COMPATIBLE SEQUENCES
AND A SLOW WINKLER PERCOLATION

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ABSTRACT. Two infinite 0-1 sequences are called *compatible* when it is possible to cast out 0's from both in such a way that they become complementary to each other. Answering a question of Peter Winkler, we show that if the two 0-1-sequences are random i.i.d. and independent from each other, with probability p of 1's, then if p is sufficiently small they are compatible with positive probability. The question is equivalent to a certain dependent percolation with a power-law behavior: the probability that the origin is blocked at distance n but not closer decreases only polynomially fast and not, as usual, exponentially.

1. INTRODUCTION

1.1. **The model.** Let us call any strictly increasing sequence $t = (t(0) = 0, t(1), \dots)$ of integers a *delay sequence*. For an infinite sequence $x = (x(0), x(1), \dots)$, the delay sequence t introduces a timing arrangement in which the value $x(n)$ occurs at time $t(n)$. For two infinite 0-1-sequences x_d ($d = 0, 1$) and corresponding delay sequences t_d we say that there is a *collision* at (d, n) if $x_d(n) = 1$, and there is no k such that $x_{1-d}(k) = 0$ and $t_d(n) = t_{1-d}(k)$. We say that the sequences x_d are *compatible* if there is a pair of delay sequences t_d without collisions. It is easy to see that this is equivalent to saying that 0's can be deleted from both sequences in such a way that the resulting sequences have no collisions in the sense that they never have a 1 in the same position.

Example 1.1. The sequences

$$\mathtt{0001100100001111}\dots,$$
$$\mathtt{1101010001011001}\dots$$

are not compatible. The sequences x, y below, are. (We insert a $\underline{\mathtt{1}}$ in x, instead of deleting the corresponding $\mathtt{0}$ of y.)

$$x = \mathtt{0000100100000111100100100100 1001}\dots,$$
$$y = \mathtt{0101010001011000000001010110 1010}\dots,$$
$$x' = \mathtt{00001001}\underline{\mathtt{1}}\mathtt{0000011110010}\underline{\mathtt{1}}\mathtt{01001001001}\cdots,$$
$$y' = \mathtt{01010100010110000000010101101}\underline{\mathtt{1}}\mathtt{010}\cdots.$$

$\diamondsuit$

Suppose that for $d = 0, 1$, $X_d = (X_d(0), X_d(1), \dots)$ are two independent infinite sequences of independent random variables where $X_d(j) = 1$ with probability p and 0 with probability $1 - p$. Our question is: are X_0 and X_1 compatible with positive probability?

Date: February 5, 2007.

The paper was written during the author's visit at CWI, Amsterdam, partially supported by a grant of NWO.

The question depends, of course, on the value of p: intuitively, it seems that they are compatible if p is small, and our result will confirm this intuition.

We can interpret the sequences $X_d(j)$ as a chat of two members of a retirement home with unlimited time on their hands for maintaining a somewhat erratic conversation. Each member can be speaking or listening in any of the time periods $[j, j+1)$ (value $X_d(j) = 1$ or 0). There is a nurse who can put a member to sleep for any of these time periods or wake him up. Speaker d wakes up to his nth action at time $t_d(n)$. The nurse wants to arrange that every time when one of the parties is speaking the other one is up and listening. She is really a fairy since she is clairvoyant: she sees the pair of infinite sequences X_d (the talking/listening decisions) in advance and can tailor her strategy t_d to it. Here are some interpretations that are either less frivolous or more hallowed by the tradition of distributed computing.

Communication: Assume that the sequences X_0, X_1 belong to two processors. Value $X_0(i) = 1$ means that in its ith turn, processor 0 wants to send some message to processor 1, and $X_0(i) = 0$ means that it is willing to receive some message. Assume that each processor is allowed to send an extra message (or, just to stay idle, to "skip turns") in any time period, postponing the rest of its actions. The goal is to achieve that with the new sequences, whenever processor d is sending a message, processor $1 - d$ is listening.

Dining: Consider the following twist on the "dining philosophers" problem (see [3]). Two philosophers, 0 and 1, sit across a table, with a fork on both sides between them. The philosophers have two possible actions: thinking and eating. Sequence $X_d(i) = 1$ says that in her ith turn, philosopher d wants to eat. Both philosophers need two forks to eat, so any one will only be able to eat when the other one is thinking. Assume that both philosophers can be persuaded to insert some extra eating periods into their sequences (or, as an equivalent but less decorous possibility, to delete some thinking periods).

Queues: A single server serves two queues, numbered by 0 and 1, where the queue d of requests is being sent by a single user d. In both queues, a sequence of requests is coming in at discrete times 0,1,2, ... Let $\tau_d(i)$ be the time elapsed between the ith and $(i+1)$th request in queue d. All variables $\tau_d(i)$ are independent of each other, with $\mathrm{Prob}\{\, \tau_d(i) = n \,\} = p(1-p)^{n-1}$ for $n > 0$ (discrete approximation of a Poisson arrival process). Suppose that the senders of the queues will accept *faster service*: they are willing to send, instead of the originally planned sequences τ_d, new sequences τ'_0, τ'_1 where $\tau'_d(i) \leqslant \tau_d(i)$ for all d, i. We want these new sequences to be served simultaneously by the single server. Equivalently, suppose that both senders are willing to accept extra services inserted into their queues.

The question in all three cases is whether a scheduler who knows both infinite sequences in advance, can make the needed synchronizations with positive probability.

Peter Winkler and Harry Kesten, independently of each other, found an upper bound smaller than $\frac{1}{2}$ on the values p for which X_0, X_1 are compatible. We reproduce here informally Winkler's argument; it would be routine to formalize it. Suppose that the infinite sequences X and Y are compatible, and let us denote by X^k, Y^k their kth initial segments. Then we can delete some 0's from these segments in such a way that one of the resulting finite sequences, X', Y', is the complement of a prefix of the other: say, X' is the complement of a prefix of Y'. Assume that both sequences contain at least $k/2 - \varepsilon k$ 1's. Then the number of deleted 0's in both sequences can be at most $2\varepsilon k$. Then we can reproduce the pair of sequences X^k, Y^k using the following information.

1. The sequence Y'.
2. A 0-1 sequence u of length k whose 1's show the positions of the deleted 0's in X^k.
3. A 0-1 sequence v of length k whose 1's show the positions of the deleted 0's in Y^k.

Using Y' and u, we can restore X^k; using Y' and v, we can restore Y^k. For $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$, the total entropy of the three sequences is at most $k + 2kh(2\varepsilon)$. But the two sequences X^k, Y^k which we restored have total entropy $2k$. This gives an implicit lower bound on ε: $h(2\varepsilon) \geqslant \frac{1}{2}$.

Computer simulations by John Tromp suggest that when $p < 0.3$, with positive probability the sequences are compatible. The following theorem answers a question of Peter Winkler.

Theorem 1.2 (Main)**.** *If p is sufficiently small then with positive probability, X_0 and X_1 are compatible.*

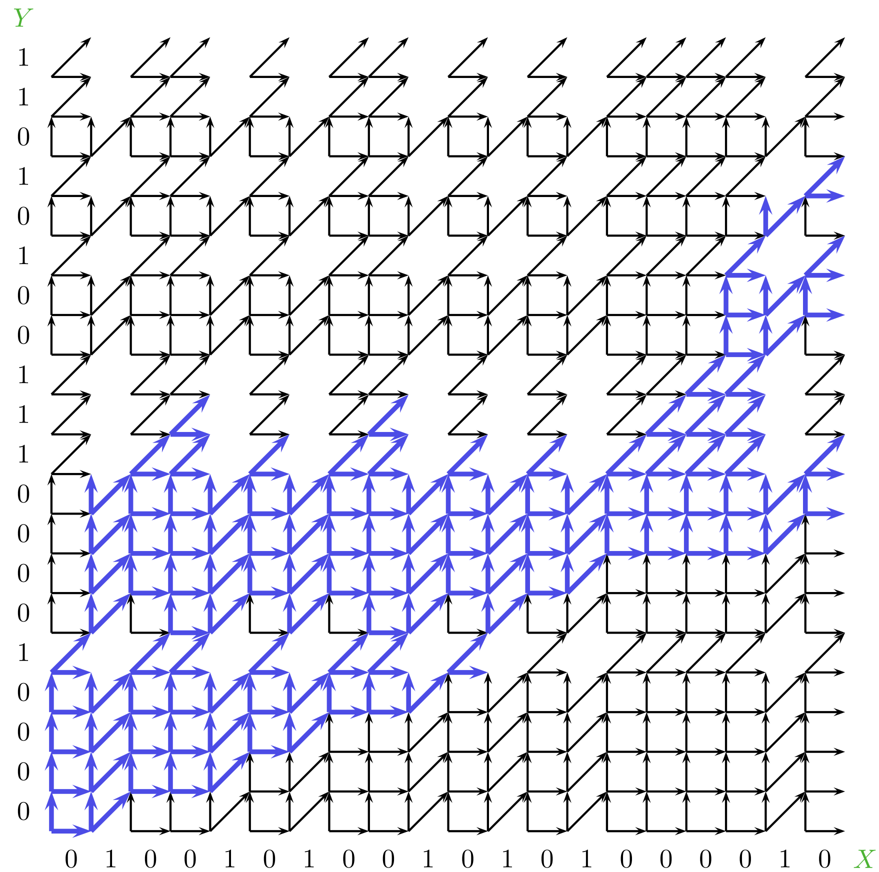
The threshold for p obtained from the proof is only 10^{-400}, so there is lots of room for improvement between this number and the experimental 0.3. It turns out that, when deciding whether to cast out a 0 in position n of a sequence, we do not have to look ahead further than position $2n^{1.5}$ (see Subsection 2.3).

1.2. **Another interpretation.** Complementing the sequence Y of the problem, we obtain an interesting equivalent problem. For infinite 0-1 sequences x, y, we will say $x \prec y$ if it is possible to obtain y from x by inserting 1's and deleting 0's. Let us be given two independent Bernoulli sequences X, Y, where the probability of 1's in X is p, and the probability of 1's in Y is $1-p$. Theorem 1.2 says that if p is sufficiently small then $X \prec Y$ with positive probability.

1.3. **A percolation.** Compatibility can also be defined for finite sequences, and we can ask then whether there is a polynomial algorithm that, given sequences X_0, X_1 of length n, decides whether they are compatible. It is easy to recognize that a dynamic programming algorithm will do it, and that the structure created by the dynamic programming leads to a useful reformulation of the original problem, too.

We define a directed graph $G = (V, E)$ as follows. $V = \mathbb{Z}_+^2$ is the set of points (i, j) where i, j are nonnegative integers. When representing the set V of points (i, j) graphically, the right direction is the one of growing i, and the upward direction is the one of growing j. The set E of edges consists of all pairs of the form $((i, j), (i+1, j))$, $((i, j), (i, j+1))$ and $((i, j), (i+1, j+1))$.

Sometimes we will write $X(i)$ for $X_0(i)$ and $Y(i)$ for $X_1(i)$. In the chat interpretation, when $X(i) = 1$ then participant 0 wants to speak in the ith turn of his waking time, which is identified with the interval $[i, i+1)$. In this case, we erase all edges of the form $((i, j), (i+1, j))$ for all j (this does not allow participant 1 to sleep through this interval). Similarly, when $Y(i) = 1$ then participant 1 wants to speak in the ith turn, and we erase all edges of the form $((j, i), (j, i+1))$. If $X(i) = Y(j)$ then we also erase edge $((i, j), (i+1, j+1))$. For $X(i) = 1$ since we do not allow the two participants to speak simultaneously, and for $X(i) = 0$ since the edge is not needed anyway, and this will allow a nicer mathematical description. This defines a graph $G(X, Y)$. For an example, see Figure 1. It is now easy to see that X and Y are compatible if and only if the graph $G(X, Y)$ contains an infinite path starting at $(0, 0)$. We will say that *there is percolation* for p if the probability at the given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parameter p that there is an infinite path is positive. We will also use graph $G(X,Y)$ for the case of finite sequences $X(0),\dots,X(m-1)$, $Y(0),\dots,Y(n-1)$, over $[0,m]\times[0,n]$.

We propose to call this sort of percolation, where two infinite random sequences X,Y are given on the two coordinate axes and the openness of a point or edge at position (i,j) depends on the pair $(X(i),Y(j))$, a *Winkler percolation*. Other examples will be seen in Section 9. Since we will talk about reachability a lot, the following notation is useful. If u,v are points of a directed graph (and the graph itself is clearly given from the context), then

$$u \rightsquigarrow v$$

denotes the fact that v is reachable from u on a directed path.

1.4. **Power-law behavior.** Our percolation problem has a power-law behavior: the probability that the origin is blocked at distance n but not closer decreases only polynomially fast, not, as usual, exponentially.

Let us indicate informally the reason: a formal proof for a similar problem is given in [4]. Let $W(n,k)$ be the event that $X(i)=1$ for i in $[n,n+k-1]$. We can view this event as the occurrence of a vertical "wall" of width k at position n. Let μ_k be the first number n with $W(n,k)$. Let $H(n,k)$ be the event that $Y(i)=0$ for i in $[n,n+k-1]$. We can view this event as the occurrence of a horizontal "hole" of width k at position n. Let ν_k be the first number n with $H(n,k)$. For a given k, $n=e^{kp}$, let

$$p_k = \text{Prob}\{\, \mu_k < pn < n < \nu_k \,\}.$$

Then with probability p_k, a vertical wall occurs at position pn but no horizontal hole appears up to height n. We can also assume that $\sum_{i\leqslant n} Y(i) > pn$. Therefore every path will have to move right $> pn$ steps while ascending to height n, but then it will hit the wall which no hole can penetrate up to height n. This gives blocking at distance n with approximately probability p_k. See Figure 2.

It can be computed that this probability is a polynomial function of n with a degree independent of k.

2. Outline of the proof

2.1. **Renormalization.** The proof method used is *renormalization*, or *multi-scale analysis*, and it is used frequently in statistical mechanics. The method is messy, laborious, and rather crude (rarely suited to the computation of exact constants). However, it is robust and well-suited to "error-correction" situations. Here is a rough first outline.

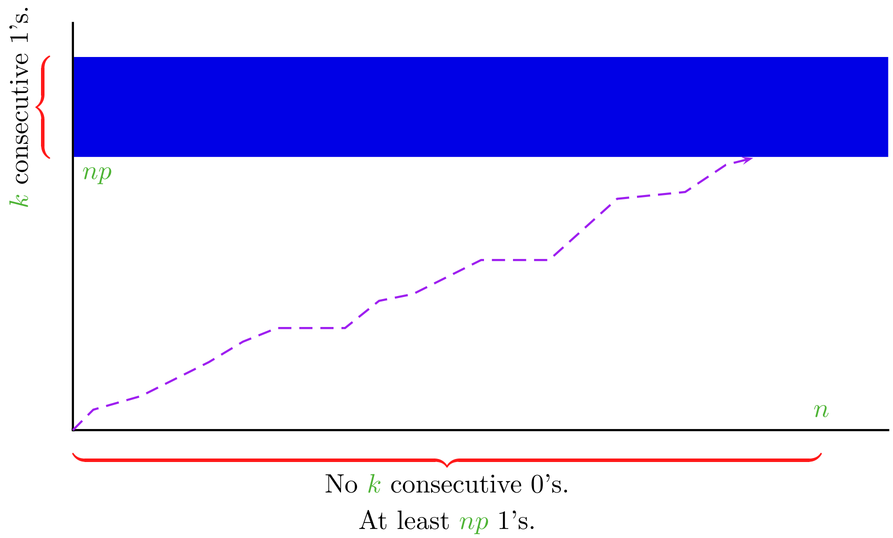
1. Fix an appropriate sequence $\Delta_1 < \Delta_2 < \cdots$, of scale parameters with $\Delta_{k+1} > 4\Delta_k$. Let

   $$\mathcal{F}_k$$

   be the event that point $(0,0)$ is blocked in $[0,\Delta_k]^2$. (In other applications, it could be some other *ultimate bad event*.) We want to prove

   $$\text{Prob}(\bigcup_k \mathcal{F}_k) < 1.$$

   This will be sufficient: if $(0,0)$ is not blocked in any finite square then by compactness (or by what is sometimes called König's Lemma), there is an infinite path starting at $(0,0)$.
2. Identify some events that you may call *bad events* and some others called *very bad events*, where the latter are much less probable.



k consecutive 1's.
np
n
No k consecutive 0's.
At least np 1's.

3. Define a series $\mathcal{M}^1, \mathcal{M}^2, \dots$ of models similar to each other, where the very bad events of $\mathcal{M}^k$ become the bad events of $\mathcal{M}^{k+1}$. Let
$$\mathcal{F}'_k$$
hold iff some bad event of $\mathcal{M}^k$ happens in $[0, \Delta_{k+1}]^2$.
4. Prove
$$\mathcal{F}_k \subset \bigcup_{i \leqslant k} \mathcal{F}'_i. \tag{2.1}$$
5. Prove $\sum_k \mathrm{Prob}(\mathcal{F}'_k) < 1$.

