls $W_{d,i}$, and whenever we mention a trap then, unless said otherwise, we mean only traps of $\mathcal{M}$ entirely within Q and not intersecting any of these walls. Let us limit the places where traps can appear in Q.

Definition 7.3 (Trap cover)**.** A set of the form $I_0 \times J$ with $|J| \leq 4\Delta$ containing the starting point of a trap of $\mathcal{M}$ will be called a ***horizontal trap cover***. Vertical trap covers are defined similarly. ⌟

In the following lemma, when we talk about the distance between two traps, we mean the distance between their starting points.

Lemma 7.4 (Trap cover)**.** *Let T_1 be a trap of $\mathcal{M}$ contained in Q. Then there is a horizontal or vertical trap cover $U \supseteq T_1$ such that the starting point of every other trap in Q is either contained in U or is at least at a distance $\Phi - \Delta$ from T_1. If the trap cover is vertical, it intersects none of the vertical walls $W_{0,i}$; if it is horizontal, it intersects none of the horizontal walls $W_{1,j}$.*

This lemma corresponds to Lemma 8.3 of [4].

Let us measure distances from the line defined by the points u, v'.

Definition 7.5 (Relations to the diagonal)**.** Define, for a point $a = (a_0, a_1)$:

$$d_{u,v'}(a) = d(a) = (a_1 - u_1) - \text{slope}(u, v')(a_0 - u_0)$$

to be the distance of a above the line of u, v', then for $w = (x, y)$, $w' = (x', y')$:

$$\begin{aligned} d(w') - d(w) &= y' - y - \operatorname{slope}(u, v')(x' - x), \\ |d(w') - d(w)| &\le |y' - y| + |x' - x|/\sigma_y. \end{aligned}$$

We define the strip

$$C^{\varepsilon}(u, v', h_1, h_2) = \{\, w \in \operatorname{Rect}^{\varepsilon}(u, v) : h_1 < d_{u,v'}(w) \le h_2 \,\},$$

a channel of vertical width $h_2 - h_1$ in $\operatorname{Rect}^{\varepsilon}(u, v)$, parallel to line of u, v'. ⌟

Lemma 7.6. *Assume that points u, v are clean for $\mathcal{M}$ in $Q = \operatorname{Rect}^{\varepsilon}(u, v)$, with*

$$\sigma_x + 4\Delta/\Gamma \le \operatorname{slope}(u, v') \le 1/(\sigma_y + 4\sigma^{-2}\Delta/\Gamma),$$

where v' relates to v as above. If $C = C^{\varepsilon}(u, v', -\Gamma, \Gamma)$ contains no traps or walls of $\mathcal{M}$ then $u \rightsquigarrow v$. (By C not containing walls we mean that its projections don't.)

This lemma corresponds to Lemma 8.4 of [4], and Figure 23 there illustrates the proof.

Proof. Let $\mu = \operatorname{slope}(u, v')$. If $|I_0| < \Gamma$ then $C = Q$, so there is no trap or wall in Q, therefore Q is a hop, and we are done via Condition 2.18.2e for $\mathcal{M}$. Suppose $|I_0| \ge \Gamma$. Let

$$n = \left\lceil \frac{|I_0|}{0.9\Gamma} \right\rceil, \quad h = \frac{|I_0|}{n}.$$

Then $\Gamma/2 \le h \le 0.9\Gamma$. Indeed, the proof of the second inequality is immediate. For the first one, if $n \le 2$, we have $\Gamma \le |I_0| = nh \le 2h$, and for $n \ge 3$:

$$\begin{aligned} &\frac{|I_0|}{0.9\Gamma} \ge n - 1, \\ &|I_0|/n \ge (1 - 1/n)0.9\Gamma \ge 0.6\Gamma. \end{aligned}$$

For $i = 1, 2, \dots, n - 1$, let

$$a_i = u_0 + ih, \quad b_i = u_1 + ih \cdot \mu, \quad w_i = (a_i, b_i), \quad S_i = w_i + [-\Delta, 2\Delta]^2.$$

