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PACKING OF CONVEX SETS IN THE PLANE WITH A GREAT NUMBER OF NEIGHBOURS

By

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Let A be a convex domain in the plane. The Newton number $n(A)$ of A is defined as the greatest number of domains congruent with A which can be packed around A so as to touch A. We call a packing of domains congruent to A a *maximal* packing [1, 2] if each domain has $n(A)$ neighbours in the packing, i.e. denoting the packing with $\mathcal{A}$ for each $B \in \mathcal{A}$ there are exactly $n(A)$ elements of $\mathcal{A}$ touching B. L. FEJES TÓTH [2] conjectured that there is a universal constant K such that in each maximal packing the number of neighbours n is less than K. Here we prove this conjecture, making use of a theorem which seems to be of interest in itself. Our method gives the very crude estimate $K < 10^8$, altough nobody knows examples with $n > 21$.

In Section 2 we give an upper bound for the number of convex domains which can be packed around a convex domain in terms of the lower bound of the widths and the upper bound of the diameteres of the domains.

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Theorem 1 intuitively says that if we have a packing of convex domains of not very different size and shape in which every domain has a great number of neighbours n, then the packing contains a "star" consisting of approximately n domains with a common boundary point.

THEOREM 1. *Let $\mathcal{A}$ be a packing of convex domains in the plane, $C_1, C_2 > 0$ positive constants for which*

(i) *every domain has at least n neighbours,*

(ii) *for every $A \in \mathcal{A}$ the area of A is at least $C_1 n$,*

(iii) *for every $A \in \mathcal{A}$ the diameter of A is at most $C_2 n$.*

Then there is a $C > 0$ and an n_0 depending only on C_1, C_2, such that for $n > n_0$ there is a point in the plane common to at least $n - C\sqrt{n}$ members of $\mathcal{A}$.

Before proving Theorem 1 we show that it implies the conjecture of Fejes Tóth.

THEOREM 2. *The conjecture of Fejes Tóth is true.*

PROOF. Let $\mathcal{A}$ be a maximal packing. We can choose the unit of measure so that the area of the elements of $\mathcal{A}$ (being congruent in this case) should be equal to the Newton number $n = n(A)$ of an element A of $\mathcal{A}$. Let us denote the diameter of A by d and the width of A by w. Then the area of A is at least $dw/2$:

$$dw/2 \leqq n.$$

On the other hand, it is obvious that the Newton number n is not less than $2d/w$:

$$2d/w \leqq n.$$

Multiplying these inequalities we obtain $d \leqq n$. Thus $\mathcal{A}$ satisfies the conditions of Theorem 1 with $C_1 = C_2 = 1$. Hence there is a C and an n_0 such that for every $n > n_0$ there exists a point in which at least $n - C\sqrt{n}$ domains meet. Thus A must have a vertex with an angle $\alpha \leqq \frac{2\pi}{n - C\sqrt{n}}$. But the Newton number of a domain having an angle α is not less than $\left[\frac{2\pi}{\alpha}\right] + \left[\frac{\pi}{\alpha}\right] - 1$, because at the respective vertex $\left[\frac{2\pi}{\alpha}\right] - 1$ congruent domains can be placed and at the point "opposite" to this vertex $\left[\frac{\pi}{\alpha}\right]$ further ones. (Fig. 1.)

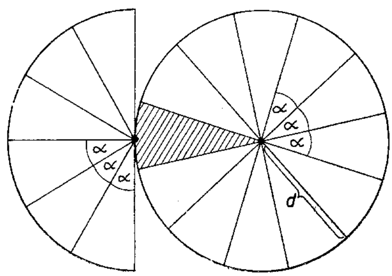

Fig. 1

Hence

$$n \geqq \frac{3}{2}(n - C\sqrt{n}) - 2 \quad \text{i.e.} \quad \sqrt{n} \leqq 3C + \frac{4}{\sqrt{n}}$$

This completes the proof of Theorem 2.

NOTE. The idea of considering the angle α is due to TIBOR ELEKES.