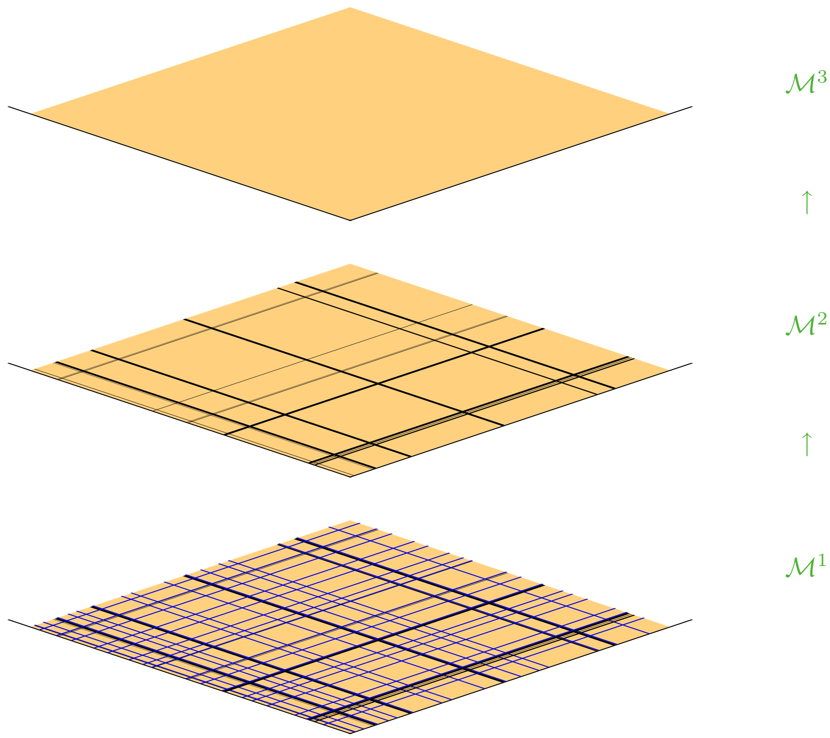
In later discussions, we will frequently delete the index k from $\mathcal{M}^k$ as well as from other quantities defined for $\mathcal{M}^k$. In this context, we will refer to $\mathcal{M}^{k+1}$ as $\mathcal{M}^*$.

2.2. **Application to our case.** In our setting, the role of the "bad events" of Subsection 2.1 will be played by *walls*. In Subsection 1.4, we have seen a simple kind of wall coming from a block of 1's. Another kind of wall comes from two blocks of 1's that are too close to each other. The same example extrapolates to a more complicated kind of wall: let $E(n, k, m)$ be the event that no hole $H(i, k)$ occurs on the segment $[n, n + m]$. For really large m, this event creates a horizontal wall. Indeed, if m/k is much larger than the typical distance between the vertical walls $W(i, k)$ then we will be able to pass through the horizontal stripe at height n only at some exceptional horizontal positions: those places where the distance between the walls $W(i, k)$ is much larger than typical. We are only saved if these exceptional positions occur significantly more frequently than the walls $E(n, k, m)$.

It seems that we need not only bad events, but also some good ones: the holes. Our proof systematizes the above ideas by introducing an abstract notion of walls and holes. We will have walls and holes of many different types. To each wall type belongs a fitting hole type. The walls of a given type occur much less frequently than their fitting holes. The whole model will be called a *mazery* $\mathcal{M}$ (a system for creating mazes). The original setting is a very simple mazery: there is just one wall type, a 1, and one hole type, a 0. Actually, we have two independent mazeries $\mathcal{M}_0$ and $\mathcal{M}_1$: for the horizontal and for the vertical line.

In any mazery, whenever it happens that walls are well separated from each other and holes are not missing, then paths can pass through. Sometimes, however, unlucky events arise. It turns out that they can be summarized in two typical examples: first, when two walls occur too close together (see Figure 3); second, when there is a large segment from which a certain hole type is missing. For any mazery $\mathcal{M}$, we will define a mazery $\mathcal{M}^*$ whose walls correspond to these typical unlucky events. A pair of uncomfortably close walls of $\mathcal{M}$ gives rise to a wall of $\mathcal{M}^*$ called a *compound wall*. A large interval without a certain type of hole of $\mathcal{M}$ gives rise to a wall of $\mathcal{M}^*$ called an *emerging wall*. Corresponding holes are defined, and it will be shown that the new mazery also has the property that its holes are much more frequent than the corresponding walls. Thus, the "bad events" of the outline in Subsection 2.1 are the walls of $\mathcal{M}$, the "very bad events" are (modulo some details that are not important now) the compound and emerging walls of $\mathcal{M}^*$. Let $\mathcal{F}, \mathcal{F}'$ be the events $\mathcal{F}_k, \mathcal{F}'_k$ formulated in Subsection 2.1. Thus, $\mathcal{F}'$ says that in either $\mathcal{M}_0$ or $\mathcal{M}_1$, a wall appears on the interval $[0, \Delta^*]$.

The idea of the scale-up construction is that on the level of $\mathcal{M}^*$ we do not want to see all the details of $\mathcal{M}$. We will not see all the walls; however, some restrictions will be inherited from them: these are distilled in the concepts of a *clean point* and of a *slope constraint*. A


$\mathcal{M}^3$
↑
$\mathcal{M}^2$
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$\mathcal{M}^1$

point (x_0, x_1) is clean for $\mathcal{M}_0 \times \mathcal{M}_1$ if x_d is clean for $\mathcal{M}_d$ for $d = 0, 1$. Let

$$\mathcal{Q}$$

be the event that point $(0, 0)$ is not clean in $\mathcal{M}_0$ or $\mathcal{M}_1$.

We would like to say that in a mazery, if points (x_0, x_1), (y_0, y_1) are such that for $d = 0, 1$ we have $x_d < y_d$ and there are no walls between x_d and y_d, then $(x_0, x_1) \rightsquigarrow (y_0, y_1)$. However, this will only hold with some restrictions. What we will have is the following, with an appropriate parameter

$$0 \leqslant \sigma < 1/2.$$

Condition 2.1. Suppose that points (x_0, x_1), (y_0, y_1) are such that for $d = 0, 1$ we have $x_d < y_d$ and there are no walls between x_d and y_d. If these points are also clean and satisfy the slope-constraint

$$\sigma \leqslant \frac{y_1 - x_1}{y_0 - x_0} \leqslant 1/\sigma$$

then $(x_0, x_1) \rightsquigarrow (y_0, y_1)$. ♢

We will also have the following condition:

Condition 2.2. Every interval of size 3Δ that does not intersect walls contains a clean point in its middle third. ♢

Lemma 2.3. *We have*

$$\mathcal{F} \subset \mathcal{F}' \cup \mathcal{Q}. \tag{2.2}$$

Proof. Suppose that $\mathcal{Q}$ does not hold, then $(0, 0)$ is clean. Suppose also that $\mathcal{F}'$ does not hold: then by Condition 2.2, for $d = 0, 1$, there is a point $x = (x_0, x_1)$ with $x_d \in [\Delta, 2\Delta]$ clean in $\mathcal{M}_d$. This x also satisfies the slope condition $1/2 \leqslant x_1/x_0 \leqslant 2$ and is hence, by Condition 2.1, reachable from $(0, 0)$. □

We will define a sequence of mazeries $\mathcal{M}^1, \mathcal{M}^2, \dots$ with $\mathcal{M}^{k+1} = (\mathcal{M}^k)^*$, with $\Delta_k \to \infty$. All these mazeries are on a common probability space, since $\mathcal{M}^{k+1}$ is a function of $\mathcal{M}^k$. Actually, we will have two independent mazeries $\mathcal{M}^k_d$ for $d = 0, 1$, constructed from $\mathcal{M}_0, \mathcal{M}_1$. All ingredients of the mazeries will be indexed correspondingly: for example, the event $\mathcal{Q}_k$ that $(0, 0)$ is not upper right clean in $\mathcal{M}_k$ plays the role of $\mathcal{Q}$ for the mazery$\mathcal{M}^k$. We will have the following property:

Condition 2.4.

$$\mathcal{Q}_k \subset \bigcup_{i<k} \mathcal{F}'_i. \tag{2.3}$$

♢

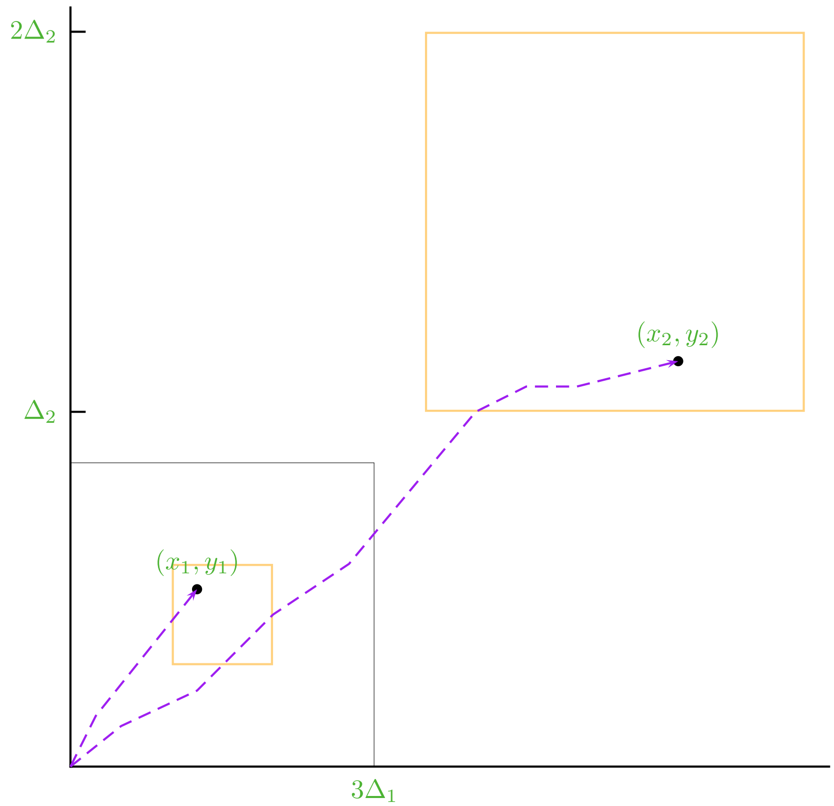
This, along with (2.2) implies $\mathcal{F}_k \subset \bigcup_{i \leqslant k} \mathcal{F}'_i$, which is inequality (2.1). Hence the theorem is implied by the following lemma, which will be proved after all the details are given:

Lemma 2.5 (Main)**.** *If p is sufficiently small then the sequence $\mathcal{M}^k$ can be constructed, in such a way that it satisfies all the above conditions and also*

$$\sum_k \mathrm{Prob}(\mathcal{F}'_k) < 1. \tag{2.4}$$

Figure 4 illustrates the proof of the theorem from the main lemma.

Remark 2.6. The first clairvoyant synchronization problem, also posed by Winkler, is more difficult than this one: it will be discussed in Section 9. ♢



$2\Delta_2$
Δ_2
(x_2, y_2)
(x_1, y_1)
$3\Delta_1$

2.3. **How much lookahead is needed?** Theorem 1.2 says that when p is small then with positive probability, the sequences X, Y are compatible: they can be synchronized by a clairvoyant demon. It is a natural question to ask how far a lookahead is needed? Suppose that sequences X, Y are given up to members $X(n), Y(n)$, and $X = 0$. Up to which members $X(f(n)), Y(f(n))$ need the sequences to be known in order to know whether to cast out member $X(n)$? The following modification of the previous argument gives an upper bound $f(n) \approx n^{1.5}$.

For point $u \in [0, \Delta_k]^2$, let

$$\mathcal{F}_k(u)$$

be the event that point u is blocked in $[0, \Delta_k]^2$. Now we assume

$$\Delta_{k+1} > 4\Delta_k,$$

and let

$$\mathcal{F}_k''$$

hold iff some wall of $\mathcal{M}^k$ or $\mathcal{M}^{k+1}$ appears in $[0, 4\Delta_{k+1}]^2$. Instead of Lemma 2.3, we have the following.

Lemma 2.7. *If $\mathcal{F}_k''$ does not hold then for each $u \in [0, \Delta_k]^2$ clean in $\mathcal{M}^k$ there is a $v \in [2\Delta_k, 3\Delta_k]^2$ clean in $\mathcal{M}^{k+1}$ and reachable from u.*

Proof. By Condition 2.2, for $d = 0, 1$, there is a point $v = (v_0, v_1)$ with $v_d \in [2\Delta_{k+1}, 3\Delta_{k+1}]$ clean in $\mathcal{M}_d^{k+1}$. This v also satisfies the slope condition $1/2 \leqslant (v_1 - u_1)/(v_0 - u_0) \leqslant 2$ and is hence, by Condition 2.1, reachable from u. □

The statement $\sum_k \text{Prob}(\mathcal{F}_k'') < 1$ will clearly be proved in just the same way as the statement $\sum_k \text{Prob}(\mathcal{F}_k') < 1$. So, with positive probability, none of the events $\mathcal{F}_k''$ occurs. Then, the above lemma allows us to find a sequence of points $(0, 0) = u_1, u_2, \ldots$ such that $u_k \subset [2\Delta_k, 3\Delta_k]^2$, and $u_k \rightsquigarrow u_{k+1}$. Finding a path from $u_k = (x_k, y_k)$ reaching u_{k+1} means deciding which elements $X(i), Y(j)$ ($x_k \leqslant i < x_{k+1}$, $y_k \leqslant j < y_{k+1}$) of the sequences X, Y should be cast out. For this, the sequences X, Y need to be known up to $4\Delta_{k+1}$. Thus, in order to decide about an index $i \geqslant 2\Delta_k$, we may need to know members with indexes $< 4\Delta_{k+1}$. Later in the paper, we will set $\Delta_{k+1} = \Delta_k^{1.5}$. This shows that lookahead up to index $2i^{1.5}$ is sufficient. Different choices of some parameters would allow us to decrease this to $i^{1+\varepsilon}$ for any $\varepsilon > 0$.

2.4. **The rest of the paper.** In Section 3, we discuss the reasons for the less obvious features of the construction that follows. The rest of the paper does not depend on this section, it can be skipped, but it may be helpful to refer back to it for better understanding.

In Section 4, mazeries will be defined. It is tough to read this section before seeing the role of each notion and assumption. The reader may want to refer forward to Section 5 for such insights.

Section 5 handles all the combinatorial details separable from the calculations.

Section 6 defines emerging and compound walls and holes precisely and estimates their probabilities as much as possible without bringing in dependence on k.

Section 7 defines the scale-up functions, substitutes them into the earlier estimates and finishes the proof of Lemma 2.5.

Section 8 gives a proof left from Section 5.

Section 9 relates the present problem to an earlier defined clairvoyant synchronization problem.

Section 10 discusses possible sharpenings of the result.

3. Some technical difficulties and their solution

In this section, we discuss the reasons for the less obvious features of the construction that follows. The rest of the paper does not depend on this section, it can be skipped, but it may be helpful to refer back to, for better understanding.

3.1. **Clean points.** Our main tool for estimating probabilities will be to attribute various events to disjoint open intervals. Cleanness is a property of a point: in order to fit it into this scheme, it will be broken up into two properties: *left-cleanness* and *right-cleanness*. A point is clean if it is both left-clean and right-clean. An interval is *inner-clean* if its left endpoint is right-clean and its right endpoint is left-clean; it is outer-clean if its left end is left-clean and its right end is right-clean. An inner-clean interval containing no walls will be called a *hop*. (Actually, we will use a slight variant of this notion, called a "jump", see below.)

3.2. **Overlapping walls.** In Section 2, we said that when two (vertical) walls $W_i = (x_i, x_i + w_i)$ $(i = 1, 2)$ of mazery $\mathcal{M}_0^k$ are too close, they will give rise to a wall of $\mathcal{M}_0^{k+1}$ of a compound type: this postpones the difficulty of dealing with this situation to a higher level. It is expected that we can get through these two walls wherever two fitting (horizontal) holes, $H_i = (y_i, y_i + h_i)$ in $\mathcal{M}_1^k$, occur at a comparable distance to each other. There is much vagueness in these ideas. What does "comparable distance" mean? What does "to be expected" mean?

Let us deal first with the issue: why is it to be expected? It is true that by the assumptions, H_1 allows us to get from (x_1, y_1), to $(x_1 + w_1, y_1 + h_1)$, but how do we get from there to (x_2, y_2)? Why can we even assume that walls W_1, W_2 are disjoint? Indeed, suppose that there were three close walls V_1, V_2, V_3 on level $\mathcal{M}^{k-1}$, then they would give rise to the compound walls $V_1 + V_2$ and $V_2 + V_3$, which are not disjoint.