Let us show $S_i \subseteq C$. For all elements w of S_i, we have $|d(w)| \le 2(1 + 1/\sigma_y)\Delta$, and we know $2(1 + 1/\sigma_y)\Delta < \Gamma$ if R_1 is sufficiently large. To see $S_i \subseteq \operatorname{Rect}^{\varepsilon}(u, v)$, we need (from the worst case $i = n - 1$) $\mu h > 2\Delta$. Using the above and the assumptions of the lemma:

$$\frac{2\Delta}{h} \le \frac{2\Delta}{\Gamma/2} = 4\Delta/\Gamma \le \mu.$$

By Remark 2.20.1, there is a clean point $w'_i = (a'_i, b'_i)$ in the middle third $w_i + [0, \Delta]^2$ of S_i. Let $w'_0 = u$, $w'_n = v'$. By their definition, each rectangle $\mathrm{Rect}^{\varepsilon}(w'_i, w'_{i+1})$ rises by at most $< \mu(0.9\Gamma + \Delta) + \Delta < \Gamma$, above or below the diagonal, hence falls into the channel C and is consequently trap-free.

If $\sigma_x \le \mathrm{slope}(w'_i, w'_{i+1}) \le 1/\sigma_y$ this will imply $w'_i \rightsquigarrow w'_{i+1}$ for $i < n-1$, and $w'_{n-1} \rightsquigarrow v$. Let $\mu' = \mathrm{slope}(w'_i, w'_{i+1})$. We know already $\mu \ge \sigma_x + 4\Delta/\Gamma$ and $1/\mu \ge \sigma_y + 4\sigma^{-2}\Delta/\Gamma$. It is sufficient to show $\mu - \mu' \le 4\Delta/\Gamma$ and $1/\mu - 1/\mu' \le 4\sigma^{-2}\Delta/\Gamma$.

The distance from w'_i to w'_{i+1} is between $h - \Delta$ and $h + \Delta$ in the x coordinate and between $\mu h - \Delta$ and $\mu h + \Delta$ in the y coordinate. We have

$$\mu - \mu' \le \mu - \frac{\mu h - \Delta}{h + \Delta} = \frac{(\mu+1)\Delta}{h+\Delta} \le \frac{(\mu+1)\Delta}{\Gamma/2 + \Delta} \le 4\Delta/\Gamma.$$

Similarly

$$\frac{1}{\mu} - \frac{1}{\mu'} \le \frac{1}{\mu} - \frac{h - \Delta}{\mu h + \Delta} = \frac{(\mu+1)\Delta}{\mu(\mu h + \Delta)} \le \frac{(\mu+1)\Delta}{\mu^2\Gamma/2}.$$

The condition of the lemma implies $\sigma \le \mu \le 1$, and this implies that the last expression is less than $4\Delta/\mu^2\Gamma \le 4\sigma^{-2}\Delta/\Gamma$. □

We introduce particular strips around the diagonal.

Definition 7.7. Let $\Psi = (\Gamma\Phi)^{1/2}$, $C = C^{\varepsilon}(u, v', -3\Psi, 3\Psi)$, where v' is defined as above. ⌟

Let us introduce the system of walls and trap covers we will have to overcome.

Definition 7.8. Let us define a sequence of trap covers $U_1, U_2, \ldots$ as follows. If some trap T_1 is in C, then let U_1 be a (horizontal or vertical) trap cover covering it according to Lemma 7.4. If U_i has been defined already and there is a trap T_{i+1} in C not covered by $\bigcup_{j \le i} U_j$ then let U_{i+1} be a trap cover covering this new trap. To each trap cover U_i we assign a real number a_i as follows. Let (a_i, a'_i) be the intersection of the diagonal of Q and the left or bottom edge of U_i (if U_i is vertical or horizontal respectively). Let (b_i, b'_i) be the intersection of the diagonal and the left edge of the vertical wall $W_{0,i}$ introduced in Lemma 7.2, and let (c'_i, c_i) be the intersection of the diagonal and the bottom edge of the horizontal wall $W_{1,i}$. Let us define the finite set

$$\{s_1, s_2, \ldots\} = \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\} \cup \{c'_1, c'_2, \ldots\}$$

where $s_i \le s_{i+1}$.

We will call the objects (trap covers or walls) belonging to the points s_i our ***obstacles***. ⌟

Lemma 7.9. *If s_i, s_j belong to the same obstacle category among the three (horizontal wall, vertical wall, trap cover) then $|s_i - s_j| \geq 0.75\varPhi$ for R_1 sufficiently large.*

This lemma corresponds to Lemma 8.5 of [4].