In what follows we denote the number of elements of a set X by $|X|$. In order to prove Theorem 1 we first prove the simple combinatorial

LEMMA 1. *Let both* $\mathfrak{D}$ and $\mathfrak{E}$ *be classes of domains (they can have common elements), D, E, γ constants, and $n \geqq 1$ an integer such that*

(1) $$|\mathfrak{D}| \geqq Dn, \qquad 2 \leqq |\mathfrak{E}| \leqq En$$

and for each $A \in \mathfrak{D}$ there are at least $\gamma n \geqq 2$ elements of $\mathfrak{E}$ touching A. Then there exist two different elements B_1, B_2 of $\mathfrak{E}$ and a subclass $\mathfrak{F}$ of $\mathfrak{D}$ such that B_1 and B_2 touch every element of $\mathfrak{F}$ and

$$|\mathfrak{F}| \geqq \frac{\gamma^2 D}{2E^2} n. \tag{2}$$

PROOF. Let us consider the following class of pairs of domains: $\mathfrak{P} = \{(A, B) : A \in \mathfrak{D}, B \in \mathfrak{E}, B \text{ touches } A\}$. Then $|\mathfrak{P}| \geqq D\gamma n^2$, but since there are only En different B's, there exist a domain $B_1 \in \mathfrak{E}$ and a subset $\mathfrak{F}_1$ of $\mathfrak{D}$ such that B_1 touches every element of $\mathfrak{F}_1$ and $|\mathfrak{F}_1| \geqq \frac{D\gamma}{E} n$. Putting $\mathfrak{D}_1 = \mathfrak{F}_1$, $\mathfrak{E}_1 = \mathfrak{E} - \{B_1\}$, $D_1 = \frac{D\gamma}{E}$, $E_1 = E$, $\gamma_1 = \gamma - \frac{1}{n}$ the same method gives us a domain $B_2 \in \mathfrak{E}_1$ and a set $\mathfrak{F}_2 \subset \mathfrak{F}_1$ such that B_1 and B_2 touch every element of $\mathfrak{F}_2$ and

$$|\mathfrak{F}_2| \geqq \frac{D}{E^2} \gamma \left(\gamma - \frac{1}{n}\right) n.$$

Thus putting $\mathfrak{F} = \mathfrak{F}_2$ and taking into consideration that $\gamma n \geqq 2$ we obtain the statement of the lemma.

The following lemma is of topogical character.

LEMMA 2. *Let $A_1, \ldots, A_k, B_0, B_1, B_2$ be a packing of convex domains in the plane, $k \geqq 3$. Suppose that every B_j touches every A_i. Then there is a finite set X of "representing" points, $|X| \leqq 4$ and an i_1 such that*

(i) *if $i \neq i_1$ then $A_i \cap X \neq 0$*

(ii) *if $x \in X$ then x is contained in at least two B's.*

PROOF. Suppose for a moment that B_1 and B_2 do not touch each other. Let x_i, y_i be common points of A_i and B_1 and A_i and B_2, respectively. Let l_i denote the line-segment connecting x_i and y_i. The segments l_i divide the part of the plane lying outside B_1 and B_2 into k (possibly empty) cells. If a cell is limited by the segments l_i and l_j, and x_i, x_j, y_j, y_i is the order of these four points on its boundary in the positive direction, then we denote it by (x_i, x_j, y_j, y_i). B_0 must lie in one of these cells $(x_{i_0}, x_{j_0}, y_{j_0}, y_{i_0})$ and if it touches an A_i other than A_{i_0} and A_{j_0} then one of the points $x_{i_0}, x_{j_0}, y_{j_0}, y_{i_0}$ is common to B_0 and A_i. In this case put $X = B_0 \cap \{x_{i_0}, x_{j_0}, y_{j_0}, y_{i_0}\}$.

If B_1 and B_2 are touching each other in a line-segment (u_1, u_2) (which can degenerate to a point) then the l_i's not terminating in any u_j divide the part of the plane lying outside B_1 and B_2 into (not necessarily k) cells of type (x_i, x_j, y_j, y_i) or of type (x_i, u_j, y_i). B_0 must lie in one of these cells. If it lies in a cell of the first type $(x_{i_0}, x_{j_0}, y_{j_0}, y_{i_0})$ then $X = B_0 \cap \{x_{i_0}, x_{j_0}, y_{j_0}, y_{i_0}\}$. If it lies in $(x_{i_0}, u_{j_0}, y_{i_0})$ then $X = (B_0 \cap \cap \{x_{i_0}, y_{i_0}\}) \cup \{u_1, u_2\}$. If all the l_i's terminate in an u_j then we choose $X = \{u_1, u_2\}$. It is easy to show that these choices satisfy (i) and (ii).