We will deal with the issue by a method used in the paper repeatedly: we "define it away". It does not exist on level 1. We require it to be solvable on level k and then use induction to show that it remains solvable on level $k + 1$. For compound holes, we simply include a requirement into the definition of the compound hole $H_1 + H_2$, that the interval between H_1 and H_2 is a hop. This will make it harder to lowerbound the probability of compound holes; see later.

We require that every interval covered by walls can also be covered by an interval spanned by a sequence of disjoint walls separated by hops. In order to prove the same property for $\mathcal{M}^{k+1}$, it is necessary to introduce *triple compound walls* (essentially, the compounding operation will be performed twice). Thus, a sequence of close walls $V_1, \dots, V_n$ on level k can be subdivided into a number of disjoint neighbor compounds. If n is even then all these compounds are of the form $V_i + V_{i+1}$. If n is odd then one of them has the form $V_i + V_{i+1} + V_{i+2}$.

This takes care of compound walls, but there are also walls of the emerging type. How can these be assumed to be disjoint from everything else and separated by hops from them? Again, we define the problem away: we will essentially introduce emerging walls one-by-one, allowing one only if it is disjoint from the others and is outer-clean.

The last step breaks an important property of our model. It is a global operation: now whether an interval I is the body of a wall of $\mathcal{M}_0^k$ is no more an event depending only on the random variables $X_0(j)$ in this interval. The events that there is a wall of a certain type on interval I_1 and of some other type on interval I_2, are not independent anymore: we seem to lose our only tool of probability estimation. Fortunately, this problem can also be defined away, since for walls, we need only probability upper bounds. We introduce another concept, the concept of a *barrier*. Every wall is a barrier, but not vice versa. The occurrence of a barrier of any given type on an open interval I will only depend on elementary events in I, and will be independent of the occurrence of a barrier of any other type on any interval J disjoint from I. Our probability bounds will be for barriers.

3.3. **Compound wall types.** When we say that walls are much less probable than the fitting holes then what we mean is that there is an overall constant

$$\chi = 0.03 \tag{3.1}$$

such that when p is the probability of a certain wall type and q the probability of a fitting hole type then $q \geqslant p^{\chi}$.

Several times, it will be important to bound the probability that any barrier at all occurs at a given point. Since our probability upper bounds apply generally to barriers of some given type, it is important to limit the number of types. We said that given two barriers W_i $(i = 1, 2)$ of types α_i at a certain small distance d, a compound barrier should arise. "Small" will mean here smaller than a certain parameter f. We cannot afford a new compound barrier type for each triple (α_1, α_2, d) for all $d < f$: this would lead to too many types. On the other hand, if we only introduce a single type for all pairs α_1, α_2 then a simple probability upper bound of this type would only be $f p(\alpha_1) p(\alpha_2)$, while the probability lower bound on the fitting compound hole type would only be $p(\alpha_1)^{\chi} p(\alpha_2)^{\chi}$, ignoring the factor f. We need a compromise between these two extremes. The parameter

$$\lambda = 2^{1/4} \tag{3.2}$$

will be introduced, and we will have compound barrier types

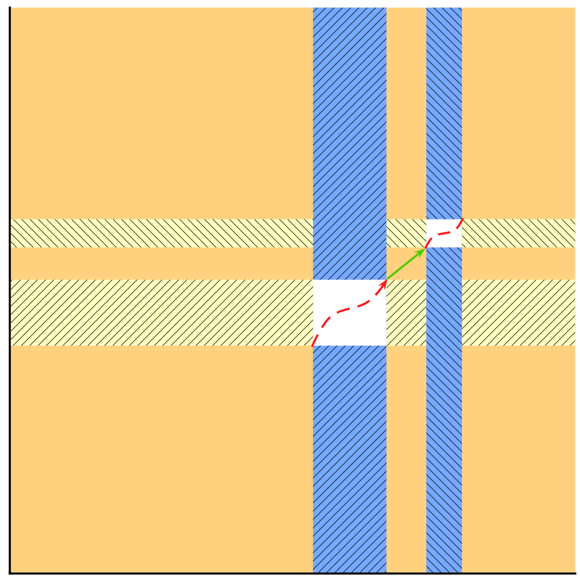
$$\beta = \langle \alpha_1, \alpha_2, i \rangle \tag{3.3}$$

for different i. A pair of barriers W_i of type α_i gives rise to a compound barrier of such a type if their distance falls between λ^i and λ^{i+1}. A fitting compound hole will have two fitting holes, whose distance is between the values λ^{i-1} and λ^i. (See Figure 5.)

With the above choice of barrier types, even barriers belonging to a given type can have different sizes. This makes it harder to prove even a probability upper bound $p(\alpha_1)p(\alpha_2)$ on the occurrence of barriers W_i of type α_i adjacent to each other. We again define the problem away: the probability bound $p(\alpha)$ on the occurrence of a barrier of type α is simply defined as the sum of bounds $p(\alpha, w)$ of the occurrence of type α and size w.

3.4. **Ranks.** Having all these different compound barrier types implies that even if the component types have probability $p(\alpha_i) = s$, the probability bound on compound barriers now ranges from s^2 to fs^2. Assuming that $f = s^{-\phi}$ for some small ϕ, the probability bound ranges from s^2 to $s^{2-\phi}$. Repeating this operation k times, the bound ranges from s^{2^k} to $s^{(2-\phi)^k}$. This makes the smallest probability of a barrier type much smaller than the probability of all barrier types.

Recall that a (vertical) barrier of the emerging type occurs in $\mathcal{M}_0^{k+1}$ on an interval I of size g (for a certain parameter g smaller than the f introduced above), if there is a barrier type α_0 of $\mathcal{M}^k$ such that no hole of fitting type α_0' appears in I. A (horizontal) hole of a fitting type of $\mathcal{M}_1^k$ occurs on an interval J of size $\approx g$ if no (horizontal) barrier of any type appears on J. It is not sufficient to exclude barriers of type α_0, since vertical holes of other types may also be missing from I. A simple probability upper bound for any barrier appearing on J is $g\overline{p} = g \sum_{\alpha} p(\alpha)$. This can only be small if $g < 1/\overline{p}$. The probability that



I is an emerging barrier is exponentially small in g but this becomes significant only if g is larger than the inverse of the probability lower bound $(p(\alpha_0))^{\chi}$ on the appearance of a hole of type α'_0. Thus, we need

$$\overline{p} < (p(\alpha_0))^{\chi}, \tag{3.4}$$

which is impossible if the difference between the smallest and the largest $p(\alpha)$ grows as shown above.

The solution is to realize that if we have barrier types with many different probability bounds, we do not have to deal with them all at the same time. We assign a number called *rank* to each barrier type, based essentially on its probability bound. The probability bound of barriers of rank r will be $p(r) \approx \lambda^{-r}$, and this will also bound approximately the probability that any barrier of rank r starts at a given point. Mazery $\mathcal{M}^k$ will have types present in it that are between ranks R_k and R_k^c for a certain constant c, where $R_k = \lambda^{\tau^k}$, $\tau = 2 - \phi$. When going from level k to level $k+1$, we eliminate only the ranks between R_k and R_{k+1} (and we add some new, higher ranks). The barrier type α_0 above in the introduction of an emerging barrier, will be restricted to have rank less than R_{k+1} (such barrier types will be called *light*). With the rank limited from above, the fitting hole probability lower bound is limited from below, and we escape the impossible requirement (3.4).

3.5. **Compound hole probability.** It is relatively easy to obtain an upper bound $(\lambda^{i+1} - \lambda^i)p(\alpha_1)p(\alpha_2)$ on the probability of a compound barrier of type β in (3.3). Indeed, we essentially sum up the probability bound $p(\alpha_1)p(\alpha_2)$ for the different possible values of the distance $d \in [\lambda^i, \lambda^{i+1} - 1]$ between the two barriers. It is less clear how to find the corresponding lower bound

$$\underline{p} = (\lambda^{i+1} - \lambda^i)^{\chi}(p(\alpha_1))^{\chi}(p(\alpha_2))^{\chi} \tag{3.5}$$

for the fitting hole. In the upper bound, we can ignore the fact that the various possibilities for distance d are not disjoint events; not so for the lower bound. We will use a combination of two methods. If i is small then we will use the main method of the paper: define the problem away, by requiring something in the definition of a mazery that will imply the result (see the hole lower bound (4.8)). If i is large, then we break up the interval considered into disjoint subintervals of size 3Δ, and use the independence of the relevant events on these along with some computation, in order to infer the same property (4.8) for $\mathcal{M}^{k+1}$.

The above tactic gives the desired lower bound on the probability that two holes of the given types appear at a distance $d \in [\lambda^{i-1}, \lambda^i - 1]$, but it still does not guarantee that the two holes are separated by an interval free of barriers (inner-cleanness of this interval is easier to achieve). The probability that an interval of size f is barrier-free, has a large lower bound: the problem is only to be able to multiply this lower bound with the lower bound $\underline{p}$ in (3.5). We will make use of the FKG inequality for this, noticing that the events whose intersection we want to form are decreasing functions of the original sequence X_0. (It is possible to avoid the reference to monotonicity, at the price of a little more sweat.)

4. WALLS AND HOLES

4.1. **Notation.** We will use

$$a \wedge b = \min(a, b), \quad a \vee b = \max(a, b).$$

The *size* of an interval $I = (a, b)$ is denoted by $|I| = b - a$. For two sets A, B in the plane or on the line,

$$A + B = \{\, a + b : a \in A,\ b \in B \,\}.$$

For two different points $u_i = (x_i, y_i)$ $(i = 0, 1)$ in the plane, when $x_0 \leqslant x_1$, $y_0 \leqslant y_1$:

$$\begin{aligned} \mathit{slope}(u_0, u_1) &= \frac{y_1 - y_0}{x_1 - x_0}, \\ \mathit{minslope}(u_0, u_1) &= \min\big(\mathit{slope}(u_0, u_1), 1/\mathit{slope}(u_0, u_1)\big). \end{aligned}$$

4.2. **Mazeries.**

4.2.1. *The whole structure.* A *mazery*

$$\mathbb{M} = (\text{Btypes}, \text{Htypes}, \mathit{Rank}(\cdot), \text{Bvalues}, \text{Hvalues}, \mathcal{M}, \Delta, \sigma, p(\cdot)) \tag{4.1}$$

has the following parts.

– Two disjoint finite sets Btypes and Htypes called the sets of *barrier types* and *hole types*.
– A function $\mathit{Rank} : (\text{Btypes} \cup \text{Htypes}) \to \mathbb{Z}_+$. Each type α has a *rank* $\mathit{Rank}(\alpha)$ that is a nonnegative integer.
– Let $\mathcal{I}$ be the set of nonempty open intervals with integer endpoints, then we have two sets:

$$\text{Bvalues} \subset \mathcal{I} \times \text{Btypes}, \quad \text{Hvalues} \subset \mathcal{I} \times \text{Htypes}.$$

Elements of Bvalues and Hvalues are called *barrier values* and *hole values* respectively. A barrier or hole value $E = (B, \alpha)$ has *body* B which is an open interval, and *type* α. We write $\mathit{Body}(E) = B$, $|E| = |B|$. We will sometimes denote the body also by E. It is not considered empty even if it has size 1. We write $\mathit{Type}(E) = \alpha$.
– The random process $\mathcal{M}$ will be detailed below.
– The parameters $\Delta > 0, \sigma \geqslant 0$ and the probability bounds $p(r)$ will also be detailed below.

The following conditions hold for the parts discussed above.

Condition 4.1.

1. Let us denote by $\underline{\Delta}(\alpha), \overline{\Delta}(\alpha)$ the infimum and supremum of the sizes of barriers or holes of type α. We require

$$\overline{\Delta}(\alpha) \leqslant \Delta.$$

2. To each barrier type α there corresponds a *fitting* hole type α' with

$$\overline{\Delta}(\alpha') \leqslant \underline{\Delta}(\alpha), \quad \mathit{Rank}(\alpha') = \mathit{Rank}(\alpha).$$

(We require a hole to have a smaller or equal size than the barrier it fits; this will bound the amount by which the minimal slope of a path must increase while passing a wall. Equality is achieved in the example where a wall of k 1's can be passed only at a hole of k 0's: thus, this example is the worst that can happen.)

♢

4.2.2. *The random processes.* Now we discuss the various parts of the random process

$$\mathcal{M} = (Z, \mathcal{B}, \mathcal{W}, \mathcal{H}, \mathcal{C}, \mathcal{S}).$$

Here, $Z = (Z(0), Z(1), \dots)$, is a sequence of independent, equally distributed 0-1-valued random variables. Since $Z(i)$ is thought of as belonging to the open interval $(i, i+1)$, we will write for any interval (a, b):

$$Z((a, b)) = (Z(a), \dots, Z(b-1)).$$

We will have the following random sets:

$$\mathcal{W} \subset \mathcal{B} \subset \text{Bvalues}, \quad \mathcal{H} \subset \text{Hvalues}, \quad \mathcal{S} \subset \mathcal{C} \subset \mathbb{Z}_+ \times (\mathbb{Z} \setminus \{0\}),$$

all of which are functions of Z. The elements of $\mathcal{B}$ are *barriers*. The elements of $\mathcal{W}$ are called *walls*: each wall is also a barrier. The elements of $\mathcal{H}$ are called *holes*.

Remarks 4.2.

1. In what follows we will refer to $\mathcal{M}$ by itself also as a *mazery*, and will mention $\mathbb{M}$ only rarely. This should not cause confusion; though $\mathcal{M}$ is part of $\mathbb{M}$, it relies implicitly on all the other ingredients of $\mathbb{M}$.
2. The distribution of $\mathcal{B}$ is simpler than that of $\mathcal{W}$, but sample sets of $\mathcal{W}$ will have a more useful structure. The parts of the paper dealing with combinatorial questions (reachability) will work mainly with walls and hops (see below), the parts containing probability calculations will work with barriers and jumps (see below).

$\diamondsuit$

4.2.3. *Cleanness.* For $x \geqslant 0, r > 0$, if $(x, -r) \in \mathcal{C}$ then we say that point x is *left r-clean*. If $(x, r) \in \mathcal{C}$ then we say that x is *right r-clean*. A point x is called left-clean (right-clean) if it is left r-clean (right r-clean) for all r. It is *clean* if it is both left- and right-clean. If the left end of an interval I is right $|I|$-clean and the right end is left $|I|$-clean then we say I is *inner-clean*. If its left end is left-clean and its right end is right-clean then we say that it is *outer-clean*. Let $(a, b), I$ be two intervals with $(a - \Delta, b + \Delta) \subset I$. We say that (a, b) is *cleanly* contained in I if it is outer-clean.

For $x \geqslant 0, r > 0$, if $(x, -r) \in \mathcal{S}$ then we say that point x is *strongly* left r-clean. Similarly, every concept concerning cleanness has a counterpart when we replace $\mathcal{C}$ with $\mathcal{S}$ and clean with strongly clean.

A closed interval is called a *hop* if it is inner clean and contains no wall. It is a *jump* if it is strongly inner clean and contains no barrier. By definition, all jumps are hops. A hop or jump may consist of a single point.

Let us call an interval *external* if it does not intersect any wall. (Remember that the bodies of walls and holes are open intervals.) Two disjoint walls or holes are called *neighbors* if the interval between them is a hop. A sequence $W_i \in \mathcal{W}$ of walls $i = 1, 2, \dots$ is called a *sequence of neighbor walls* if for all i, W_i is a left neighbor of W_{i+1}.

4.2.4. *Conditions on the process $\mathcal{M}$.* There are many conditions on the distribution of process $\mathcal{M}$, but most of them are fairly natural. We would like to call attention to a crucial condition that is not clearly motivated: Condition 4.3.3d. It implies, as a special case with $c = b + 1$, that through every wall, at every position, there is a fitting hole with sufficiently large probability. The general case has been formulated carefully and it is crucial for the inductive proof that the hole lower bound will also hold on compound walls after renormalization (going from $\mathcal{M}^k$ to $\mathcal{M}^{k+1}$).

The function

$$p(r, w) \tag{4.2}$$

is defined as the supremum of probabilities (over all points) that any barrier with rank r and size w starts at a given point. The function $p(r)$ will be an upper bound on $\sum_w p(r, w)$. Let

$$p(\alpha, w) = p(r, w), \quad p(\alpha) = p(r)$$

for any type α of rank r.