It follows that for every i at least one of the three numbers $(s_{i+1} - s_i)$, $(s_{i+2} - s_{i+1})$, $(s_{i+3} - s_{i+2})$ is larger than $0.25\varPhi$.

7.2 Passing through the obstacles

The remark after Lemma 7.9 allows us to break up the sequence of obstacles into groups of size at most three, which can be dealt with separately. So the main burden of the proof of the Approximation Lemma is carried by following lemma.

Lemma 7.10. *There is a constant $\varLambda$ with the following properties. Let u, v be points with*

$$\begin{aligned} \sigma_x + (\varLambda - 1)\sigma^{-3}\varDelta/\varGamma &\leq \mathrm{slope}(u, v'), \\ \sigma_y + (\varLambda - 1)\sigma^{-3}\varDelta/\varGamma &\leq 1/\mathrm{slope}(u, v'), \end{aligned} \tag{7.2}$$

where v' is related to v as above. Assume that the set $\{s_1, s_2, \ldots\}$ defined above consists of at most three elements, with the consecutive elements less than $0.25\varPhi$ apart. Assume also

$$v_0 - s_i,\ s_i - u_0 \geq 0.1\varPhi. \tag{7.3}$$

Then if $\mathrm{Rect}^{\rightarrow}(u, v)$ or $\mathrm{Rect}^{\uparrow}(u, v)$ is a hop of $\mathcal{M}^*$ then $u \rightsquigarrow v$.

Proof. Let $\mu = \mathrm{slope}(u, v')$, and note that the conditions imply $\mu \leq 1$. We can assume without loss of generality that there are indeed three points s_1, s_2, s_3. By Lemma 7.9, they must then come from three obstacles of different categories: $\{s_1, s_2, s_3\} = \{a, b, c'\}$ where b comes from a vertical wall, c' from a horizontal wall, and a from a trap cover. There is a number of cases.

If the index $i \in \{1, 2, 3\}$ of a trap cover is adjacent to the index of a wall of the same orientation, then this pair will be called a ***parallel pair***. A parallel pair is either horizontal or vertical. It will be called a ***trap-wall pair*** if the trap cover comes first, and the ***wall-trap pair*** if the wall comes first.

We will call an obstacle i ***free***, if it is not part of a parallel pair. Consider the three disjoint channels

$$C(u, v', K - \varPsi, K + \varPsi), \text{ for } K = -2\varPsi,\ 0,\ 2\varPsi.$$

The three lines (bottom or left edges) of the trap covers or walls corresponding to s_1, s_2, s_3 can intersect in at most two places, so at least one of the above channels does not contain such an intersection. Let K belong to such a channel. Its middle is the line $C(u,v',K,K)$. For $i\in\{1,2,3\}$, let

$$w_i=(x_i,y_i)$$

be the intersection point of the starting edge of obstacle i with this line. These points will guide us to define the rather close points

$$w_i'=(x_i',y_i'),\quad w_i''=(x_i'',y_i'')$$

in the channel $C(u,v',K-\Psi,K+\Psi)$ through which an actual path will go. Not all these points will be defined, but they will always be defined if i is free. Their role in this case is the following: w_i' and w_i'' are points on the two sides of the trap cover or wall with $w_i'\rightsquigarrow w_i''$. We will have

$$|x-x_i|+|y-y_i|=O(\sigma^{-1}\Gamma) \tag{7.4}$$

for $x=x_i',x_i''$ and $y=y_i',y_i''$.
We will make use of the following relation for arbitrary $a=(a_0,a_1)$, $b=(b_0,b_1)$:

$$\operatorname{slope}(a,b)=\mu+\frac{d(b)-d(a)}{b_0-a_0}. \tag{7.5}$$

For the analysis that follows, note that all points within distance $\Psi/2$ of any points w_i are contained in the channel C, and hence also in the rectangle Q.
The following general remark will also be used several times below. Suppose that for one of the (say, vertical) trap covers with starting point x_i, we determine that the rectangle $[x_i,x_i+5\Delta]\times I$ intersecting the channel C, where $|I|<\Psi$, contains no trap. Then the much largest rectangle $[x_i-\Phi,x_i+\Phi]\times I$ contains no trap either. Indeed, there is a trap somewhere in the intersection of the channel with the trap cover C (this is why the trap cover is needed), and then the trap cover property implies that there is no other trap outside the trap cover within distance $\Phi\gg\Psi$ of this trap.