COROLLARY. *If the conditions of the lemma hold then there exists a point x contained in at least two B's and* $\frac{k-1}{4}$ *A's. Thus if $k \geqq 5$, then x is contained in* $\frac{k}{5}$ *A's.*

PROOF OF THEOREM 1. Let $B_0 \in \mathcal{A}$. We call $B \in \mathcal{A}$ a *neighbour* of B_0 if it touches B_0. B is said to be a *second neighbour* of B_0 if it is a neighbour of a neighbour of B_0 other than B_0. In the first step of the proof we show that there is a point x common to "many" members of $\mathcal{A}$. In the second step we show that the number of the domains containing x is at least $n - C\sqrt{n}$.

1. Let $\mathfrak{D}$ be the class of all neighbours of B_0 and $\mathfrak{S}$ the class of the second neighbours of B_0. Then $|\mathfrak{D}| \geqq n$ and every element of $\mathfrak{D}$ has at least $n-1$ neighbours in $\mathfrak{S}$. Evidently, all the elements of $\mathfrak{S}$ are contained in a certain circle of radius $3C_2n$ (see (iii)). So we have

$$|\mathfrak{S}| \leqq \frac{(3\,C_2 n)^2 \pi}{C_2 n} = E'n. \tag{3}$$

Hence by Lemma 1 for a sufficiently large n there are two domains $B_1 \in \mathfrak{S}$ and $B_2 \in \mathfrak{S}$, and a set $\mathfrak{F} \subset \mathfrak{D}$ such that every element of $\mathfrak{F}$ touches B_0, B_1 and B_2 and

$$|\mathfrak{F}| \geqq n \cdot \frac{(1-1/n)^2 \cdot 1}{2E'^2} \geqq \frac{1}{3E'^2} n.$$

Then, by Corollary of Lemma 2, for a sufficiently large n there is a point x contained in at least

$$n \cdot \frac{1}{15E'^2} = n \cdot D'$$

elements of $\mathfrak{F}$.

2. Suppose that the number of domains containing x is less than $n - \gamma n$. Let $\mathfrak{G}$ denote the set of elements of $\mathfrak{D}$ containig x and $\mathfrak{G}'$ its complementary in $\mathfrak{S}$. Then $|\mathfrak{G}| \geqq D'n$ and every element of $\mathfrak{G}$ must have at least $\gamma n - 1 = \left(\gamma - \frac{1}{n}\right) n$ neighbours in $\mathfrak{G}'$. Since $|\mathfrak{G}'| \leqq E'n$, in case $\gamma n - 1 \geqq 2$ Lemma 1 guarantees the existence of two domains $B_1', B_2' \in \mathfrak{G}'$ and a set $\mathfrak{F}' \subset \mathfrak{G}$ such that every element of $\mathfrak{F}'$ touches B_0, B_1' and B_2', and $|\mathfrak{F}'| \geqq \frac{(\gamma - 1/n)^2 D'}{2E'^2} \cdot n$. If γn is not very small, the last result assures that $|\mathfrak{F}'| \geqq \frac{\gamma^2 D'}{3E'^2} n$. If $|\mathfrak{F}'| \geqq 5$ then we can apply the Corollary of Lemma 2 obtaining a point x' contained at least $\frac{|\mathfrak{F}'|}{5} \geqq \gamma^2 \frac{D'}{15E'^2} n$ elements of $\mathfrak{F}'$. Furthermore, we know that x' is contained in at least two B's, so it is different rom x. But the number of the elements containing both x and x' cannot be more

than 2. Hence

$$\gamma^2 \frac{D'}{15E'^2} n \leqq 2, \qquad \gamma n \leqq \sqrt{\frac{30\,E'^2}{D^2}} \sqrt{n} = C\sqrt{n}.$$

This completes the proof of the theorem.