Let us introduce the constants

$$c_1 = 6, \quad c_0 \approx 7.41, \tag{4.3}$$

where the requirement $c_1 > 5$ comes from inequality (7.22) below, while the value for c_0 will be motivated in (7.4). For each rank r, let us define the function

$$h(r) = c_0 r^{c_1 \chi} (p(r))^{\chi}, \quad h(\alpha) = h(\mathit{Rank}(\alpha)). \tag{4.4}$$

The exponent χ has been introduced in (3.1). Its choice will be motivated in Section 7. The factor $c_0 r^{c_1 \chi}$ will absorb some nuisance terms as they arise in the estimates. The function $h(r)$ will be used as a lower bound for the probability of holes of rank r.

Condition 4.3.

1. (Dependencies and monotonicity)
   a. For a barrier value E, the event $\{ E \in \mathcal{B} \}$ is an increasing function of $Z(\mathit{Body}(E))$. Thus, any set of events of the form $E_i \in \mathcal{B}$ where E_i are disjoint, is independent.
   b. For a hole value E, the event $\{ E \in \mathcal{H} \}$ is a decreasing function of $Z(\mathit{Body}(E))$.
   c. For every point x and integer r, the events $\{ (x, -r) \in \mathcal{S} \}$, $\{ (x, r) \in \mathcal{S} \}$ are decreasing functions of $Z((x - r, x))$ and $Z((x, x + r))$ respectively.
   When Z is fixed, strong and not strong left (right) r-cleanness are decreasing as functions of r. These functions reach their minimum at $r = \Delta$: thus, if x is (strongly) left (right) Δ-clean then it is (strongly) left (right)-clean.
2. (Combinatorial requirements)
   a. A maximal external interval of size $\geqslant \Delta$ is inner-clean (and hence is a hop).
   b. If an interval I is surrounded by maximal external intervals of size $\geqslant \Delta$ then it is spanned by a sequence of neighbor walls. (Thus, it is covered by the union of the neighbor walls of this sequence and the hops between them.)
   c. If an interval of size $\geqslant 3\Delta$ contains no walls then its middle third contains a clean point.
3. (Probability bounds)
   a. For all r we have

$$p(r) \geqslant \sum_{l} p(r, l). \tag{4.5}$$

   b. The following requirement imposes an implicit upper bound on $p(\alpha)$:

$$\Delta^{\chi} h(\alpha) < 0.6. \tag{4.6}$$

   c. For $\overline{q} = \sup_x \mathrm{Prob}\{ x \text{ is not strongly clean} \}$, we have

$$\overline{q} < 0.25. \tag{4.7}$$

   d. Let α be a barrier type, let $a \leqslant b < c$ and $b - a, c - b \leqslant 6\Delta$, and let $E(a, b, c, \alpha)$ be the event that there is a $d \in [b, c - 1]$ such that $[a, d]$ is a jump and a hole of type α' starts at d. Then

$$\mathrm{Prob}(E(a, b, c, \alpha)) \geqslant (c - b)^{\chi} h(\alpha). \tag{4.8}$$

   e. The functions $p(r)$ and $h(r)$ are monotonically decreasing.

♢

Remark 4.4. These conditions imply that the property that an interval I is strongly cleanly contained in some interval J depends only on $Z(J)$. ♢

4.2.5. *Reachability.* Take two random processes

$$\mathcal{M}_d \quad (d = 0, 1),$$

independent, and distributed like $\mathcal{M}$. To the pair of mazeries $\mathcal{M}_0, \mathcal{M}_1$ belongs a random graph

$$\mathcal{V} = \mathbb{Z}_+^2, \quad \mathcal{G} = (\mathcal{V}, \mathcal{E})$$

where $\mathcal{E}$ is determined by the above random processes as in Subsection 1.3. From now on, *reachability* is always understood in the graph $\mathcal{G}$. Just as the random sets $\mathcal{W}, \mathcal{H}$ of walls and holes were introduced as parts of what makes up a mazery $\mathcal{M}$, for example the random set $\mathcal{W}_0$ is the corresponding part of mazery $\mathcal{M}_0$. If $u = (x_0, x_1)$ and $v = (y_0, y_1)$ are points of $\mathcal{V}$ such that for $d = 0, 1$, $x_d < y_d$, and (x_d, y_d) is a hop then we will say that (u, v) is a *hop*.

The graph $\mathcal{G}$ is required to satisfy the following conditions.

Condition 4.5 (Reachability).

1. Let $u = (x_0, x_1)$. Suppose that a wall W of $\mathcal{W}_0$ starts at x_0, and a fitting hole H of $\mathcal{H}_1$ starts at x_1. Then $u \rightsquigarrow u + (|W|, |H|)$. The same holds when we interchange the indexes 0 and 1.
2. If u, v are points of $\mathcal{V}$ such that (u, v) is a hop and $\mathit{minslope}(u, v) \geqslant \sigma$, then $u \rightsquigarrow v$. We require

$$0 \leqslant \sigma < 0.5. \tag{4.9}$$

♢

Example 4.6. The compatible sequences problem can be seen as a special case of such a mazery. We define this mazery as $\mathcal{M}^1$, the first one of a series of mazeries $\mathcal{M}^k$ to be defined later. There is only one barrier type, of rank R_1 where the number R_1 will be chosen conveniently (sufficiently large) later, and one hole type. Let D_1 be any positive integer. We have $\Delta = D_1$ and $\sigma = 0$. The reader should check that if the probability $p = p(R_1) = p(R_1, 1)$ of the barrier type is chosen sufficiently small then the hole lower bound (4.8) and the bound (4.6) will still be satisfied. There is a wall of size 1 starting at j if $Z(j) = 1$, and a hole of size 1 if $Z(j) = 0$. Barriers are walls, and every point is strongly clean. ♢

5. The scaled-up structure

In this section, we will define the structural parts of the scaling-up operation $\mathbb{M} \mapsto \mathbb{M}^*$: we still postpone the definition of various parameters and probability bounds for $\mathbb{M}^*$. We start by specifying when we will consider walls to be sufficiently sparse and holes sufficiently dense, and what follows from this. Let Λ be a constant and f, g be some parameters with

$$\Lambda = 48, \tag{5.1}$$

$$g > 6\Delta, \tag{5.2}$$

$$\Lambda g / f < 0.5 - \sigma. \tag{5.3}$$

Let

$$\sigma^* = \sigma + \Lambda g / f. \tag{5.4}$$

We will say that the process $(\mathcal{M}_0, \mathcal{M}_1)$ satisfies the *grate condition* over the rectangle $I_0 \times I_1$ with parameters f, g if the following holds.

Condition 5.1 (Grate).

1. For $d = 0, 1$, for some $n_d \geqslant 0$, there is a sequence of neighbor walls $W_{d,1}, \dots, W_{d,n_d}$ and hops around them such that taking the walls along with the hops between and around them, the union is I_d (remember that a hop is a closed interval). Also, the hops between the walls have size $\geqslant f$, and the hops next to the ends of I_d (if $n_d > 0$) have size $\geqslant f/3$.
2. For every wall W occurring in I_d, for every subinterval J of size g of the hops between and around walls of $\mathcal{W}_{1-d}$ there is a hole of $\mathcal{H}_{1-d}$ fitting W, with its body cleanly contained in J.

$\diamondsuit$

Lemma 5.2 (Grate). *Recall the constant Λ from (5.1). Suppose that $(\mathcal{M}_0, \mathcal{M}_1)$ satisfies the grate condition over the rectangle $I_0 \times I_1$ with $I_d = [u_d, v_d]$, $u = (u_0, u_1)$, $v = (v_0, v_1)$. If*

$$\mathit{minslope}(u, v) \geqslant \sigma^*$$

then $u \rightsquigarrow v$ in $\mathcal{G}$.

We postpone the proof of this lemma to Section 8.

After defining the mazery $\mathbb{M}^*$, eventually we will have to prove the required properties. To prove Condition 4.5.2 for $\mathbb{M}^*$, we will invoke the Grate Lemma 5.2: for that, we will need the Grate Condition 5.1, with appropriate parameters f, g. To ensure these conditions in $\mathcal{M}^*$, we will introduce some new barrier types whenever they would fail; this way, they will be guaranteed to hold in a wall-free interval of $\mathcal{M}^*$. The set Btypes* will thus contain a new, so-called *emerging* barrier type, and several new, so-called *compound* barrier types, as defined below, and Htypes* will contain the corresponding hole types. The following algorithm creates the barriers, walls and holes of these new types out of $\mathcal{M}$. Let us denote by R a lower bound on all possible ranks, and by $R^* > R$ the number that will become the lower bound in $\mathcal{M}^*$. We will also make use of parameter λ defined in (3.2) with the property

$$R^* \leqslant 2R - \log_\lambda f. \tag{5.5}$$

Types of rank lower than R^* are called *light*, the other ones are called *heavy*.

1. (Cleanness) For an integer $r > 0$, a point x of $\mathcal{M}^*$ will be called r-left-clean if it is r-left-clean in $\mathcal{M}$ and there is no wall of $\mathcal{M}$ in $[x - r, x]$ whose right end is closer to x than $f/3$. Right-cleanness is defined similarly.

   The point x will be called strongly r-left-clean in $\mathcal{M}^*$ if it is strongly r-left-clean in $\mathcal{M}$ and there is no barrier of $\mathcal{M}$ in $[x - r, x]$ whose right end is closer to x than $f/3$. Strong right-cleanness is defined similarly.
2. (Emerging type) There is a new barrier type $\langle g \rangle$ in Btypes*, called the *emerging type*, with rank R^*, so it will be heavy. Suppose that there are $a \leqslant a' < b' \leqslant b$ with $a' - a \leqslant \Delta$, $b' - a' = g - 4\Delta$, $b - b' \leqslant \Delta$ and a light barrier type α such that no hole of type α' is cleanly contained in $[a', b']$. Then $((a, b), \langle g \rangle)$ is a barrier of $\mathcal{M}^*$.

   Let us list all barriers of type $\langle g \rangle$ in a (typically infinite) sequence $B_1, B_2, \dots$. We will mark some of these barriers as walls of $\mathcal{M}^*$ as follows. Let us go through the sequence $B_1, B_2, \dots$ in order. Suppose that we have already decided which of the barriers B_i, $i < n$ are walls. Then we mark B_n with body $[a, b]$ as a wall if and only if the following conditions hold:
   - $[a, b]$ does not intersect any of the barriers B_i, $i < n$ already marked as walls;
   - $[a, b]$ is a hop of $\mathcal{M}$, cleanly contained in a hop.

   There is a fitting hole type $\langle g \rangle'$. Any interval of size $g - 4\Delta$ is a hole of this type in $\mathcal{M}^*$, if it is a jump of $\mathcal{M}$.

3. (Compound types) We make use of a certain sequence of integers d_i, which is defined by the following formula, but only for $d_i \leqslant f$:

$$d_i = \begin{cases} i & \text{if } 0 \leqslant i < 17, \\ \lfloor \lambda^i \rfloor & \text{if } i \geqslant 17. \end{cases} \tag{5.6}$$

For any pair α_1, α_2 of barrier types where α_1 is light, and any $0 \leqslant i$ with $d_i \leqslant f$, we introduce a new type $\beta = \langle \alpha_1, \alpha_2, i \rangle$ in Btypes*. (The type α_2 can be not only a heavy type of $\mathcal{M}$ but also the new emerging type $\langle g \rangle$ just defined above.) Such types will be called *compound types*. The rank of this type is defined by

$$Rank(\beta) = Rank(\alpha_1) + Rank(\alpha_2) - i. \tag{5.7}$$

It follows from (5.5) that these new types are heavy. A barrier of type β occurs in $\mathcal{M}^*$ wherever disjoint barriers W_1, W_2 of types α_1, α_2 occur (in this order) at a distance $d \in [d_i, d_{i+1} - 1]$. (If d_{i+1} is not defined then $d \in [d_i, f - 1]$.) Thus, a shorter distance gives higher rank. The body of this compound barrier is the union of the bodies of W_1, W_2 and the interval between them. We denote the new compound barrier by

$$W_1 + W_2.$$

It is also a wall if W_1, W_2 are walls of $\mathcal{M}$ or $\mathcal{M}^*$ (we have already introduced some walls of $\mathcal{M}^*$, of the emerging type) and the interval between the W_i is a hop of $\mathcal{M}$. To make sure this construction always succeeds, we require

$$2f + \Delta < \Delta^*. \tag{5.8}$$

Compound hole types will be defined similarly, using a sequence e_i, $i = 1, 2, \ldots$ defined by the following formula, but only if $d_i \leqslant f$:

$$e_i = \begin{cases} i & \text{if } 0 \leqslant i \leqslant 17, \\ d_{i-1} & \text{if } i > 17. \end{cases} \tag{5.9}$$

Thus, the distance between the component holes of a compound hole of type $\beta' = \langle \alpha'_1, \alpha'_2, i \rangle$ varies in the interval $[e_i, e_{i+1} - 1]$. This hole type fits the barrier type $\beta = \langle \alpha_1, \alpha_2, i \rangle$. A hole of type β' occurs in $\mathcal{M}^*$ if holes H_1, H_2 of types α'_1, α'_2 occur, connected by a jump of $\mathcal{M}$ of size $d \in [e_i, e_{i+1} - 1]$.

Now we repeat the whole compounding step, introducing compound types in which now α_2 is required to be light. The type α_1 can be any type introduced until now, also a compound type introduced in the first compounding step. So, the walls that will occur as a result of the compounding operation are of the type L-$*$, $*$-L, or L-$*$-L, where L is a light wall of $\mathcal{M}$ and $*$ is any wall of $\mathcal{M}$ or an emerging wall of $\mathcal{M}^*$.

Note that emerging walls were made to be disjoint from each other, but compound walls were not.

4. (Finish) Remove all light types and all the corresponding barriers, walls and holes. Moreover, if a heavy wall W of $\mathcal{M}$ is contained in an outer clean light wall W' of $\mathcal{M}$ then with W', remove also W. (We do not know whether such heavy walls can really occur but their removal will not hurt.)

The graph $\mathcal{G}$ does not change in the scale-up: $\mathcal{G}^* = \mathcal{G}$.

Let us prove some of the required properties of $\mathbb{M}^*$.

Lemma 5.3. *The new mazery $\mathcal{M}^*$ satisfies Condition 4.3.1.*

Proof. We will see that all the properties in the condition follow essentially from the form of our definitions. Note that when an existential or universal quantifier is applied to a family of monotonically increasing (decreasing) events, the result is monotonically increasing (decreasing) event (as a function of the sequence Z). Indeed, these quantifiers are just the maximum and minimum operations.

Condition 4.3.1a says that for a barrier value E, the event $\{ E \in \mathcal{B} \}$ is an increasing function of $Z(Body(E))$. To check this, consider all possible barriers of $\mathcal{M}^*$. We have the following kinds:

– Heavy barrier values E of $\mathcal{M}$: all heavy barriers of $\mathcal{M}$ remained barriers of $\mathcal{M}^*$, so monotonicity holds, and the event still depends only on $Z(Body(E))$.
– Barrier values E of $\mathcal{M}^*$ of the emerging type: such a barrier is defined by the absence of some holes of $\mathcal{M}$ on some subintervals of $Body(E)$. Since the presence of a hole in $\mathcal{M}$ is a monotonically decreasing event, the presence of an emerging barrier is an increasing event depending only on $Z(Body(E))$.
– Barrier values E of some compound type. Such a barrier appears in $\mathcal{M}^*$ if certain barriers appear in $\mathcal{M}$ in $Body(E)$ in certain positions. Since barrier events are increasing in $\mathcal{M}$, compound barrier events of $\mathcal{M}^*$ depend only on $Z(Body(E))$, in an increasing way.

Condition 4.3.1b says that for a hole value E, the event $\{ E \in \mathcal{H} \}$ is a decreasing function of $Z(Body(E))$. To check this, consider all possible holes of $\mathcal{M}^*$. We have the following kinds:

– Heavy hole values E of $\mathcal{M}$: all holes of the heavy type of $\mathcal{M}$ remained holes of $\mathcal{M}^*$, so the event still depends only on $Z(Body(E))$ in a decreasing way.
– Hole values E of $\mathcal{M}^*$ of the emerging type: such a hole is defined by the property that $Body(E)$ is a jump. The jump property is a decreasing function of $Z(Body(E))$, and therefore so is the property of being a hole of the emerging type.
– Hole values E of some compound type. Such a hole appears in $\mathcal{M}^*$ if two holes and a jump appear in $\mathcal{M}$ in $Body(E)$ in certain positions. Since hole and jump events are decreasing functions of $Z(Body(E))$ in $\mathcal{M}$, compound barrier events of $\mathcal{M}^*$ depend only on $Z(Body(E))$, in an increasing way.