1. Consider crossing a free vertical trap cover.

   Recall the definition of L_2 in (4.3). We apply Lemma 4.10 to vertical correlated traps $J\times I'$, with $J=[x_i,x_i+5\Delta]$, $I'=[y_i,y_i+L_2]$. The lemma is applicable since $w_i\in C(u,v',K-\Psi,K+\Psi)$ implies $u_1<y_i-L_2-7\Delta<y_i+2L_2+7\Delta<v_1$. Indeed, formula (7.3) implies, using (7.2):

   $$y_i>u_1+0.1\mu\Phi\ge u_1+7\Delta+L_2$$

   for sufficiently large R_1, using $L_2\ll\Phi$. The inequality about v_1 is similar, using the other inequality of (7.3).

Lemma 4.10 implies that there is a region $[x_i, x_i + 5\Delta] \times [y, y + 2.2\sigma^{-1}\Gamma]$ containing no traps, with $[y, y + 2.2\sigma^{-1}\Gamma) \subseteq [y_i, y_i + L_2)$. Thus, there is a y in $[y_i, y_i + L_2 - 2.2\sigma^{-1}\Gamma)$ such that $[x_i, x_i + 5\Delta] \times [y, y + 2.2\sigma^{-1}\Gamma]$ contains no traps. (In the present proof, all other arguments finding a region with no traps in trap covers are analogous, so we will not mention Lemma 4.10 explicitly again.) Since all nearby traps must start in a trap cover, the region $[x_i - 2\Delta, x_i + \Gamma] \times [y, y + 2.2\sigma^{-1}\Gamma]$ contains no trap either. Thus there are clean points w_i' in $(x_i - \Delta, y + \Delta) + [0, \Delta]^2$ and w_i'' in $(x_i + \Gamma - 2\Delta, y + \sigma_x\Gamma + \Delta) + [0, \Delta]^2$. Let us estimate slope$(w_i', w_i'')$. We have

$$\begin{aligned} \Gamma - 2\Delta \le x_i'' - x_i' &\le \Gamma, \\ \sigma_x\Gamma \le y_i'' - y_i' &\le \sigma_x\Gamma + 2\Delta, \\ \sigma_x \le \mathrm{slope}(w_i', w_i'') &\le \frac{\sigma_x\Gamma + 2\Delta}{\Gamma - 2\Delta} \le \sigma_x + \frac{4\Delta}{\Gamma - 4\Delta} \\ &\le \sigma_x^* \le 1/\sigma_y \end{aligned} \tag{7.6}$$

if R_1 is large, where we used Definition 4.2 and (2.2). So the pair w_i', w_i'' satisfies the slope conditions. The rectangle between them is also trap-free, due to $\sigma_x\Gamma + 2\Delta \le 2\Gamma$, hence $w_i' \rightsquigarrow w_i''$.

The point w_i' is before the trap cover defined by w_i, while w_i'' is after. Their definition certainly implies the relations (7.4).

2. Consider crossing a free horizontal trap cover.

   There is an x in $[x_i - L_2, x_i - 7\Delta)$ such that $[x, x + 2.2\sigma^{-1}\Gamma] \times [y_i, y_i + 5\Delta]$ contains no trap. Thus there are clean points w_i' in $(x + \Delta, y_i - \Delta) + [0, \Delta]^2$ and w_i'' in $(x + \sigma_x^{-1}\Gamma, y_i + \Gamma) + [0, \Delta]^2$. Now estimates similar to (7.6) hold again, so $w_i' \rightsquigarrow w_i''$. The point w_i' is before the trap cover defined by w_i, while w_i'' is after. Their definition implies the relations (7.4).

3. Consider crossing a free vertical wall.

   Let us apply Lemma 7.2(b), with $I' = [y_i, y_i + \Gamma]$. The lemma is applicable since by $w_i \in C(u, v', K - \Psi, K + \Psi)$ we have $u_1 \le y_i - \Gamma - 7\Delta < y_i + 2\Gamma + 7\Delta < v_1$. It implies that our wall contains an outer upward clean hole $(y_i', y_i''] \subseteq y_i + (\Delta, \Gamma - \Delta]$ passing through it. (In the present proof, all other arguments finding a hole through walls are analogous, so we will not mention Lemma 7.2(b) explicitly again.) Let $w_i' = (x_i, y_i')$, and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. This definition implies the relations (7.4).