NOTE. It is interesting to observe that an analogue of Theorem 1 in the space does not hold. To show this, let us consider the following example. Put together $2n^2$ quadratic prisms with lengths of edges n, n, $\frac{1}{n}$ according to Figure 2. They

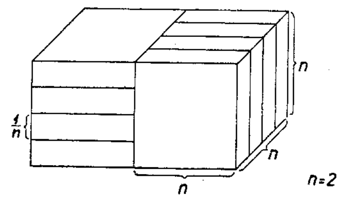

Fig. 2

have volume n and number of neighbours n^2 and this is the order of magnitude of the number of neighbours allowed by considerations of volume.

2.

We shall prove

THEOREM 3. *Let E be a bounded convex domain with perimeter l. Let $\mathcal{B}$ be a set of convex domains touching E which have no inner points in common with each other and with E. For every element B of $\mathcal{B}$ we assume that*

(i) *the width of B is at least* 1,

(ii) *the diameter of B is at most $h > 1$.*

Then

1. *there are universal constants $\bar{K}$, $\bar{C}$, $\bar{D} > 0$ such that*

$$|\mathcal{B}| \leqq \bar{K} l \log h + \bar{C} h + \bar{D} l, \tag{4}$$

2. *there are universal constants K, C, $D > 0$ such that for every E there exists a $\mathcal{B}$ satisfying the conditions of the theorem, for which*

$$|\mathcal{B}| \geqq K l \log h + C h + D l. \tag{5}$$

PROOF. 1. Let us divide $\mathcal{B}$ into the following subclasses $\mathcal{B}_0, \mathcal{B}_1, \ldots$: $\mathcal{B}_i = \{B \in \mathcal{B}$: $h \cdot 2^{-i} \geqq$ the diameter of $B > h \cdot 2^{-i-1}\}$. Obviously there are at most $[\log_2 h] + 1$ such classes. The area of the elements of $\mathcal{B}_i$ is at least $h \cdot 2^{-i-2}$. The elements of $\mathcal{B}_i$ are contained in a parallel domain of E of radius $h \cdot 2^{-i}$. The area of this domain is $l \cdot h \cdot 2^{-i} + \pi \cdot h^2 \cdot 2^{-2i}$. Hence $|\mathcal{B}_i| \leqq 4l + 4\pi h \cdot 2^{-i}$ showing that

$$|\mathcal{B}| \leqq 4l([\log_2 h] + 1) + 8\pi h \leqq 4l \log_2 h + 8\pi h + 4l. \tag{6}$$

2. Let E be an arbitrary domain. Let AB be one of its longest diameters. Writing $\overline{AB} = d$ we obviously have $d \geqq \frac{l}{4\pi}$. At the points A, B approximately $4\pi h$ domains can be placed in fan-form. Perpendicularly to the diameter AB approximately $\frac{2d}{3} \leqq \frac{l}{6\pi}$ rectangles of side-length h, 3 can be placed, each of them containing

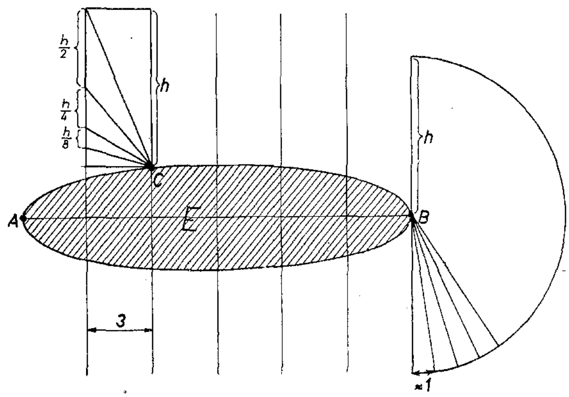

Fig. 3

max (3, $[\log_2 h]$) triangles (as at the point C in Figure 3). So we can place around E approximately

$$4\pi h + \frac{l}{6\pi}\left(\frac{1}{2}[\log_2 h] + 1\right) \geqq \frac{l}{12\pi}\log_2 h + \frac{l}{12\pi} + 4\pi h$$

domains of the allowed length and width.

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