Condition 4.3.1c says first that for every point x and integer r, the events $\{ (x, -r) \in \mathcal{S} \}$, $\{ (x, -r) \in \mathcal{S} \}$ are decreasing functions of $Z((x - r, x))$ and $Z((x, x + r))$ respectively. The property that x is strongly left r-clean in in $\mathcal{M}^*$ is defined in terms of strong left-cleanness of x in $\mathcal{M}$ and the absence of certain barriers in $[x - r, x]$. Strong left r-cleanness in $\mathcal{M}$ is a decreasing function of $Z((x - r, x))$, so is the absence of barriers in $[x - r, x]$. Therefore strong left r-cleanness of x in $\mathcal{M}^*$ is a decreasing function of $Z((x - r, x))$.

Since both strong and regular left r-cleanness in $\mathcal{M}$ are decreasing functions of r, and the properties stating the absence of barriers/walls are decreasing functions of r, both strong and regular left r-cleanness are also decreasing functions of r in $\mathcal{M}^*$. The inequality $f/3 + \Delta < \Delta^*$, implies that these functions reach their minimum for $r = \Delta^*$. Similar relations hold for right-cleanness. □

Lemma 5.4. *The new mazery $\mathcal{M}^*$ defined by the above construction satisfies Conditions 4.3.2a and 4.3.2b.*

Proof. Let $E_1, E_2, \dots$ be the sequence of maximal external intervals of $\mathcal{M}$, of size $\geqslant f/3 (> \Delta)$. (We consider $(-\infty, 0) \subseteq E_1$, so that E_1 is automatically of size $\geqslant f$.) Let $I_1, I_2, \dots$ be the intervals betwen them. By Condition 4.3.2 of $\mathcal{M}$, each I_j can be covered by a sequence of neighbors W_{jk} in $\mathcal{M}$. Every wall of $\mathcal{M}$ intersects an element of this sequence. Each pair of these neighbors will be closer than f to each other. Indeed, each point of the hop between them belongs either to a wall intersecting one of the neighbors, or to a maximal external interval of size $\leqslant f/3$, so the distance between the neighbors is at most $2\Delta + f/3 \leqslant f$.

Now operation 2 above puts some new walls of the emerging type between the intervals I_j or into some of the hops, all of them disjoint from all existing walls and each other. If one of these new walls W comes closer than f to some interval I_j then we add W to the sequence W_{jk}; otherwise, we start a new sequence with W. If as a result of these additions (or originally), some of these sequences come closer than f to each other then we unite them. By the properties of $\mathcal{M}$ and the construction of walls of emerging type, the external intervals between the sequences are hops. Between the resulting sequences, the distance is $> f$. Within these new sequences, every pair of neighbors is closer than f.

Consider one of the above sequences, let us call its elements $W_1, W_2, \ldots$. If it consists of a single light wall W_1 then it is farther than f from all other sequences and operation 4 removes it. Since W_1 is surrounded by maximal intervals, any potential heavy wall W of $\mathcal{M}$ intersecting W_1 is contained in W_1 and the same operation removes W as well.

Let us show that the operations of forming compound walls can be used to create a sequence of consecutive neighbors W'_i of $\mathcal{M}^*$ spanning the same interval as $W_1, W_2, \ldots$. Assume that walls W_i for $i < j$ have been processed already, and a sequence of neighbors W'_i for $i < j'$ has been created in such a way that

$$\bigcup_{i<j} W_i \subset \bigcup_{i<j'} W'_i,$$

and W_j is not a light wall which is the last in the series. (This condition is satisfied when $j = 1$; indeed, each light wall W_i is part of some new wall via one of the compounding operations, since the ones that are not, would have been removed in operation 4.) We show how to create $W'_{j'}$.

If W_j is the last element of the series then it is heavy, and we set $W'_{j'} = W_j$. Suppose now that W_j is not last.

Suppose that it is heavy. If W_{j+1} is also heavy, or light but not last then $W'_{j'} = W_j$. Else $W'_{j'} = W_j + W_{j+1}$,

Suppose now that W_j is light: then it is not last. If W_{j+1} is last or W_{j+2} is heavy then $W'_{j'} = W_j + W_{j+1}$. Suppose that W_{j+2} is light. If it is last then $W'_{j'} = (W_j + W_{j+1}) + W_{j+2}$; otherwise, $W'_{j'} = W_j + W_{j+1}$. □

The following lemma is needed among others, for condition 4.5.2 in $\mathcal{M}^*$.

Lemma 5.5. *Suppose that interval I is a hop of $\mathcal{M}^*$. Then it is either also a hop of $\mathcal{M}$ or it contains a sequence $W_1, \ldots, W_n$ of light walls of $\mathcal{M}$ separated from each other by hops of $\mathcal{M}$ of size $\geqslant f$, and from the ends by hops of $\mathcal{M}$ of size $\geqslant f/3$.*

Proof. If I contains no walls of $\mathcal{M}$ then it is a hop of $\mathcal{M}$ and we are done. Let U be the union U of all walls of $\mathcal{M}$ in I. The inner cleanness of I in $\mathcal{M}^*$ implies that U is farther than $f/3$ from its ends. Condition 4.3.2 applied to U implies that U is spanned by a sequence of neighbor walls $W_1, W_2, \ldots$ of $\mathcal{M}$. Since I contains no walls of $\mathcal{M}^*$ (and thus no compound walls), these neighbor walls are farther than f from each other. None of these walls W_i is a wall of $\mathcal{M}^*$ therefore each is contained in an outer clean light wall W'_i. The sequence W'_i satisfies our condition: its members are still separated by hops of size $\geqslant f$, for the same reason as the W_i were. □

Lemma 5.6. *The new mazery $\mathcal{M}^*$ defined by the above construction satisfies Condition 4.3.2c.*

Proof. We will use

$$15\Delta < f, \tag{5.10}$$

which follows from (4.9), (5.2) and (5.3). Consider an interval I of size $3\Delta^*$ containing no walls of $\mathcal{M}^*$. Let I' be the middle third of I. By Condition 4.3.2a, it is contained in a hop of $\mathcal{M}^*$. Lemma 5.5 implies that I' is covered by a sequence $W_1, \dots, W_n$ of light neighbor walls of $\mathcal{M}$ separated from each other by hops of $\mathcal{M}$ of size $\geqslant f$, and surrounded by hops of $\mathcal{M}$ of size $\geqslant f/3$. By (5.8) we have $|I'| > 2f + \Delta$, and removing the W_i from I' leaves a subinterval $(a,b) \subseteq I'$ of size at least f. (If at least two W_i intersect I' take the interval between consecutive ones, otherwise I' is divided into two pieces of total length at least $2f$.) Now $K = (a + \Delta + f/3, b - \Delta - f/3)$ is an interval of length at least $f/3 - 2\Delta > 3\Delta$ which has distance at least $f/3$ from any wall of $\mathcal{M}$. There will be a clean point in the middle of K which will then be clean in $\mathcal{M}^*$. □

Let us look at the reachability conditions 4.5. Condition 4.5.1 will be satisfied for walls of the emerging type by the (easy) Lemma 6.5. For compound walls, it will be shown to be be satisfied in Subsection 6.3.

Lemma 5.7. *Consider the mazery in the middle of the scaling-up operation, after the construction of all the barriers, walls and holes of the emerging type. Let $[a, a+g]$ be an interval contained in a hop of $\mathcal{M}$ that contains no emerging walls. Then for all light barrier types α of $\mathcal{M}$, some hole of type α' is cleanly contained in $[a + 2\Delta, a + g - 2\Delta]$.*

Proof. Suppose that this is not the case. Then the operation of creating emerging barriers would turn every interval of the form $[a + \Delta + x, a + g - \Delta - y]$ with $0 \leqslant x, y \leqslant \Delta$ into an emerging barrier. Both $a + \Delta + x$ and $a + g - \Delta - y$ can be chosen to be clean. This choice would define an emerging wall. Since we assumed that we are at the point of the construction when no more emerging wall can be added, this is not possible. □

To check condition 4.5.2, we will proceed as follows. Let $u = (x_0, x_1)$ and $v = (y_0, y_1)$ be points of $\mathcal{V}^*$ with $\mathit{minslope}(u, v) \geqslant \sigma^*$, such that for $d = 0, 1$, the interval (x_d, y_d) is a hop in $\mathcal{M}^*$. We need to prove $u \rightsquigarrow v$ in $\mathcal{G}^*$. Since the intervals (x_d, y_d) are hops in $\mathcal{M}^*$, Lemmas 5.7 and 5.5 show that they satisfy the conditions of Lemma 5.2.

6. Emerging and compound types

In this section, we give bounds on the probabilities of emerging and compound barriers and holes, as much as this is possible without indicating the dependence on k.

6.1. **General bounds.** We start with some estimates that are needed for both kinds of types. Let $\overline{p}$ be some upper bound on the sum of all barrier probabilities:

$$\overline{p} \geqslant \sum_{r \geqslant R} p(r). \tag{6.1}$$

Let α be a barrier type, $a \leqslant b < c$ and $(b - a), (c - b) \leqslant 6\Delta$, and let $E(a, b, c, \alpha)$ be defined as in Condition 4.3.3d. We extend the bound (4.8) of this condition in several ways. These extension lemmas rely explicitly on Condition 4.3.3d. For the following lemma, remember the definition of $\overline{q}$ before (4.7).

Lemma 6.1. *Let $F(a, b, c, \alpha)$ be the event that $E(a, b, c, \alpha)$ occurs and also the end of the hole is strongly right-clean.*

$$\operatorname{Prob}(F(a, b, c, \alpha)) \geqslant (1 - \overline{q}) \operatorname{Prob}(E(a, b, c, \alpha)). \tag{6.2}$$

Proof. For $b \leqslant x \leqslant c + \Delta$, let E_x be the event that $E(a, b, c, \alpha)$ is realized by a hole ending at x but is not realized by any hole ending at any $y < x$. Let F_x be the event that x is strongly right-clean. Then $E(a, b, c, \alpha) = \bigcup_x E_x$, $F(a, b, c, \alpha) \supset \bigcup_x (E_x \cap F_x)$, the events E_x and F_x are independent for each x, and the events E_x are mutually disjoint. Hence

$$\begin{aligned}\operatorname{Prob}(F(a, b, c, \alpha)) &\geqslant \sum_x \operatorname{Prob}(E_x) \operatorname{Prob}(F_x) \geqslant (1 - \overline{q}) \sum_x \operatorname{Prob}(E_x) \\ &= (1 - \overline{q}) \operatorname{Prob}(E(a, b, c, \alpha)).\end{aligned}$$

□

We extend the hole lower bound (4.8) here to cases when $b - a > 6\Delta$, though with a different constant.

Lemma 6.2. *Let $a \leqslant b < c$ with $c - b \leqslant 6\Delta$: then we have*

$$\operatorname{Prob}(E(a, b, c, \alpha)) \geqslant (1 - \overline{q} - (c - a)\overline{p})(c - b)^{\chi} h(\alpha).$$

Proof. If $b - a \leqslant 6\Delta$ then we can apply the hole lower bound (4.8); suppose therefore that this does not hold. Let $a' = b - \Delta$. Then (4.8) is applicable to (a', b, c), and we get

$$\operatorname{Prob}(E(a', b, c, \alpha)) \geqslant (c - b)^{\chi} h(\alpha).$$

Consider the event C that a is strongly right-clean and the interval (a, c) contains no barriers. Then $C \cap E(a', b, c, \alpha) \subset E(a, b, c, \alpha)$. Event C is decreasing with $\operatorname{Prob}(C) \geqslant 1 - \overline{q} - (c - a)\overline{p}$. By the FKG inequality, we have $\operatorname{Prob}(E(a, b, c, \alpha)) \geqslant \operatorname{Prob}(C) \operatorname{Prob}(E(a', b, c, \alpha))$. □

Finally, we extend the hole lower bound (4.8) to cases when $c - b > 6\Delta$.

Lemma 6.3. *Let $a \leqslant b < c$.*

1. *We have*

$$\operatorname{Prob}(E(a, b, c, \alpha)) \geqslant (1 - \overline{q} - (c - a)\overline{p})(0.6 \wedge (0.1(c - b)^{\chi} h(\alpha))). \tag{6.3}$$

2. *If the event $E^*(a, b, c, \alpha)$ is defined like $E(a, b, c, \alpha)$ except that the jump in question must be a jump of $\mathcal{M}^*$ then, assuming $\overline{p}^* \leqslant \overline{p}$, we have*

$$\operatorname{Prob}(E^*(a, b, c, \alpha)) \geqslant (1 - \overline{q}^* - 2(c - a)\overline{p})(0.6 \wedge (0.1(c - b)^{\chi} h(\alpha))). \tag{6.4}$$

Proof. We will use the following inequality, which can be checked by direct calculation. Let $v = 1 - 1/e = 0.632\ldots$, then for $x > 0$ we have

$$1 - e^{-x} \geqslant v \wedge vx. \tag{6.5}$$

In view of Lemma 6.2, for the first statement of the lemma, we only need to consider the case $c - b > 6\Delta$.

Let $n = \left\lfloor \frac{c-b}{3\Delta} \right\rfloor - 1$, then we have

$$n\Delta \geqslant (c - b)/9. \tag{6.6}$$

Let

$$\begin{aligned} &a_i = b + 3i\Delta, \quad E_i = E(a_i, a_i + \Delta, a_i + 2\Delta, \alpha), \quad \text{for } i = 1, \ldots, n, \\ &E = \bigcup_i E_i, \quad s = \Delta^{\chi} h(\alpha). \end{aligned}$$

Inequality (4.8) is applicable to E_i and also $s \leqslant 0.6$ by (4.6). We have $\operatorname{Prob}(E_i) \geqslant s$, hence $\operatorname{Prob}(\neg E_i) \leqslant 1 - s \leqslant e^{-s}$. The events E_i are independent, so

$$\operatorname{Prob}(E) = 1 - \prod_i \operatorname{Prob}(\neg E_i) \geqslant 1 - e^{-ns} \geqslant 0.6 \wedge (0.6ns), \tag{6.7}$$

where in the last step we used (6.5). We have

$$ns = n\Delta^{\chi} h(\alpha). \tag{6.8}$$

Now, by (6.6), we have $n\Delta^{\chi} \geqslant (\Delta n)^{\chi} \geqslant 9^{-\chi}(c-b)^{\chi}$. Substituting into (6.7), (6.8):

$$\mathrm{Prob}(E) \geqslant 0.6 \wedge (0.6 \cdot 9^{-\chi}(c-b)^{\chi} h(\alpha)) \geqslant 0.6 \wedge (0.1(c-b)^{\chi} h(\alpha)),$$

where we used $\chi < \log_9 6$, which follows from (3.1). The event E implies that a hole of type α' starts in $[b, c-1]$ and that the left end of the hole is strongly left-clean. Let C be the event that a is strongly right-clean and that there is no barrier in (a, c). If also C holds then there is a jump from a to our hole, which we need. The event C is decreasing, so the FKG inequality implies that $\mathrm{Prob}(C) \geqslant 1 - \overline{q} - (c-a)\overline{p}$ can be multiplied with $\mathrm{Prob}(E)$ for a lower bound. For the second statement, the requirements must be added that there are no barriers of $\mathcal{M}^*$ in (a, c), and that a must be strongly right-clean in $\mathcal{M}^*$. We can replace $\overline{q}$ with the larger $\overline{q}^*$, and we can replace $\overline{p}$ with $2\overline{p} \geqslant \overline{p} + \overline{p}^*$. □

Remark 6.4. It may seem that the proof of Lemmas 6.2 and 6.3 would go through even with $(c-b)h(\alpha)$ in place of $(c-b)^{\chi}h(\alpha)$. However, we used inequality (4.6) (the smallness needed for the approximation by e^x). Even if we assume that this inequality holds for $\mathcal{M}$ without χ, we can prove it for $\mathcal{M}^*$ only with χ. ◇

6.2. **The emerging type.** Recall the definition of an emerging type in part 2 of the scale-up algorithm in Section 5. Recall also that dimension 0 is the direction of the X sequence and dimension 1 the direction of the Y sequence, thus $\mathcal{W}_0$ is the set of walls defined by X.

Lemma 6.5. *If a wall of the emerging type $\langle g \rangle$ and size w_1 begins at x in $\mathcal{W}_0$ and a hole of type $\langle g \rangle'$ and size w_2 begins at y in $\mathcal{H}_1$ then in graph $\mathcal{G}$ there is a path of slope $\leqslant 1$ from (x, y) to $(x + w_1, y + w_2)$.*

Proof. This follows directly from the reachability condition 4.5.2, if we observe that $w_2 \leqslant w_1 \leqslant 2w_2$, so that the the minslope is $\geqslant 1/2$. □

Lemma 6.6. *Let $n = \left\lfloor \frac{g - 5\Delta}{3\Delta} \right\rfloor$. For any point x, the expression*

$$2\Delta |\mathrm{Btypes}| e^{-(1-\overline{q})nh(R^*)} \tag{6.9}$$

is an upper bound on the sum, over all w, of the probabilities that an emerging barrier of type $\langle g \rangle$, with rank R^* and size w, starts at x.