4. Consider crossing a free horizontal wall.

   Similarly to above, this wall contains an outer rightwards clean hole $(x_i', x_i''] \subseteq x_i + (-\Gamma + \Delta, -\Delta]$ passing through it. Let $w_i' = (x_i', y_i)$ and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w'_i. This definition implies the relations (7.4).

For a trap-wall or wall-trap pair, we first find a big enough hole in the trap cover, and then locate a hole in the wall that allows to pass through the big hole of the trap cover. There are cases according to whether we have a trap-wall pair or a wall-trap pair, and whether it is vertical or horizontal, but the results are all similar. Figure 24 of [4] illustrates the similar construction in that paper.

5. Consider crossing a vertical trap-wall pair $(i, i+1)$.

   Recall $x_i = s_i$, $x_{i+1} = s_{i+1}$. Let us define $x = x_i - \varGamma$. Find a $y^{(1)}$ in $[y_i, y_i + L_2 - 2.2\sigma^{-1}\varGamma)$ such that the region $[x_i, x_{i+1}] \times [y^{(1)}, y^{(1)} + 2.2\sigma^{-1}\varGamma] \cap C$ contains no trap.

   Let $\tilde{w} = (x_{i+1}, \tilde{y})$ be defined by $\tilde{y} = y^{(1)} + \mu(x_{i+1} - x_i) + 1.1\sigma^{-1}\varGamma$. Thus, it is the point on the left edge of the wall if we intersect it with a slope μ line from $(x_i, y^{(1)})$ and then move up $1.1\sigma^{-1}\varGamma$. Similarly to the forward crossing in Part 3, the vertical wall starting at x_{i+1} is passed through by an outer upward clean hole $(y'_{i+1}, y''_{i+1}] \subseteq \tilde{y} + (\varDelta, \varGamma - \varDelta]$. Let $w'_{i+1} = (x_{i+1}, y'_{i+1})$, and let $w''_{i+1} = (x''_{i+1}, y''_{i+1})$ be the point on the other side of the wall reachable from w'_{i+1}. Define the line E of slope μ going through the point w'_{i+1}. Let $w = (x, y^{(2)})$ be the intersection of E with the vertical line defined by x, then $y^{(2)} = y'_{i+1} - \mu(x_{i+1} - x)$. The channel of (vertical) width $2.2\varGamma$ around the line E intersects the trap cover in a trap-free interval (that is smallest rectangle containing this intersection is trap-free).

   There is a clean point $w'_i \in (x - \varDelta, y^{(2)}) + [0, \varDelta]^2$. (Point w''_i is not needed.) We have

   $$-\varDelta \le d(w'_i) - d(w'_{i+1}) \le \varDelta. \tag{7.7}$$

   The relation (7.4) is easy to prove. Let us show $w'_i \rightsquigarrow w'_{i+1}$. Given the trap-freeness of the channel mentioned above, it is easy to see that the channel $C^{\varepsilon}(w'_i, w'_{i+1}, -\varGamma, \varGamma)$ is also trap-free. We can apply Lemma 7.6 after checking its slope condition. We get using (7.5), (7.7) and $x'_{i+1} - x \ge \varGamma$:

   $$\mu - \varDelta/\varGamma \le \operatorname{slope}(w'_i, w'_{i+1}) \le \mu + \varDelta/\varGamma.$$

6. Consider crossing a horizontal trap-wall pair $(i, i+1)$.

   Let us define $y = y_i - \varGamma$. There is an $x^{(1)}$ in $[x_i, x_i + L_2 - 2.2\sigma^{-1}\varGamma)$ such that the region $[x^{(1)}, x^{(1)} + 2.2\sigma^{-1}\varGamma] \times [y_i, y_{i+1}] \cap C$ contains no trap. Let $\tilde{w} = (\tilde{x}, y_{i+1})$ be defined by $\tilde{x} = x^{(1)} + \mu^{-1}(y_{i+1} - y_i) + 1.1\sigma^{-1}\varGamma$. The horizontal wall starting at y_{i+1} is passed through by an outer rightward clean hole $(x'_{i+1}, x''_{i+1}] \subseteq \tilde{x} + (\varDelta, \varGamma -$

$\Delta]$. Let $w'_{i+1} = (x'_{i+1}, y_{i+1})$, and $w''_{i+1} = (x''_{i+1}, y''_{i+1})$. Define the line E of slope μ going through the point w'_{i+1}. Let $w = (x^{(2)}, y)$ be the intersection of E with the horizontal line defined by y, then $x^{(2)} = x'_{i+1} - \mu^{-1}(y_{i+1} - y)$. The channel of horizontal width $2.2\mu^{-1}\Gamma$ and therefore vertical width 2.2Γ around the line E intersects the trap cover in a trap-free interval. There is a clean point $w'_i \in (x^{(2)}, y - \Delta) + [0, \Delta]^2$. The proof of (7.4) and $w'_i \rightsquigarrow w'_{i+1}$ is similar to the one for the vertical trap-wall pair.

7. Consider crossing a vertical wall-trap pair $(i-1, i)$.

   This part is somewhat similar to Part 5: we are again starting the construction at the trap cover.

   Let us define $x = x_i + \Gamma$. Find a $y^{(1)}$ in $[y_i, y_i + L_2 - 2.2\sigma^{-1}\Gamma)$ such that the region $[x_i, x_i + 5\Delta] \times [y^{(1)}, y^{(1)} + 2.2\sigma^{-1}\Gamma] \cap C$ contains no trap. Let $\tilde{w} = (x_{i-1}, \tilde{y})$ be defined by $y_{i-1} = y^{(1)} - \mu(x_i - x_{i-1}) + 1.1\sigma^{-1}\Gamma$. The vertical wall starting at x_{i-1} is passed through by an outer upward clean hole $(y'_{i-1}, y''_{i-1}] \subseteq \tilde{y} + (\Delta, \Gamma - \Delta]$. We define w'_{i-1}, and w''_{i-1} accordingly. Define the line E of slope μ going through the point w''_{i-1}. Let $w = (x, y^{(2)})$ be the intersection of E with the vertical line defined by x, then $y^{(2)} = y''_{i-1} + \mu(x - x''_{i-1})$. The channel of (vertical) width 2.2Γ around the line E intersects the trap cover in a trap-free interval. There is a clean point $w''_i \in (x, y^{(2)}) + [0, \Delta]^2$. The proof of (7.4) and $w''_{i-1} \rightsquigarrow w''_i$ is similar to the corresponding proof for the vertical trap-wall pair.

8. Consider crossing a horizontal wall-trap pair $(i-1, i)$.

   This part is somewhat similar to Parts 6 and 7. Let us define $y = y_i + \Gamma$. There is an $x^{(1)}$ in $[x_i, x_i + L_2 - 2.2\sigma^{-1}\Gamma)$ such that the region $[x^{(1)}, x^{(1)} + 2.2\sigma^{-1}\Gamma] \times [y_i, y_i + 5\Delta] \cap C$ contains no trap. Let $\tilde{w} = (\tilde{x}, y_{i-1})$ be defined by $\tilde{x} = x^{(1)} - \mu^{-1}(y_i - y_{i-1}) + 1.1\mu^{-1}\Gamma$. The wall starting at y_{i-1} contains an outer rightward clean hole $(x'_{i-1}, x''_{i-1}] \subseteq \tilde{x} + (\Delta, \Gamma - \Delta]$ passing through it. We define w'_{i-1}, w''_{i-1} accordingly. Define the line E of slope μ going through the point w''_{i-1}. The point $x^{(2)} = x''_{i-1} + \mu^{-1}(y - y''_{i-1})$ is its intersection with the horizontal line defined by y. The channel of horizontal width $2.2\mu^{-1}\Gamma$ and therefore vertical width 2.2Γ around the line E intersects the trap cover in a trap-free interval. There is a clean point $w''_i \in (x^{(2)}, y) + [0, \Delta]^2$. The proof of (7.4) and $w''_{i-1} \rightsquigarrow w''_i$ is similar to the corresponding proof for the vertical trap-wall pair.

9. We have $u \rightsquigarrow v$.

   Proof. If there is no parallel pair then $w'_i \rightsquigarrow w''_i$ is proven for $i = 1, 2, 3$. Suppose that there is a parallel pair. If it is a trap-wall pair $(i, i+1)$, then instead of $w'_i \rightsquigarrow w''_i$ we proved $w'_i \rightsquigarrow w'_{i+1}$; if it is a wall-trap pair $(i-1, i)$, then instead of $w''_{i-1} \rightsquigarrow w'_i$ we proved $w''_{i-1} \rightsquigarrow w''_i$.