Proof. Recall the definition of an emerging barrier. Suppose that there are $a \leqslant a' < b' \leqslant b$ with $a' - a \leqslant \Delta$, $b' - a' = g - 4\Delta$, $b - b' \leqslant \Delta$ and a light barrier type α such that no hole of type α' is cleanly contained in $[a', b']$. Then $((a, b), \langle g \rangle)$ is a barrier of $\mathcal{M}^*$. This definition implies that if (x, y) is an emerging barrier then there is a light barrier type α such that no hole of type α' is cleanly contained in both $[x + 2\Delta, x + g - 6\Delta]$ and $[x + \Delta, x + g - 5\Delta]$.

Let us fix α, and for any $i = 0, \dots, n-1$, let $\mathcal{A}(i, \alpha)$ be the event that no strongly outer-clean hole of type α' starts at $x + (3i+1)\Delta$. For each fixed α, these events are independent. Indeed, according to Condition 4.3.1, event $\mathcal{A}(i, \alpha)$ only depends on $Z((x + 3i\Delta), (x + 3(i+1)\Delta))$.

Recall the hole lower bound: Condition 4.3.3d. It says the following. For $a \leqslant b < c$ and $b - a, c - b \leqslant 6\Delta$, and let $E(a, b, c, \alpha)$ be the event that there is a $d \in [b, c-1]$ such that $[a, d]$ is a jump and a hole of type α' starts at d. Then

$$\mathrm{Prob}(E(a, b, c, \alpha)) \geqslant (c-b)^{\chi} h(\alpha).$$

With $a = x + 3i\Delta$, $b = x + (3i + 1)\Delta$, $c = x + (3i + 1)\Delta + 1$, this implies that with probability at least $h(\alpha) \geqslant h(R^*)$, a strongly left-clean hole starts at $x + (3i + 1)\Delta$. Lemma 6.1 implies that with probability at least $h(R^*)(1 - \overline{q})$, this hole is also strongly right-clean, and so it is strongly outer-clean. Hence, each of the independent events $\mathcal{A}(i, \alpha)$ has a probability upper bound $1 - (1 - \overline{q})h(R^*)$. The probability that all the events $\mathcal{A}(i, \alpha)$ hold is bounded by $(1 - (1 - \overline{q})h(R^*))^n \leqslant e^{-n(1-\overline{q})h(R^*)}$. The probability that one of these events holds for some α is at most |Btypes| times larger. The sizes of emerging barriers vary in the range $[g - 4\Delta, g - 2\Delta]$, hence the factor 2Δ. □

6.3. **Compound types.**

Lemma 6.7. *For $i = 1, 2$, let W_i be two neighboring walls of $\mathcal{W}_0$ starting at points $x_1 < x_2$, with sizes w_i and at distance d. Let H_i, for $i = 1, 2$ be two disjoint holes of $\mathcal{H}_1$ with sizes h_i, with starting points y_i where H_i fits W_i. Suppose that the interval between these holes is a hop. Let e be the distance between H_i.*

(a) *The reachability relation $(x_1, y_1) \rightsquigarrow (x_2 + w_2, y_2 + h_2)$ is implied by the inequality*

$$\sigma d \leqslant e \leqslant d. \tag{6.10}$$

(b) *Let $\underline{d} < \overline{d}$, $\underline{e} < \overline{e}$, be integers satisfying*

$$\sigma(\overline{d} - 1) \leqslant \underline{e} \leqslant \overline{e} - 1 \leqslant \underline{d}. \tag{6.11}$$

If $\underline{d} \leqslant d < \overline{d}$ and $\underline{e} \leqslant e < \overline{e}$ then inequality (6.10) is satisfied.

Proof. Assume (6.10). As W_1 is passed by H_1, there is a path from $u_1 = (x_1, y_1)$ to

$$u_2 = (x_1 + w_1,\ y_1 + h_1).$$

Due to (6.10) there is a path from u_2 to $u_3 = (x_2, y_2)$. As W_2 is passed by H_2, there is a path from u_3 to $(x_2 + w_2,\ y_2 + h_2)$. The total slope of the combination of these three paths is clearly at most 1. (For reachability, the total slope condition is not important; but, a compound hole will need to satisfy Condition 4.1.2.) The proof of statement (b) is immediate. □

Recall the definition of compound (barrier and wall) types given in part 3 of the scale-up algorithm of Section 5. In (5.6), we defined a sequence of integers d_i. For any pair α_1, α_2 of barrier types where α_1 is light, and any $0 \leqslant i$ with $d_i \leqslant f$, there is a new type $\beta = \langle \alpha_1, \alpha_2, i \rangle$ in Btypes*. A barrier of type β occurs in $\mathcal{M}^*$ wherever disjoint barriers W_1, W_2 of types α_1, α_2 occur (in this order) at a distance $d \in [d_i, d_{i+1} - 1]$. Let the barriers W_i have starting points $x_1 < x_2$, and sizes w_i. The body of the new barrier is $(x_1, x_2 + w_2)$. The barrier becomes a wall if the component barriers are walls and the interval between them is a hop.

For compound hole types, we used the sequence e_i defined in (5.9). A hole of type β' occurs in $\mathcal{M}^*$ if holes H_1, H_2 of types α'_1, α'_2 occur, connected by a jump of $\mathcal{M}$ of size $d \in [e_i, e_{i+1} - 1]$.

Lemma 6.8. *For each i if we set $\underline{d} = d_i$, $\overline{d} = d_{i+1}$, $\underline{e} = e_i$, $\overline{e} = e_{i+1}$ then inequality (6.11) is satisfied. Therefore for each compound type β, if W is a vertical compound wall of type β with body (x_1, x_2) and and H is a horizontal wall of type β' with body (y_1, y_2) then $(x_1, y_1) \rightsquigarrow (x_2, y_2)$: the hole "passes" through the wall.*

Proof. Assume $i < 16$. then $d_i = i = e_i$, and $d_{i+1} = e_{i+1} = i + 1$. Inequalities (6.11) turn into the true inequalities $\sigma i \leqslant i \leqslant i \leqslant i$. Assume $i = 16$, then $d_i = e_i = 16$, and $d_{i+1} = 19$, $e_{i+1} = 17$. Inequalities (6.11) will turn into the true inequalities $\sigma \cdot 18 \leqslant 16 \leqslant 16 \leqslant 16$. Assume $i = 17$, then $d_i = 19$, $d_{i+1} = 22$, $e_i = 17$, $e_{i+1} = 19$. Inequalities (6.11) will turn into the true inequalities $\sigma \cdot 21 \leqslant 17 \leqslant 18 \leqslant 19$. Assume $i > 17$, then $d_i = \lfloor \lambda^i \rfloor$, $e_i = \lfloor \lambda^{i-1} \rfloor$. Given $\sigma \leqslant \frac{1}{2}$ and $\lambda = 2^{1/4}$, what we need to check from (6.11) is

$$\frac{1}{2}(\lfloor \lambda^{i+1} \rfloor - 1) \leqslant \lfloor \lambda^{i-1} \rfloor \leqslant \lfloor \lambda^i \rfloor - 1,$$

which is true for $i > 17$. □

Lemma 6.9. *For any r_1, r_2, the sum, over all w, of the probabilities for the occurrence of a compound barrier of type $\langle \alpha_1, \alpha_2, i \rangle$ with $Rank(\alpha_j) = r_j$ and width w at a given point x_1 is bounded above by*

$$(d_{i+1} - d_i)p(r_1)p(r_2). \tag{6.12}$$

Proof. For fixed r_1, r_2, x_1, d, let $B(d, w)$ be the event that a compound barrier of any type $\langle \alpha_1, \alpha_2, i \rangle$ with $Rank(\alpha_j) = r_j$, distance d between the component barriers, and size w appears at x_1. For any w, let $A(x, r, w)$ be the event that a barrier of rank r and size w starts at x. We can write

$$B(d, w) = \bigcup_{w_1 + d + w_2 = w} A(x_1, r_1, w_1) \cap A(x_1 + w_1 + d, r_2, w_2),$$

where events $A(x_1, r_1, w_1)$, $A(x_1 + w_1 + d,\ r_2, w_2)$ are independent. By (4.2):

$$\mathrm{Prob}(B(d, w)) \leqslant \sum_{w_1 + d + w_2 = w} p(r_1, w_1)p(r_2, w_2).$$

Hence by (4.5):

$$\sum_w \mathrm{Prob}(B(d, w)) \leqslant \sum_{w_1} p(r_1, w_1) \sum_{w_2} p(r_2, w_2) \leqslant p(r_1)p(r_2).$$

□

Lemma 6.10. *Consider the compound hole type β' where $\beta = \langle \alpha_1, \alpha_2, i \rangle$. For $a \leqslant b < c$, let $E_2(a, b, c, \beta)$ be the event that there is a $d \in [b, c-1]$ such that $[a, d]$ is a jump of $\mathcal{M}^*$, and a compound hole of type β starts at d. Assume*

$$c - a \leqslant 12\Delta^*, \quad (\Delta^*)^\chi h(\alpha_i) \leqslant 0.5, \quad \overline{p}^* \leqslant \overline{p}. \tag{6.13}$$

Then we have

$$\mathrm{Prob}(E_2(a, b, c, \beta)) \geqslant 0.01(1 - 2\overline{q}^* - 25\Delta^*\overline{p})(c - b)^\chi (e_{i+1} - e_i)^\chi h(\alpha_1)h(\alpha_2). \tag{6.14}$$

Proof. Let $\underline{e} = e_i$, $\overline{e} = e_{i+1}$. For each $x \in [b, c + \Delta - 1]$, let A_x be the event that there is a $d_1 \in [b, c-1]$ such that $[a, d_1]$ is a jump of $\mathcal{M}^*$, and a hole of type α_1' starts at d_1 and ends at x, and that x is the smallest possible number with this property. Let B_x be the event that there is a $d_2 \in [x + \underline{e}, x + \overline{e})$ such that $[x, d_2]$ is a jump of $\mathcal{M}$, and a hole of type α_2' starts at d_2. Then $E_2(a, b, c, \beta) \supset \bigcup_x (A_x \cap B_x)$, and for each x, the events A_x, B_x are independent. We have, using the notation of Lemma 6.3:

$$\begin{aligned}\sum_x \mathrm{Prob}(A_x) &= \mathrm{Prob}(E^*(a, b, c, \alpha_1)) \\ &\geqslant (1 - \overline{q}^* - 2(c - a)\overline{p})(0.6 \wedge (0.1(c - b)^\chi h(\alpha_1))).\end{aligned}$$

Further, using the same lemma:

$$\mathrm{Prob}(B_x) = \mathrm{Prob}(E(x, x + \underline{e}, x + \overline{e}, \alpha_2)) \geqslant (1 - \overline{q} - \overline{e}\overline{p})(0.6 \wedge (0.1(\overline{e} - \underline{e})^\chi h(\alpha_2))).$$

By the assumptions (6.13): $0.1(c-b)^{\chi}h(\alpha_1) \leqslant 0.1(12\Delta^*)^{\chi}h(\alpha_1) \leqslant 0.6$, hence the operation $0.6\wedge$ can be deleted. The same reasoning applies to the second application of $0.6\wedge$. Combining these, using $\overline{q}^* > \overline{q}$, $\overline{p}^* < \overline{p}$:

$$\begin{aligned}\operatorname{Prob}(E_2(a,b,c,\beta)) &\geqslant \sum_x \operatorname{Prob}(A_x)\operatorname{Prob}(B_x)\\ &\geqslant (1-2\overline{q}^* - (2(c-a)+\overline{e})\overline{p})0.01(c-b)^{\chi}(\overline{e}-\underline{e})^{\chi}h(\alpha_1)h(\alpha_2).\end{aligned}$$

□

7. The scale-up functions

7.1. **Parameters.** Lemma 2.5 says, with $p = \operatorname{Prob}\{\, Z(i) = 1\,\}$, that there is a p_0 such that if $p < p_0$ then the sequence $\mathcal{M}^k$ can be constructed in such a way that (2.4) holds. Our construction has several parameters. If we computed p_0 explicitly then all these parameters could be turned into constants: but this is unrewarding work and it would only make the relationships between the parameters less intelligible. We prefer to name all these parameters, point out the necessary inequalities among them, and finally show that if p is sufficiently small then all these inequalities can be satisfied simultaneously.

Recall that the slope lower bound σ must satisfy

$$\sigma < \sigma_0 = 1/2. \tag{7.1}$$

The parameter $\lambda > 0$ was introduced in (3.2). We will use a parameter R (not necessarily integer) for a lower bound on all ranks in the mazery.

It can be seen from the definition of compound ranks in (5.7) and from Lemma 6.9 that the probability bound $p(r)$ of a barrier type should be approximately λ^{-r}. We introduce an upper bound that is a little smaller:

$$p(r) = c_2 r^{-c_1}\lambda^{-r} \tag{7.2}$$

where c_1 has been defined in (4.3) above, and

$$0 < c_2 = 0.15 < 1. \tag{7.3}$$

The inequality requiring this definition is (7.6) below. The term $c_2 r^{-c_1}$, just like the factor in the function $h(r)$ defined in the hole lower bound (4.8), serves for absorbing some lower-order factors that arise in estimates like (6.14). We define c_3 and then c_0 by

$$c_0 c_2^{\chi} = c_3 = 436, \tag{7.4}$$

this gives

$$h(r) = c_3\lambda^{-r\chi}.$$

The value of c_3 is required by the inequality (7.26) below. This defines c_0 implicitly using the values of χ from (3.1) and c_2 from (7.3).

By these definitions, we can give a concrete value to the upper bound $\overline{p}$ introduced in (6.1):

$$\overline{p} = \lambda^{-R}, \quad T = 1/\overline{p} = \lambda^R \tag{7.5}$$

where we have also introduced its inverse, T.

Lemma 7.1. *The above definition of $\overline{p}$ satisfies (6.1).*

Proof. We have

$$\sum_{r\geqslant R} p(r) < c_2 \sum_{r\geqslant R} \lambda^{-r} = \lambda^{-R}\frac{c_2}{1-1/\lambda} < \lambda^{-R} \quad \text{if} \quad c_2 < 1 - 1/\lambda, \tag{7.6}$$

which is satisfied by the choice c_2 in (7.3). □

Several other parameters of $\mathcal{M}$ and the scale-up are expressed conveniently in terms of T:

$$\Delta = T^{\delta}, \quad f = T^{\phi}, \quad g = T^{\gamma}, \quad 0 < \delta < \gamma < \phi < 1. \tag{7.7}$$

To obtain the new rank lower bound, we multiply R by a constant:

$$R = R_k = R_1\tau^{k-1}, \quad R_{k+1} = R^* = R\tau, \quad 1 < \tau, R_1. \tag{7.8}$$

It is convenient to introduce $R_0 = R_1/\tau$, to be able to write $R_k = R_0\tau^k$. Setting R_0 large enough will enable to us to satisfy any inequality of the form $cT^{-x} < 1$ for each mazery in the sequence as long as the constants c and x are strictly positive. We will collect the required bounds on R_0 as we go along. A few more consequences of these definitions:

$$T^* = \lambda^{R^*} = \lambda^{R\tau} = T^{\tau}, \quad \Delta^* = \Delta^{\tau}, \quad \log_\lambda f = R\phi.$$

A bound on τ has been indicated in the requirement (5.5) which will be satisfied if

$$\tau \leqslant 2 - \phi. \tag{7.9}$$

Let us make sure that Condition 4.1.1 is satisfied by $\mathcal{M}^*$. Barriers of the emerging type have size at most $g - 2\Delta$, and at the time of their creation, they are the largest existing ones. We get the largest new barriers when the compound operation combines these with light barriers on both sides, leaving the largest gap possible, so the largest new barrier size is $g - 2\Delta + 2(f + \Delta) \leqslant g + 2f \leqslant 3f$, where we used (5.3), (5.2). Hence any value larger than $3f$ can be chosen as $\Delta^* = \Delta^{\tau}$. With R_0 large enough, we always get this if

$$\phi < \delta\tau < 1 \tag{7.10}$$

(where the second inequality will also be needed).[1] We can satisfy (5.2) similarly, if

$$R \geqslant 130. \tag{7.11}$$

The exponent χ has been part of the definition of a mazery: it is the power by which, roughly, hole probabilities are larger than barrier probabilities. We require

$$0 < \chi < (\gamma - \delta)/\tau. \tag{7.12}$$

Lemma 7.2. *The exponents $\delta, \phi, \gamma, \tau, \chi$ can be chosen to satisfy the inequalities* (7.7), (7.8), (7.9), (7.10), (7.12).