In both cases, it remains to prove $w_i'' \rightsquigarrow w_{i+1}'$ whenever $(i, i+1)$ is not a parallel pair and $i = 1, 2$, further $u \rightsquigarrow w_1'$, $w_3'' \rightsquigarrow v$.

The rectangle $\mathrm{Rect}(w_i'', w_{i+1}')$ is a hop by definition. We just need to check that it satisfies the slope condition of $\mathcal{M}$. Since $(i, i+1)$ is not a parallel pair they intersect, and by the choice of the number K, their intersection is outside the channel $C(u, v', K - \Psi, K + \Psi)$. This implies $x_{i+1} - x_i \geq \Psi$ since $\mathrm{slope}(u, v) \leq 1$. On the other hand, by (7.4), the points w_i'', w_{i+1}' differ from w_i, w_{i+1} by at most $O(\sigma^{-1}\Gamma)$. It is easy to see from here that

$$|\mathrm{slope}(w_i'', w_{i+1}') - \mu| \leq c_0 \sigma^{-1} \Gamma / \Psi = c_0 \sigma^{-1} \Delta / \Gamma,$$
$$|1/\mathrm{slope}(w_i'', w_{i+1}') - 1/\mu| \leq c_0 \sigma^{-3} \Delta / \Gamma$$

for some absolute constant c_0 that can be computed. Choosing $\Lambda > c_0 + 1$, the definition of σ_i^* and the assumption on μ imply that the slope condition $\sigma_x \leq \mathrm{slope}(w_i'', w_i') \leq 1/\sigma_y$ is satisfied.

The proof of $u \rightsquigarrow w_1'$ and $w_3'' \rightsquigarrow v$ is similar, taking into account $x_1 - u_0 \geq 0.1\Phi$ and $v_0 - x_3 \geq 0.1\Phi$.

□

Proof of Lemma 7.1(Approximation). Recall that the lemma says that if a rectangle $Q = \mathrm{Rect}^{\varepsilon}(u, v)$ contains no walls or traps of $\mathcal{M}^*$, is inner clean in $\mathcal{M}^*$ and satisfies the slope condition $\sigma_x^* \leq \mathrm{slope}(u, v') \leq 1/\sigma_y^*$ with v' related to v as above, then $u \rightsquigarrow v$.

The proof started by recalling, in Lemma 7.2, that walls of $\mathcal{M}$ in Q can be grouped to a horizontal and a vertical sequence, whose members are well separated from each other and from the sides of Q. Then it showed, in Lemma 7.4, that all traps of $\mathcal{M}$ are covered by certain horizontal and vertical stripes called trap covers. Walls of $\mathcal{M}$ and trap covers were called obstacles.

Next it showed, in Lemma 7.6, that in case there are no traps or walls of $\mathcal{M}$ in Q then there is a path through Q that stays close to the diagonal.

Next, a series of obstacles (walls or trap covers) was defined, along with the points $s_1, s_2, \ldots$ that are obtained by the intersection points of the obstacle with the diagonal, and projected to the x axis. It was shown in Lemma 7.9 that these obstacles are well separated into groups of up to three. Lemma 7.10 showed how to pass each triple of obstacles. It remains to conclude the proof.

For each pair of numbers s_i, s_{i+1} with $s_{i+1} - s_i \geq 0.22\Phi$, define its midpoint $(s_i + s_{i+1})/2$. Let $t_1 < t_2 < \cdots < t_n$ be the sequence of all these midpoints. With $\mu = \mathrm{slope}(u, v')$, let us define the square

$$S_i = (t_i, u_1 + \mu(t_i - u_0)) + [0, \Delta] \times [-\Delta, 0].$$

By Remark 2.20.1, each of these squares contains a clean point p_i.

1. For $1 \le i < n$, the rectangle $\mathrm{Rect}(p_i, p_{i+1})$ satisfies the conditions of Lemma 7.10, and therefore $p_i \rightsquigarrow p_{i+1}$. The same holds also for $\mathrm{Rect}^{\varepsilon}(u, p_1)$ if the first obstacle is a wall, and for $\mathrm{Rect}(p_n, v)$ if the last obstacle is a wall. Here $\varepsilon = \uparrow, \rightarrow$ or nothing, depending on the nature of the original rectangle $\mathrm{Rect}^{\varepsilon}(u, v)$.

   Proof. By Lemma 7.9, there are at most three points of $\{s_1, s_2, \dots\}$ between t_i and t_{i+1}. Let these be $s_{j_i}, s_{j_i+1}, s_{j_i+2}$. Let t'_i be the x coordinate of p_i, then $0 \le t'_i - t_i \le \Delta$. The distance of each t'_i from the closest point s_j is at most $0.11\Phi - \Delta \ge 0.1\Phi$. It is also easy to check that p_i, p_{i+1} satisfy (7.2), so Lemma 7.10 is indeed applicable.

2. We have $u \rightsquigarrow p_1$ and $p_n \rightsquigarrow v$.

   Proof. If $s_1 \ge 0.1\Phi$, then the statement is proved by an application of Lemma 7.10, so suppose $s_1 < 0.1\Phi$. Then s_1 belongs to a trap cover.

   If s_2 belongs to a wall then $s_2 \ge \Phi/3$, so $s_2 - s_1 > 0.23\Phi$. If s_2 also belongs to a trap cover then the reasoning used in Lemma 7.9 gives $s_2 - s_1 > \Phi/4$. In both cases, a midpoint t_1 was chosen between s_1 and s_2 with $t_1 - s_1 > 0.1\Phi$, and there is only s_1 between u and t_1.

   If the trap cover belonging to s_1 is closer to u than $\Gamma - 6\Delta$ then the fact that u is clean in $\mathcal{M}^*$ implies that it contains a large trap-free region where it is easy to get through.

   If it is at a distance $\ge \Gamma - 6\Delta$ from u then we will pass through it, going from u to p_1 similarly to Part 1 of the proof of Lemma 7.10, but using case $j = 1$ of Lemma 4.10, in place of $j = 2$. This means using $L_1 = 29\sigma^{-1}\Delta$ in place of L_2. As a consequence, we will have $|x - x_1| + |y - y_1| = O(\sigma^{-1}\Delta)$ in place of (7.4). This makes a change of slope by $O(\sigma^{-1}\Delta/\Gamma)$, so an appropriate choice of the constant Λ finishes the proof just as in part 9 of the proof of Lemma 7.10.

   The relation $p_n \rightsquigarrow v$ is shown similarly.

□

8 Proof of the main lemma

Lemma 3.2 asserts the existence of a sequence of mazeries $\mathcal{M}^k$ such that certain inequalities hold. The construction of $\mathcal{M}^k$ is complete by the definition of $\mathcal{M}^1$ in Example 2.21 and the scale-up algorithm of Section 4, after fixing all parameters in Section 5.

We will prove, by induction, that every structure $\mathcal{M}^k$ is a mazery. Lemma 2.22 shows this for $k = 1$. Assuming that it is true for all $i \le k$, we prove it for $k + 1$.

The dependency properties in Condition 2.18.1 are satisfied according to Lemma 4.5. The combinatorial properties in Condition 2.18.2 have been proved in Lemmas 4.6 and 4.11. The reachability property in Condition 2.18.2e is satisfied via Lemma 7.1.

The trap probability upper bound in Condition 2.18.3a has been proved in Lemma 6.6. The wall probability upper bound in Condition 2.18.3b has been proved in Lemma 6.11. The cleanness probability lower bounds in Condition 2.18.3c have been proved in Lemma 6.21. The hole probability lower bound in Condition 2.18.3d has been proved in Lemmas 6.16, 6.18 and 6.19.

Inequality (3.1) of Lemma 3.2 is proved in Lemma 6.20.

9 Conclusions

The complex hierarchical technique has been used now to prove three results of the dependent percolation type: those in [3], [4] and the present one. Each of these proofs seems too complex for the result proved, and to give only a very bad estimate of the bound on the critical value of the respective parameter. In this, they differ from the related results on the undirected percolation in [8] and [1] (on the other hand, all three directed percolations exhibit power-law behavior).

For the other two problems, given that their original form relates to scheduling, it was natural to ask about possible extensions of the results to more than two sequences. I do not see what would be a natural extension of the embedding problem in this direction.

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