Proof. We can choose χ last, to satisfy (7.12), so consider just the other inequalities. Choose $\tau = 2 - \phi$ to satisfy (7.9); then (7.7) and (7.10) will be satisfied if $\delta < \gamma < \phi < \delta(2 - \phi) < 1$. This is achieved by

$$\delta = 0.4, \quad \gamma = 0.45, \quad \phi = 0.5, \quad \chi = 0.03, \text{ hence } \tau = 1.5. \tag{7.13}$$

□

Let us fix now all these exponents. In order to satisfy all our requirements also for small k, we fix c_1 next; then we fix c_0 and finally R_0. Each of these last three parameters just has to be chosen sufficiently large as a function of the previous ones.

We need upper bounds on the largest ranks, and on the number of types.

[1] More exactly, we need $R \geqslant 64$.

Lemma 7.3.

1. *The quantity* $R\frac{2\tau}{\tau-1}$ *is an upper bound on all existing ranks in a mazery. Hence every rank exists in* $\mathcal{M}^k$ *for at most* $\log_\tau \frac{2\tau}{\tau-1}$ *values of* k*.*
2. *We have, denoting for the moment* $c = \log_\tau 3$*:*

$$|\text{Btypes}| < \frac{R_0^{(R/R_0)^c}}{\tau R} = \frac{R_0^{3^k}}{\tau R} \tag{7.14}$$

Proof. For the moment, let us denote the largest existing rank by $\overline{R}$. Emerging types got a rank equal to R^*, and the largest rank produced by the compound operation is at most $\overline{R} + 2R^*$ (since the compound operation is applied twice), hence $\overline{R}_{k+1} \leqslant \overline{R}_k + 2R_{k+1}$. Since also $\overline{R}_1 \leqslant 2R_1$ (since there is only one rank in $\mathcal{M}^1$), we have for $k \geqslant 1$:

$$\overline{R}_k \leqslant 2\sum_{i=1}^{k} R_i = 2R_0\tau\frac{\tau^k - 1}{\tau - 1} \leqslant R_k\frac{2\tau}{\tau - 1}. \tag{7.15}$$

Now for the number of types. There is only one emerging type. The operation of forming compound types multiplies the number of types at most by the number of values i in the definition (5.6): this is $\leqslant f$ for $f < 17$ and $\leqslant \log_\lambda f = R\phi$ otherwise. For the moment, let N denote the number of barrier types. The operation of forming compound types once results in multiplying N by at most $NR\phi$ and adding the result to N. We have to repeat this operation twice, and use $\phi = 0.5$:

$$N^* \leqslant 1 + N(1 + NR\phi)^2 = R^2N^3(R^{-2}N^{-3} + R^{-2}N^{-2} + R^{-1}N^{-1} + 0.25) < R^2N^3$$

if $R \geqslant 3$. This recursive inequality leads to the estimate (7.14). This is straightforward with the recursion $N^* \leqslant N^3$ since what we are proving is $N_k \leqslant R_0^{3^k-1}/\tau^{k+1}$. The divisor R in (7.14) absorbs the effect of the factor R^2 in the recursion. □

7.2. **Probability bounds after scale-up.** The structures $\mathcal{M}^k$ are now defined but we have not proved yet that they are mazeries, since not all inequalities required in the definition of mazeries have been verified yet.

Lemma 7.4. *If* R_0 *is sufficiently large then for each* k*, for the structure* $\mathcal{M}^k$*, for any barrier type* α*, inequality* (4.6) *holds; we also have*

$$\sum_k \Delta_{k+1}\overline{p}_k < 0.5. \tag{7.16}$$

Proof. Let us prove (4.6), which says $\Delta^\chi h(R) < 0.6$. We have $\Delta^\chi h(R) = T^{\delta\chi}c_3T^{-\chi} = c_3T^{-\chi(1-\delta)}$ which is smaller than 0.6 if R_0 is sufficiently large.[2] For inequality (7.16), note that

$$\sum_k \Delta_{k+1}\overline{p}_k = \sum_k \lambda^{-R_0\tau^k(1-\delta\tau)}$$

which because of (7.10), is clearly less than 0.5 if R_0 is large.[3] □

Recall the constant Λ defined in (5.1). Note that for R_0 large enough, the relations

$$\Delta^*\overline{p} < 0.5(0.25 - \overline{q}), \tag{7.17}$$

$$\Lambda g/f < 0.5(\sigma_0 - \sigma). \tag{7.18}$$

[2] More exactly, we need $R \geqslant 2113$.

[3] More exactly, we need at most $R_0 \geqslant 31$.

hold for $\mathcal{M} = \mathcal{M}^1$.[4] Further, since $\sigma_1 = 0$, for (7.18) we need

$$\sigma_0/2 > \Lambda g/f = \Lambda T^{-(\phi-\gamma)}, \tag{7.19}$$

satisfied if $R \geqslant 1517$. The following lemma establishes Condition 4.3.3c and inequality (5.3) for all k.

Lemma 7.5. *Suppose that the structure $\mathcal{M} = \mathcal{M}^k$ is a mazery and it satisfies (7.17) and (7.18). Then $\mathcal{M}^* = \mathcal{M}^{k+1}$ also satisfies these inequalities.*

Proof. The probability that a point a is strongly clean in $\mathcal{M}$ but not in $\mathcal{M}^*$ is clearly upperbounded by $\Delta^*\overline{p}$, which upperbounds the probability that a barrier of $\mathcal{M}$ appears in $[a - f/3 - \Delta, a + f/3 + \Delta]$:

$$\overline{q}^* - \overline{q} \leqslant \Delta^*\overline{p} = T^{\delta\tau-1}.$$

For sufficiently large R_0, we will always have $\Delta^{**}\overline{p}^* < 0.5\Delta^*\overline{p}$. Indeed, this says $(T^{\delta\tau-1})^\tau < 0.5T^{\delta\tau-1}$, which is satisfied if $R \geqslant 21$. This implies that if (7.17) holds for $\mathcal{M}$ then it also holds for $\mathcal{M}^*$. For the inequality (7.18), since the scale-up definition (5.4) says $\sigma^* - \sigma = \Lambda g/f$, the inequality

$$\Lambda g^*/f^* < 0.5(\sigma_0 - \sigma^*)$$

will be guaranteed if R_0 is large.[5] □

Lemma 7.6. *If R_0 is sufficiently large then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery.*

1. *For any point x, the sum, over all w, of the probabilities that a barrier of the emerging type of rank r and size w starts at x is at most $p(r)/2$.*
2. *For the emerging barrier type the fitting emerging holes satisfy the hole lower bound (4.8).*

Proof. Recall Lemma 6.6. Let $n = \left\lfloor \frac{g-5\Delta}{3\Delta} \right\rfloor$. For any point x, the expression

$$2\Delta|\text{Btypes}|e^{-(1-\overline{q})nh(R^*)}$$

is an upper bound on the sum, over all w, of the probabilities that an emerging barrier of type $\langle g \rangle$ (with rank R^*) starts at x. We have

$$\begin{aligned} n &> g/(3\Delta) - 8/3 = T^{\gamma-\delta}/3 - 8/3, \\ h(R^*) &= c_3 T^{-\tau\chi}, \\ (1-\overline{q})nh(R^*) &> T^{\gamma-\delta-\tau\chi}c_3/6 - 1. \end{aligned}$$

Due to (7.12), this expression grows exponentially in R, and $e^{-(1-\overline{q})nh(R^*)}$ decreases double exponentially in R. It follows from (7.14) that its multiplier $2\Delta|\text{Btypes}|$ only grows exponentially in a power of R. Hence for large enough R_0, the product decreases double exponentially in R. So, for sufficiently large R_0, claim 1 follows.[6]

To prove claim 2, let $\alpha = \langle g \rangle$ be the emerging barrier type, let $a \leqslant b < c$ and $b - a, c - b \leqslant 6\Delta^*$, and let $E(a, b, c, \alpha)$ be the event that there is a $d \in [b, c-1]$ such that (a, d) is a jump, and a hole of type α' starts at d. We will be done if we prove

$$\text{Prob}(E(a, b, c, \alpha)) \geqslant (c-b)^\chi h(\alpha). \tag{7.20}$$

[4] More exactly, since $\overline{q}_1 = 0$, for (7.17) we need $R \geqslant 31$.

[5] More exactly, if $R_0 \geqslant 401$.

[6] More exactly, we need $R_0 \geqslant 1000$.

Let $\mathcal{F}$ be the event that a is strongly right-clean in $\mathcal{M}$, that b is strongly clean and $b+g-4\Delta$ is strongly left-clean in $\mathcal{M}$ and that no barrier of $\mathcal{M}$ occurs in $[a, b+\Delta^*]$. By the definition of emerging holes, $\mathcal{F}$ implies the event $E(a,b,c,\alpha)$, since $[b, b+g-4\Delta]$ will be an emerging hole. Clearly,

$$\operatorname{Prob}(\mathcal{F}) \geqslant 1 - 3\overline{q} - 7\Delta^*\overline{p}. \tag{7.21}$$

Lemma 7.5 implies $\overline{q} < 0.25$, and we have

$$7\Delta^*\overline{p} = 7T^{\delta\tau - 1}.$$

By (7.10), this is < 0.1 if R_0 is sufficiently large.[7] Hence the right-hand side of (7.21) can be lowerbounded by 0.1. The required lower bound of (4.8) is

$$(c-b)^\chi h(\alpha) \leqslant (6\Delta^*)^\chi h(R^*) = (6T^{\tau\delta})^\chi h(R^*) = c_3 6^\chi T^{-\tau\chi(1-\delta)} < 0.1$$

if R_0 is sufficiently large.[8] □

Lemma 7.7. *For sufficiently large R_0, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. After one operation of forming compound types, for any rank r and any point x, the sum, over all w, of the probabilities for the occurrence of a compound barrier of rank r and size w at point x is at most $p(r)R^{-c_1/2}$.*

Proof. Let α_1, α_2 be two types with ranks r_1, r_2. Assume without loss of generality that $r_1 \leqslant r_2$ and that α_1 is light: $r_1 < R^* = R^\tau$. With these, according to part 3 of the scale-up algorithm, we can form compound barrier types $\langle \alpha_1, \alpha_2, i\rangle$, as long as $d_i < f$. This gives a type of rank $r_1 + r_2 - i$, for all $i \leqslant \log_\lambda f = R\phi$. The bound (6.12) and the definition of $p(r)$ in (7.2) shows that the contribution by this term to the sum (over w) of probabilities that a barrier of size w and rank $r = r_1 + r_2 - i$ starts at x is at most

$$(d_{i+1} - d_i)p(r_1)p(r_2) \leqslant \lambda^{i+1} p(r_1)p(r_2) = c_2^2 \lambda^{-(r_1+r_2-i-1)}(r_1 r_2)^{-c_1}.$$

Now we have $r_1 r_2 \geqslant R r_2 \geqslant (R/2)(r_1+r_2) \geqslant rR/2$, hence the above bound reduces to $c_2^2\lambda^{-r+1}(rR/2)^{-c_1}$. The total contribution to the sum for rank r is therefore at most

$$\begin{aligned} &c_2^2\lambda^{-r+1}(rR/2)^{-c_1} |\{\, (i, r_1) : i \leqslant R\phi,\ r_1 < R^\tau \,\}| \leqslant c_2^2\lambda^{-r+1}(rR/2)^{-c_1}\phi R^{\tau+1} \\ &= p(r)R^{-c_1/2} c_2 2^{c_1}\lambda\phi R^{-(c_1/2-\tau-1)} < p(r)R^{-c_1/2}, \end{aligned}$$

where in the last step we used

$$R > (c_2 2^{c_1}\lambda\phi)^{\frac{1}{c_1/2-\tau-1}}, \tag{7.22}$$

satisfied if $R_0 \geqslant 26$. □

Lemma 7.8. *Suppose that each structure $\mathcal{M}^i$ for $i \leqslant k$ is a mazery. Then inequality (4.5) holds for $\mathcal{M}^{k+1}$.*

Proof. By Lemma 7.3, each rank r occurs for at most a constant number $n = \log_\tau \frac{2\tau}{\tau-1}$ values of k. For every such value but possibly the last one, the probability sum can only be increased as a result of the two operations of forming compound types. According to Lemma 7.7, the increase is upperbounded by $p(r)R^{-c_1/2}$. After these increases, the probability becomes at most $2np(r)R^{-c_1/2}$. The last contribution, due to the emerging type, is at most $p(r)/2$ by Lemma 7.6; clearly, if R_0 is sufficiently large, the total is still less than $p(r)$.[9] □

[7] More exactly, we need $R_0 \geqslant 71$.
[8] More exactly, we need $R_0 \geqslant 1803$.
[9] More exactly, we need $R_0 \geqslant 3$.

Lemma 7.9. *After choosing c_1, c_0, R_0 sufficiently large in this order, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery: then every compound hole type β' satisfies the hole lower bound (4.8).*

Proof. We will show that compound hole types in $\mathcal{M}^*$ satisfy (4.8) if their component types do (they are either in $\mathcal{M}$ or are formed in the process of going from $\mathcal{M}$ to $\mathcal{M}^*$). Consider the compound hole type β' where

$$\beta = \langle \alpha_1, \alpha_2, i \rangle.$$

Let $r_j = \mathit{Rank}(\alpha_j)$, then $r = \mathit{Rank}(\beta) = r_1 + r_2 - i$. Let $a \leqslant b < c$ and $b - a, c - b \leqslant 6\Delta^*$. Following the notation of Lemma 6.10, let $E_2(a, b, c, \beta)$ be the event that there is a $d \in [b, c-1]$ such that $[a, d]$ is a jump of $\mathcal{M}^*$, and a compound hole of type β' starts at d. That lemma assumes $c - a \leqslant 12\Delta^*$, which holds in our case. Let us check the condition $(\Delta^*)^\chi h(\alpha_i) \leqslant 0.5$. We have

$$h(\alpha_i) = c_3 \lambda^{-\chi r_i} \leqslant c_3 T^{-\chi}, \quad (\Delta^*)^\chi h(\alpha_i) \leqslant c_3 T^{-\chi(1-\delta\tau)}$$

which, due to (7.10), is always smaller than $1/2$ if R_0 is sufficiently large.[10] The condition $\overline{p}^* \leqslant \overline{p}$ of the lemma is satisfied automatically by the definitions. Hence all conditions of the lemma are satisfied. The conclusion is

$$\mathrm{Prob}(E_2(a, b, c, \beta)) \geqslant 0.01(1 - 2\overline{q}^* - 25\Delta^*\overline{p})(c-b)^\chi (e_{i+1} - e_i)^\chi h(\alpha_1) h(\alpha_2). \tag{7.23}$$

Le us show that for c_0 and then R_0 chosen sufficiently large, this is always larger than $(c - b)^\chi h(\beta)$. First we show

$$e_{i+1} - e_i \geqslant \lambda^i / 17. \tag{7.24}$$

Indeed, recall the definition of e_i in (5.9). For $i > 17$, we have

$$e_{i+1} - e_i = \lfloor \lambda^i \rfloor - \lfloor \lambda^{i-1} \rfloor \geqslant \lambda^i - \lambda^{i-1} - 1 = \lambda^i (1 - \lambda^{-1} - \lambda^{-i}) > 0.1\lambda^i.$$

For $i \leqslant 17$, we have $e_{i+1} - e_i \geqslant 1 \geqslant \lambda^i/17$. This proves (7.24). Using (7.24) gives

$$\begin{aligned} h(\alpha_i) &= c_3 \lambda^{-r_i \chi}, \\ (e_{i+1} - e_i)^\chi h(\alpha_1) h(\alpha_2) &\geqslant 17^{-\chi} c_3^2 \lambda^{-\chi(r_1 + r_2 - i)} = 17^{-\chi} c_3^2 \lambda^{-r\chi}. \end{aligned} \tag{7.25}$$

Note also that $25\Delta^*\overline{p} = 25 T^{\tau\delta - 1} < 0.25$ if R_0 is large enough.[11] Thus, the second factor on the right-hand side of (7.23) is $\geqslant 1 - 0.5 - 0.25 = 1/4$. Substituting into (7.23), we get the lower bound $\frac{1}{400 \cdot 17^\chi} c_3$ for the factors of $(c-b)^\chi h(r)$. This is $\geqslant 1$ if c_3 is sufficiently large. More exactly, we need

$$c_3 > 435.5, \tag{7.26}$$

satisfied by the choice in (7.4). □

Proof of Lemma 2.5. The construction of $\mathcal{M}^k$ is complete by the algorithm of Section 5, and the fixing of all parameters in the present section. The largest of all the lower bounds on R_0 in the footnotes was 3257, so we can choose $R_1 = 1.5R_0 = 5000$. This gives, via (7.2), the upper bound 10^{-400} on the initial wall probability.

We have to prove that every structure $\mathcal{M}^k$ is a mazery. The proof is by induction. We already know that the statement is true for $k = 1$: it was handled in Example 4.6. Assuming that it is true for all $i \leqslant k$, we prove it for $k + 1$.

Condition 4.1.1 is satisfied by the argument before Lemma 7.2. Condition 4.1.2 is satisfied by the form of the definition of the new types.

[10] More exactly, we need $R_0 \geqslant 3257$.

[11] More exactly, we need $R_0 \geqslant 67$.

Condition 4.3.1 is satisfied as shown in Lemma 5.3.

Condition 4.3.2 has been proved in Lemmas 5.4 and 5.6.

In Condition 4.3.3, inequality (4.5) has been proved in Lemma 7.8. Inequality (4.6) has been proved in Lemma 7.4. Inequality (4.7) has been proved in Lemma 7.5. Inequality (4.8) is proved for emerging walls in Lemma 7.6, and for compound walls in Lemma 7.9.

Condition 4.5.1 is satisfied trivially for the emerging type, (as pointed out in Lemma 6.5), and proved for the compound type in Lemma 6.8.

Condition 4.5.2 is satisfied via Lemma 5.2 (the Grate Lemma), as discussed after Lemma 5.5. There are some conditions on f, g, Δ required for this lemma. Of these, (5.2) follows from (7.11), while (5.3) follows from Lemma 7.5.

Let us show that the conditions preceding the Main Lemma 2.5 hold. Condition 2.1 is implied by Condition 4.5.2. Condition 2.2 is implied by Condition 4.3.2c. Condition 2.4 follows immediately from the definition of cleanness.

Finally, inequality (2.4) follows from (7.16). □

8. Proof of Lemma 5.2

Let $b_d(j)$ denote the starting point of wall $W_{d,j}$ for $j = 1, \dots, n_d$. Let $b_d(0), b_d(n_d + 1)$ denote the beginning and end of interval I_d. For convenience, sometimes we will write

$$b(j) = b_0(j), \quad c(j) = b_1(j).$$

Without loss of generality, assume $b_d(0) = 0$. Let

$$\begin{aligned} L_d(j) &= [b_d(j) + 2g, b_d(j+1)], \\ P_{i,j} &= (\{b_0(i+1)\} \times L_1(j)) \cup (L_0(i) \times \{b_1(j+1)\}) \end{aligned}$$

for $i = 0, \dots, n_0$, $j = 0, \dots, n_1$. Imagine the rectangle $I_0 \times I_1$ with the direction 0 running horizontally and the direction 1 vertically, divided into subrectangles by the vertical lines $x_0 = b(j)$ ($1 \leqslant j \leqslant n_0$), and the corresponding horizontal lines. The set $P_{m,n}$ is almost the whole upper right rim of the (m, n)th subrectangle: a segment of size $2g$ is missing from the left end of the top rim and from the bottom of the right rim. For induction purposes, we will prove a statement slightly stronger than the lemma. Let

$$\Gamma_0 = \bigcup_{m,n} \{ (x, y) \in P_{m,n} : y - \sigma x \geqslant 8g(m + n) \},$$

$$\Gamma_1 = \bigcup_{m,n} \{ (x, y) \in P_{m,n} : x - \sigma y \geqslant 8g(m + n) \}.$$

Let H be the set of left-clean points (x_0, x_1) (both x_0 and x_1 must be left-clean). We will show that

$$(\bigcup_{m,n} P_{m,n}) \cap \Gamma_0 \cap \Gamma_1 \cap H$$

is reachable. Let us first see that this is sufficient. Note that if $(x, y) \in P_{m,n}$ then $mf/3 < x$, and therefore $m + n < 3(x + y)/f$. Suppose that for $(x, y) \in \bigcup_{m,n} P_{m,n}$ with $x \geqslant y$ we have $(x, y) \notin \Gamma_0$. Then we have

$$y/x < \sigma + 24(x + y)g/(fx) \leqslant \sigma + 48g/f,$$

saying that $\mathit{minslope}(x, y) < \sigma^*$.

Our claim says that the reachable region is somewhat decreased from the “cone” $\{ u : \mathit{minslope}((0, 0), u) \geqslant \sigma \}$. Every time the lower side of the reachable region crosses a vertical line $x = b(i)$ or a horizontal line $y = c(j)$, it continues in the same direction, but after an upward shift by at most $2g$. The upper side gets shifted down similarly. The two conditions Γ_0 and Γ_1 are symmetric: the first one limits the lower half of the set (where $x \geqslant y$), the other one the upper half (where $y \geqslant x$). We will prove the claim by induction on $m+n$; the case $m+n=0$ is immediate from the reachability condition 4.5.2. Consider a point

$$u_1 = (x_1, y_1)$$

in $P_{m,n} \cap \Gamma_0 \cap \Gamma_1 \cap H$. Without loss of generality, assume $y_1 \leqslant x_1$. In the interval $b(m) \leqslant x \leqslant b(m+1)$, define the function

$$K(x) = \min(x, y_1 + (x - x_1)\sigma),$$

whose graph is a broken line below the diagonal $x = y$ made up of a part (maybe of length 0) of slope 1 followed by a part (maybe of length 0) of slope $\sigma < 1$, and ending in point u_1. We define the stripe

$$\{\, (x, K(x) - y) : b(m) \leqslant x \leqslant b(m+1),\ 0 \leqslant y \leqslant 8g \,\}$$

of vertical width $8g$, below this broken line. (See Figure 6.) It intersects the set $(\{b(m)\} \times L_1(n)) \cup (L_0(m) \times \{c(n)\})$. Assume that the intersection with $\{b(m)\} \times L_1(n)$ is longer. The size of this intersection segment is at least $2g$ (it is not $4g$ since $L_1(n)$ starts only at $c(n) + 2g$, not at $c(n)$. Its top edge is the point $(b(m), K(b(m)))$. Its subsegment

$$\{b(m)\} \times (K(b(m)) + [-1.5g, -0.5g])$$

contains therefore the starting point

$$u_0 = (b(m), y_0)$$

of an outer-clean hole fitting the wall $W_{0,m}$. We have

$$0.5g \leqslant K(b(m)) - y_0 \leqslant 1.5g. \tag{8.1}$$

The point u_0 is left-clean since $b(m)$ is the start of a wall and y_0 is the start of an outer-clean hole. Let

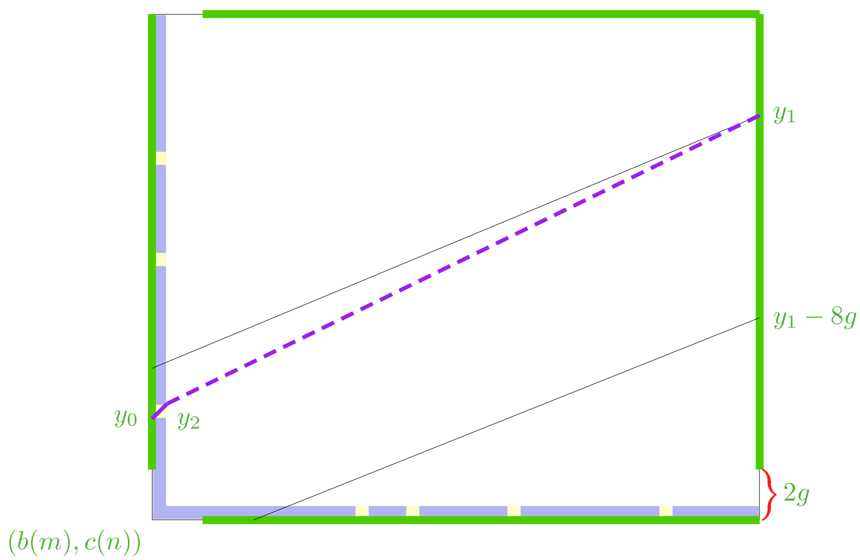
$$w_0, w_1$$

be the size of the wall $W_{0,m}$ and the size of the hole at y_0 respectively. Let us show that u_0 is reachable. Since $u_0 \in P_{m-1,n}$, if we show that $u_0 \in \Gamma_0 \cap \Gamma_1$ then the statement follows from the inductive assumption. By the definition of our stripe, we have $y_0 \leqslant b(m)$, hence $b(m) - \sigma y_0 \geqslant y_0 - \sigma b(m)$. Therefore for $u_0 \in \Gamma_0 \cap \Gamma_1$, we only need to show

$$y_0 - \sigma b(m) \geqslant 8g(m + n - 1). \tag{8.2}$$

By our assumptions, $u_1 \in \Gamma_0$, that is $y_1 - \sigma x_1 \geqslant 8g(m+n)$. We passed from u_0 to u_1 by moving horizontally by the amount w_0, moving up by at most $8g$ and then ascending with slope at least σ. From this, inequality (8.2) follows.

Then $u_0 \rightsquigarrow u_2 = (b(m) + w_0, y_0 + w_1) \stackrel{\text{def}}{=} (x_2, y_2)$. Also, x_2, y_2 are right-clean. If we show that $\mathit{minslope}(u_2, u_1) \geqslant \sigma$ then $u_2 \rightsquigarrow u_1$ follows. This can be done using the assumptions



y_1
$y_1 - 8g$
y_0
y_2
$2g$
$(b(m), c(n))$

$g > 2\Delta \geqslant 2w_i$ and (8.1):

$$\begin{aligned}
y_1 - K(b(m)) + 0.5g \leqslant y_1 - y_0 &\leqslant y_1 - K(b(m)) + 1.5g,\\
\sigma(x_1 - b(m)) \leqslant y_1 - K(b(m)) &\leqslant x_1 - b(m),\\
\sigma(x_1 - b(m)) + 0.5g \leqslant y_1 - y_0 &\leqslant x_1 - b(m) + 1.5g,\\
y_1 - y_2 &= y_1 - y_0 - w_1,\\
\sigma(x_1 - x_2) &\leqslant \sigma(x_1 - b(m)) + 0.5g - w_1\\
&\leqslant y_1 - y_2 \leqslant x_1 - b(m) + 1.5g - w_1\\
&= x_1 - x_2 + 1.5g - w_1 + w_0 \leqslant x_1 - x_2 + 2g.\\
\sigma \leqslant \frac{y_1 - y_2}{x_1 - x_2} &\leqslant 1 + \frac{2g}{x_1 - x_2} \leqslant 2 \leqslant 1/\sigma.
\end{aligned}$$

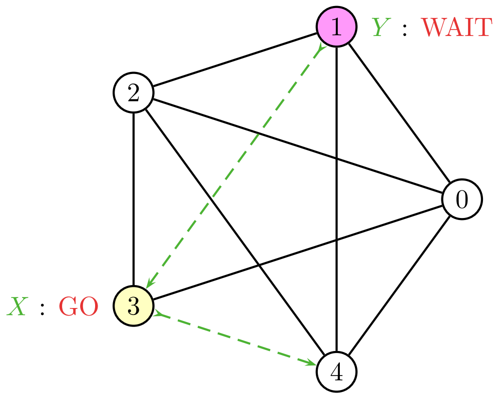
The case when the larger part of the width of the stripe intersects the horizontal segment $L_0(m) \times \{c(n)\}$, is similar.

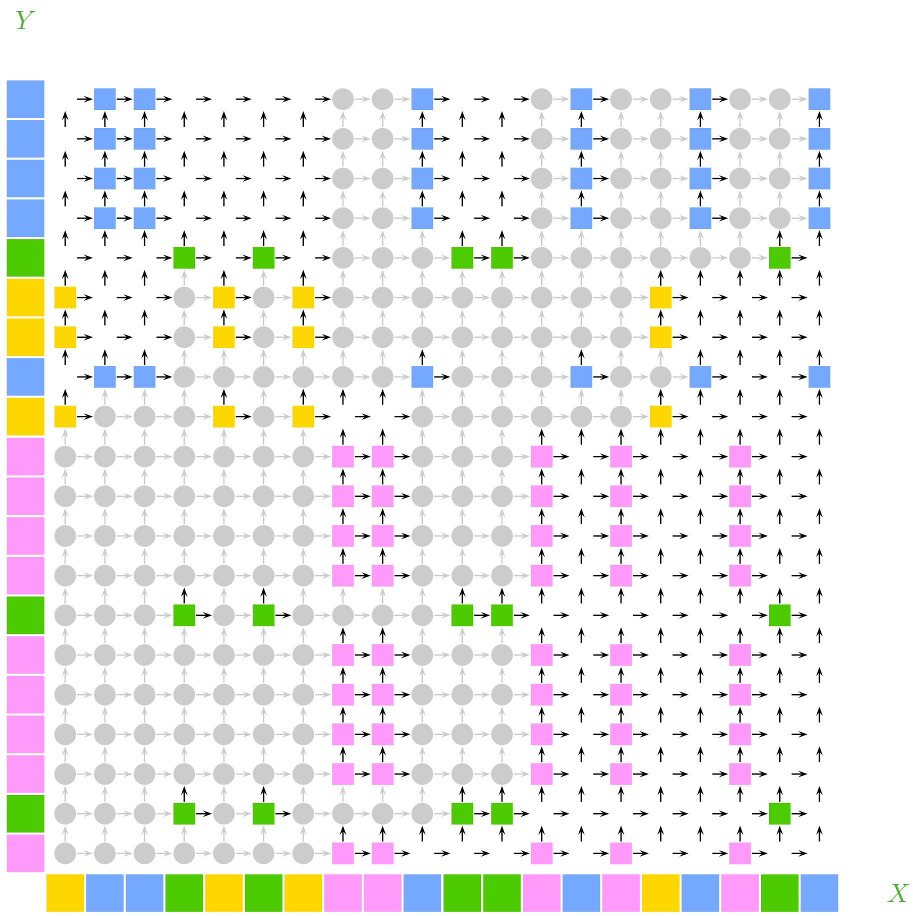
9. Related synchronization problems

The clairvoyant synchronization problem that has appeared first has also been introduced by Peter Winkler. We again have two infinite random sequences X_d for $d = 0, 1$ independent from each other. Now, both of them are random walks on the same graph. (See Figure 7.) Given delay sequences t_d, we say that there is a collision at (d, n, k) if $t_d(n) \leqslant t_{1-d}(k) < t_d(n+1)$ and $x_d(k) = x_{1-d}(n)$. Here, the delay sequence t_d can be viewed as causing the sequence X_d to stay in state $x(n)$ between times $t_d(n)$ and $t_d(n+1)$. A collision occurs when the two delayed walks enter the same point of the graph. This problem, called the clairvoyant demon problem, arose originally from a certain leader-election problem in distributed computing. Consider the case when the graph is the complete graph of size m. Since we have now just a random walk on a graph, there is no real number like p in the compatible sequences problem, that we can decrease in order to give a better chance of a solution. But, $1/m$ serves the same purpose. Simulations suggest that the walks do not collide if $m \geqslant 5$, and it is known that they do collide for $m = 3$. In paper [5], we prove that for sufficiently large m, the walks do not collide. The proof relies substantially on the technique developed here.

The clairvoyant demon problem also has a natural translation into a percolation problem, this time site percolation rather than edge percolation. (See Figure 8.) Consider the lattice $\mathbb{Z}_+^2$, and a graph obtained from it in which each point is connected to its right and upper neighbor. For each i, j, let us "color" the ith vertical line by the state $X_0(i)$, and the jth horizontal line by the state $X_1(j)$. A point (i, j) will be called blocked if $X_0(i) = X_1(j)$, if its horizontal and vertical colors coincide. The question is whether there is, with positive probability, an infinite directed path (moving only right and up) starting from $(0, 0)$ and avoiding the blocked points.

This problem permits an interesting variation: undirected percolation, allowing arbitrary paths in the graph, not only directed ones. This variation has been solved, independently, in [6] and [2]. On the other hand, the paper [4] shows that the directed problem has a different nature, since if there is percolation, it has power-law convergence (the undirected percolations have the usual exponential convergence).


1
Y : WAIT
2
0
X : GO
3
4



Y
X

10. Conclusions

One of the claims to interest in this dependent percolation problem is its power-law behavior over a whole range of parameter values p, not only at a critical point. Let us call $b(n)$ the probability that there is a path from $(0,0)$ to distance n but not further. As we indicated in Subsection 1.4, one can prove that

$$b(n) > n^{k_1 \frac{\log p}{p}}$$

for some k_1. Implicitly, one can see that our paper gives an upper bound $n^{-k_2/\delta} > b(n)$ for some k_2. Here, δ is clearly not smaller than p, but we do not know whether our proof can be refined to make δ approach p.

We could allow $\mathrm{Prob}\{ X_d(i) = 1 \} = p_d$ to be different in the two sequences, say $p_0 < p_1$. This describes the chat situation when one of the two speakers is more likely to speak than the other. It does not seem difficult to generalize the methods of the present paper to show that synchronization is possible if $p_0/(1-p_1)$ is small.

Acknowledgement. The author is grateful to Peter Winkler, Marton Balazs and John Tromp for valuable comments, and particularly the anonymous referee for his very careful reading that caught many errors.

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